14. From the following data find $\frac{dy}{dx}$ at $x = 0.09$.

x	0	0.05	0.10	0.15	0.20	2.0
y	0.0000	0.1001	0.2013	0.3045	0.4107	0.5211

[Ans. 1.99999]

15. Find the first and second derivatives of the function $y = f(x)$ tabulated below at the point $x = 1.1$

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0.00	0.1280	0.5440	1.2960	2.4320	4.00

[Ans. 0.630; 6.6]

❑ NUMERICAL INTEGRATION

The term Numerical integration is the numerical evaluation of a definite integral

$$A = \int_a^b f(x)\, dx$$

where 'a' and 'b' are given constants and $f(x)$ is a function given analytically by a formula or empirically by a table of values. Geometrically, A is the area under the curve of $f(x)$ between the ordinates $x = a$ and $x = b$.

But in engineering problems we frequently come across the integrals whose integrand is an empirical function given by a table. In these cases we may use a numerical method for approximate integration. When we apply numerical integration to a function of a single variable, the process is sometimes called **mechanical quadrature;** when we apply numerical integration to the computation of a double integral involving a function of two independent variables it is called **mechanical cubature.**

The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrand by an interpolation formula and then integrating this formula between the given limits. Thus, to find the value of the definite integral $\int f(x)\,dx$ (or) $\int_a^b y\,dx$ we replace the function $f(x)$ (or y) by an interpolation formula, usually one involving differences, and then integrate this formula between the limits a and b. In this way we can derive quadrature formulae for the approximate integration of any function for which numerical values are known.

Of the many possible quadrature formulae, here we shall derive some of the simplest and most useful one.

Quadrature Formula for Equidistant Ordinates

Consider the Newton's forward difference formula

$$y(x) = y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula can also be written by replacing n by u as

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots (1)$$

Let $y = y(x)$ ... (2) be the given function.

Let us now integrate (2) over n equidistant intervals of width $h (= \Delta x)$.

i.e., $$\int_{x_0}^{x_0+nh} y(x)\,dx = ?$$

$$\text{Let } x = x_0 + uh$$

$$\therefore dx = h\,du$$

TRAPEZOIDAL RULE

Putting $n = 1$ in (A), we get

$$\int_{x_0}^{x_0+h} y(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

(neglecting higher order differences)

$$= \frac{h}{2}[2y_0 + \Delta y_0] = \frac{h}{2}[y_0 + (y_0 + \Delta y_0)]$$

$$= \frac{h}{2}[y_0 + y_1] \quad \ldots (1)$$

In the interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} y(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$

$$= \frac{h}{2}[2y_1 + \Delta y_1] = \frac{h}{2}[y_1 + (y_1 + \Delta y_1)]$$

$$= \frac{h}{2}[y_1 + y_2] \quad \ldots (2)$$

....

.... etc

....

$$\int_{x_0+(n-1)h}^{x_0+nh} y(x)\,dx = \frac{h}{2}[y_{n-1} + y_n] \quad \ldots (3)$$

Adding (1), (2) and (3), we get

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right] \quad \ldots (A)$$

This is called the **Trapezoidal Rule.**

Note: The trapezoidal rule is the simplest of the formulae for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval *h*.

TRUNCATION ERROR IN THE TRAPEZOIDAL RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \quad \ldots (1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'' \ldots$ etc are the values of y', y'', etc at $x = x_1$.

$$\therefore \int_{x_1}^{x_2} y\, dx = \int_{x_1}^{x_2} \left[y_1 + \frac{(x-x_1)}{1!} y_1' + \frac{(x-x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x-x_1)^2}{2!} y_1' + \frac{(x-x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_2}$$

$$= y_1 (x_2 - x_1) + \frac{(x_2 - x_1)^2}{2!} y_1' + \frac{(x_2 - x_1)^3}{3} y_1'' + \ldots$$

$$= h y_1 + \frac{h^2}{2!} y_1' + \frac{h^3}{3!} y_1'' + \quad \ldots (2)$$

where $h = x_2 - x_1$

Now, A_1 = area of the trapezium in the interval (x_1, x_2)

$$= \frac{1}{2} h (y_1 + y_2) \quad \ldots (3)$$

Putting $x = x_2$ and $y = y_2$ in (1), we get

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \ldots \quad \ldots (4)$$

where $h = x_2 - x_1$

Substituting (4) in (3), we get

$$A_1 = \frac{h}{2}\left[2y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \dots\right]$$

$$= hy_1 + \frac{h^2}{2!}y_1' + \frac{h^3}{2 \times 2!}y_1'' + \dots \quad \dots (5)$$

(2) − (5) ⇒

$$\int_{x_1}^{x_2} y\,dx - A_1 = \left(\frac{1}{3!} - \frac{1}{2 \times 2!}\right) h^3 y_1'' + \dots$$

$$= \frac{-h^3}{12} y_1'' + \dots$$

i.e., Principal part of the error in (x_1, x_2)

$$= \frac{-h^3}{12} y_1''$$

Similarly principal part of the error in the interval (x_2, x_3)

$$= \frac{-h^3}{12} y_2'' \text{ and so on.}$$

Hence the total error $E = \frac{-h^3}{12}\left[y_1'' + y_2'' + \dots + y_n''\right]$

$$\therefore E < \frac{-nh^3}{12} y''(\xi)$$

Where $y''(\xi)$ is the largest of the n quantities $y_1'', y_2'', \dots, y_n''$.

i.e., $E < \frac{-nh^3}{12} y''(\xi) = -\frac{(b-a)h^2}{12} y''(\xi) \quad \left[\because n = \frac{b-a}{h}\right]$

$\therefore$ **Error in the trapezoidal rule is of the order h^2.**

Example 1

Compute the value of the definite integral $\int_4^{5.2} \log_e x \, dx$ **or** $\int_4^{5.2} \ln x dx$ **using trapezoidal rule.**

Solution

Divide the interval of integration into six equal parts each of width 0.2 *i.e.*, $h = 0.2$. The values of the function $y = \ln x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\ln x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By **Trapezoidal rule,** we have

$$\int_4^{5.2} \ln x \, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}[(1.386294 + 1.648658) + 2(1.435084 + 1.481604 + 1.526056 + 1.568616 + 1.609437)]$$

$$= (0.1)[3.034952 + 15.241562]$$

$$\int_4^{5.2} \ln x \, dx = \mathbf{1.8276544}$$

Example 2

Evaluate $\int_0^1 e^{-x^2}\,dx$ ***by dividing the range of integration into 4 equal parts using trapezoidal rule.*** *[Nov. '91, Nov. '89]*

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By **Trapezoidal rule** we have

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}[1.3678 + 2(2.2876)] = (0.125)(5.943)$$

$$\int_0^1 e^{-x^2}\,dx = \mathbf{0.7428}$$

Example 3

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By **Trapezoidal rule** we have,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4) \right]$$

$$= \frac{0.2}{2}[1.5 + 2(3.1687)] = (0.1)(7.8374)$$

$$\int_0^1 \frac{dx}{1+x^2} = \mathbf{0.78374}$$

We know that

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4} \qquad \therefore \pi = 4\int_0^1 \frac{dx}{1+x^2}$$

$$= 4(0.78374) \qquad \text{[From Trapezoidal Rule]}$$

$$\therefore \quad \pi = \mathbf{3.13496}$$

Example 4

Using Trapezoidal rule evaluate $\int_{0.6}^{2} y\, dx$ ***from the following table.***

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

Solution

Here $h = 0.2$

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Trapezoidal rule, we have

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\left[(y_0 + y_7) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)\right]$$

$$= \frac{0.2}{2}\left[13.68 + 2\,(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)\right]$$

$$= (0.1)\,[79.22]$$

$$\int_{0.6}^{2} y\,dx = 7.922$$

SIMPSON'S $\frac{1}{3}$ RULE

Putting $n = 2$ in the above relation (A) (Refer Pg. No. 3.37) and neglecting all differences above the second we get,

$$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6\,(y_1 - y_0) + y_2 - 2y_1 + y_0}{6}\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\therefore \int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \qquad \ldots (1)$$

Similarly for the next two intervals $x_0 + 2h$ to $x_0 + 4h$ we get,

$$\int_{x_0+2h}^{x_0+4h} y(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] \qquad \ldots (2)$$

In general,

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\, dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right] \qquad \ldots (3)$$

Adding all the above integrals (1), (2), (3) we get,

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots) + y_n \right]$$

$$= \frac{h}{3}\left[y_0 + y_n + 4\,(\text{sum of odd ordinates}) + 2\,(\text{sum of even ordinates}) \right]$$

This is called Simpson's one third rule or Simpson's $\frac{1}{3}$ rule.

Note 1 : **When using this formula the student must bear in mind that the interval of integration must be divided into an even number of subintervals of width h.**

Note 2 : **Simpson's $\frac{1}{3}$ rule is also called a closed formula, since the end point y_0 and y_1 are also included in the formula.**

❑ SIMPSON'S THREE – EIGHTH RULE :

Putting $n = 3$ in (A) (Refer Pg. No. 3.37) and neglecting the higher order differences above the third we get

$$\int_{x_0}^{x_0+nh} y(x)\, dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3}) \right].$$

This is known as **Simpson's three - eighth rule.**

Note : **This rule can be applied only if the number of subintervals is a multiple of 3.**

TRUNCATION ERROR IN SIMPSON'S RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \quad \ldots (1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'', \ldots$ *etc.* are the values of $y', y'', \ldots$ *etc.* at $x = x_1$.

Hence

$$\int_{x_1}^{x_3} y\,dx = \int_{x_1}^{x_3} \left[y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x - x_1)^2}{2!} y_1' + \frac{(x - x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_3}$$

$$= y_1 (x_3 - x_1) + \frac{(x_3 - x_1)^2}{2!} y_1' + \frac{(x_3 - x_1)^3}{3!} y_1'' + \ldots$$

$$= 2hy_1 + \frac{(2h)^2}{2!} y_1' + \frac{(2h)^3}{3!} y_1'' + \frac{(2h)^4}{4!} y_1''' + \frac{(2h)^5}{5!} y_1^{iv} + \ldots$$

$$[\because x_2 - x_1 = h; \therefore x_3 - x_1 = 2h]$$

$$= 2hy_1 + 2h^2 y_1' + \frac{4h^3}{3} y_1'' + \frac{2h^4}{3} y_1''' + \frac{4h^5}{15} y_1^{iv} + \ldots \quad \ldots (2)$$

Now, Area A_1 = area over the first double strip by Simpson's $\frac{1}{3}$ rule

$$= \frac{1}{3} h (y_1 + 4y_2 + y_3) \quad \ldots (3)$$

Putting $x = x_2$ and therefore $y = y_2$ in (1), we get,

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \quad \ldots (4)$$

where $h = x_2 - x_1$

Example 2

Evaluate $\int_0^1 e^{-x^2}\,dx$ ***by dividing the range of integration into 4 equal parts using Simpson's rule.*** [Nov. '91, Nov. '89]

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1 y_0	0.9394 y_1	0.7788 y_2	0.5698 y_3	0.3678 y_4

By **Simpson's rule** we have

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{3}\left[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)\right]$$

$$= \frac{0.25}{3}\left[1.3678 + 1.5576 + 6.0368\right]$$

$$\int_0^1 e^{-x^2}\,dx = \mathbf{0.7468}$$

Example 3

Find the value of $\int_0^{\pi/2} \sqrt{1 - 0.162 \sin^2 x}\,dx$***, using Simpson's one third rule.***

Solution

Let us divide the interval of integration into 6 equal subintervals

$$\text{i.e., } h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12} = 15°$$

The values of the function $y = \sqrt{1 - 0.162 \sin^2 x}$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1.0000	0.9946	0.9795	0.9586	0.9373	0.9213	0.9154
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^{\pi/2} y\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^{\pi/2} \sqrt{1 - 0.162 \sin^2 x}\,dx = \frac{\pi}{36}[(1.0000 + 0.9154) + 4(0.9946 + 0.9586 + 0.9213) + 2(0.9795 + 0.9373)]$$

$$\int_0^{\pi/2} \sqrt{1 - 0.162 \sin^2 x}\,dx = 1.5051$$

Example 4

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\left[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)\right]$$

$$= \frac{0.25}{3}\left[(0+0.5) + 2(0.22222) + 4(0.06154 + 0.39560)\right]$$

$$= \frac{0.25}{3}\left[0.5 + 0.44444 + 1.82856\right]$$

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = 0.231083$$

We know that,

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}(\log 2 - \log 1) = \frac{1}{3}\log_e 2$$

$$\therefore \log 2^{\frac{1}{3}} = \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$\log 2^{\frac{1}{3}} = 0.231083$$

Example 5

When a train is moving at 30 metres per second steam is shut off and brakes are applied. The speed of the train (V) in metres per second after t seconds is given by

t	0	5	10	15	20	25	30	35	40
V	30	24	19.5	16	13.6	11.7	10.0	8.5	7.0

Using Simpson's rule determine the distance moved by the train in 40 secs.

Solution

We know that velocity is the rate of change displacement.

$V = \frac{ds}{dt}$ or $ds = V\,dt$

Here we have to find the total distance moved by the train in 40 secs.

$\therefore S = \int_0^{40} V\,dt$

The given table is

t	0	5	10	15	20	25	30	35	40
V	30	24	19.5	16	13.6	11.7	10.0	8.5	7.0
	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8

By Simpson's rule we have

$$S = \int_0^{40} V\,dt$$

$$= \frac{h}{3}[(V_0 + V_8) + 2(V_2 + V_4 + V_6) + 4(V_1 + V_3 + V_5 + V_7)]$$

$$= \frac{5}{3}[37 + 2(19.5 + 13.6 + 10.0) + 4(24 + 16 + 11.7 + 8.5)]$$

$$= \frac{5}{3}[37 + 86.2 + 240.8] = \textbf{606.66 metres.}$$

$\therefore$ Distance moved by the train in 40 secs = 606.66 *m*.

Example 6

Given $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$. Use Simpson's rule to find an approximate value of $\int_0^4 e^x\,dx$. Also compare your result with the exact value of the integral.

[AMIE, S '88]

Solution

The given values can be arranged in the form of table as given below.

x	0	1	2	3	4
$y = e^x$	1	2.72	7.39	20.09	54.60
	y_0	y_1	y_2	y_3	y_4

By Simpson's rule, we have

$$\int_0^4 e^x\,dx = \frac{h}{3}[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)]$$

$$= \frac{1}{3}[55.60 + 14.78 + 4(2.72 + 20.09)]$$

$$= \frac{1}{3}[70.38 + 91.24]$$

$$\int_0^4 e^x\,dx = 53.8733$$

Now by ordinary integration we get

$$\int_0^4 e^x\,dx = (e^x)_0^4 = e^4 - e^0 = 54.598 - 1$$

$$\int_0^4 e^x\,dx = 53.598$$

Example 7

A river is 80 feet wide. The depth 'd' in feet at a distance x feet from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross section of the river using Simpson's rule. *[AMIE S' 76]*

Solution

Here $h = 10$. The given table is

x	0	10	20	30	40	50	60	70	80
$y = d$	0	4	7	9	12	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By **Simpson's $\frac{1}{3}$ rule**, we have

$$\text{Area of cross-section} = \int_0^{80} y\,dx$$

$$= \frac{h}{3}\left[(y_0 + y_8) + 2\,(y_2 + y_4 + y_6) + 4\,(y_1 + y_3 + y_5 + y_7)\right]$$

$$= \frac{10}{3}\,[3 + 2\,(33) + 4\,(36)]$$

Area of cross section = 710 sq. feet.

Example 8

Evaluate $\int_{0.2}^{1.4} (\sin x - \ln x + e^x)\, dx$ *by Simpson's* $\frac{1}{3}$ *rule.*

Solution

Let us divide the interval of integration into twelve equal parts by taking $h = 0.1$. Now the table of values of the given function $y = \sin x - \ln x + e^x$ at each point of subdivision is as given below.

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	3.02951	2.84936	2.79754	2.82130	2.89754	3.01465	3.16604
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
x	0.9	1.0	1.1	1.2	1.3	1.4	
y	3.34830	3.55935	3.80007	4.06984	4.37050	4.70418	
	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	

By **Simpson's $\frac{1}{3}$ rule**, we have,

$$\int_{0.2}^{1.4} y\,dx = \frac{h}{3}\left[(y_0 + y_{12}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11})\right]$$

$$= \frac{0.1}{3}\left[7.73369 + 2(16.49077) + 4(20.20418)\right]$$

$$= 4.05106$$

$$\therefore \int_{0.2}^{1.4} (\sin x - \ln x + e^x)\,dx = 4.05106$$

Example 9

Use Simpson's $\frac{1}{3}$ rule to estimate the value of $\int_{1}^{5} f(x)\,dx$ given

x	1	2	3	4	5
$y = f(x)$	13	50	70	80	100
	y_0	y_1	y_2	y_3	y_4

Solution

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{1}^{5} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{1}{3}\left[(13 + 100) + 2(70) + 4(50 + 80)\right]$$

$$= \frac{1}{3}\left[113 + 140 + 520\right]$$

$$\int_{1}^{5} f(x)\,dx = 257.67$$

Example 10

Evaluate $\int_1^4 f(x)\,dx$ *from the following table by Simpson's* $\frac{3}{8}$ *rule.*

x	1	2	3	4
$y=f(x)$	1	8	27	64
	y_0	y_1	y_2	y_3

Solution

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_1^4 f(x)\,dx = \frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right]$$

$$= \frac{3(1)}{8}\left[1+3(8)+3(27)+64\right]$$

$$= \frac{3}{8}\left[1+24+81+64\right] = \frac{3}{8}[170]$$

$$\int_1^4 f(x)\,dx = 63.75$$

Example 11

Evaluate $\int_0^{\pi/2} \sin x\,dx$*, using Simpson's* $\frac{3}{8}$ *rule.*

Solution

To use Simpson's $\frac{3}{8}$ rule the number of subintervals should be a multiple of 3. Hence we divide the interval of integration $\left(0,\frac{\pi}{2}\right)$ into 9 subintervals of width $\frac{\pi}{18}$. Let $y=\sin x$. The values of the function $y=\sin x$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
y	0	0.1736	0.3420	0.5000	0.6428	0.7660	0.8660	0.9397	0.9848	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_0^{\pi/2} y\,dx = \frac{3h}{8}\left[(y_0 + y_9) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) + 2(y_3 + y_6)\right]$$

$$\therefore \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{48}\left[(0 + 1) + 3(0.1736 + 0.3420 + 0.6428 + 0.7660 + 0.9397 + 0.9848) + 2(0.5 + 0.8660)\right]$$

$$= \frac{\pi}{48}(15.2787)$$

$$\int_0^{\pi/2} \sin x\,dx = 0.999988$$

Checking : $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$

Example 12

The velocity V of a particle at distances from a point on its path is given by the table :

S (feet)	0	10	20	30	40	50	60
V (feet/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's one-third rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

[AMIE S'9?]

Solution

We know that the rate of change of displacement is velocity.

i.e., $\frac{ds}{dt} = V$

(or) $ds = V\,dt$

i.e., $dt = \frac{1}{V}\,ds$ ... (1)

Here we want to find the time taken to travel 60 feet. Therefore integrate (1) from 0 to 60, we get $\int_0^{60} dt = \int_0^{60} \frac{1}{V}\,ds$

The time taken to travel 60 feet is

$$t = \int_0^{60} \frac{1}{V}\,ds = \int_0^{60} y\,dx$$

The given table can be written as given below.

$x\ (= s)$	0	10	20	30	40	50	60
$y = \frac{1}{V}$	0.02127	0.01723	0.01563	0.01538	0.01639	0.01923	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By **Simpson's one third rule,** we have

$$\int_0^{60} y\,dx = \frac{h}{3}\,[(y_0 + y_6) + 2\,(y_2 + y_4) + 4\,(y_1 + y_3 + y_5)]$$

$$= \frac{10}{3}\,[(0.02127 + 0.0263) + 2\,(0.01563 + 0.01639)$$
$$+ 4\,(0.01724 + 0.01538 + 0.01923)]$$

$$= \frac{10}{3}\,[0.04757 + 0.06404 + 0.2074] = 1.063 \text{ secs.}$$

Hence time taken to travel 60 feet is 1.063 secs.

By **Simposon's $\frac{3}{8}$ rule**

$$\int_0^{60} y\,dx = \frac{3h}{8}\,[(y_0 + y_6) + 3\,(y_1 + y_2 + y_4 + y_5) + 2\,(y_3)]$$

$$= \frac{3 \times 10}{8} [(0.02127 + 0.02630) + 3 (0.01723 + 0.01563 + 0.01639 + 0.01923) + 2 (0.01538)]$$

$$= 3.75 [0.04757 + 0.20544 + 0.03076]$$

$$\int_0^{60} y\,dx = 1.064 \text{ secs.}$$

Example 13

By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by using Simpson's $\frac{1}{3}$ rule. Is it possible to evaluate the same by Simpson's $\frac{3}{8}$ th rule. Justify your answer.

Solution

Here range $= \pi - 0 = \pi$

$$\therefore \quad h = \frac{\pi}{10}$$

The values of the function $y = \sin x$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$
y	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9

By Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3} [(y_0 + y_{10}) + 2 (y_2 + y_4 + y_6 + y_8) + 4 (y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$= \frac{\frac{\pi}{10}}{3} [(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1.0 + 0.8090 + 0.3090)]$$

$$\int_0^{\pi} \sin x \, dx = 2.00091$$

Note : Here we cannot use Simpson's $\frac{3}{8}$ th rule since the subintervals is not a multiple of 3.

ROMBERG'S METHOD

In trapezoidal formula the error for an interval of size h is

$$E = -\frac{(b-a)}{12} h^2 y''(\xi) \qquad [a < \xi < b]$$

$$= ch^2, \quad c = \frac{-(b-a)}{12} y''(\xi) \qquad \dots (1)$$

Let us evaluate the definite integral

$I = \int_a^b y \, dx$, using Trapezoidal with two different sub-intervals say h_1 and h_2.

Let I_1, I_2 be the values of the given integral with corresponding errors E_1 and E_2.

Clearly $\quad I = I_1 + E_1 = I_1 + ch_1^2 \qquad \dots (2)$

Also $\quad I = I_2 + E_2 = I_2 + ch_2^2 \qquad \dots (3)$

From (2) and (3), we get,

$$I_1 + ch_1^2 = I_2 + ch_2^2$$

$$c(h_2^2 - h_1^2) = I_1 - I_2$$

16. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}\, dx$ using Simpson's rule by dividing the range into 8 equal parts. [AMIE] [Ans. 3.1058]

17. Evaluate $\int_0^{\pi} \sin^3 x\, dx$ using Simpson's $\frac{1}{3}$ rule by dividing the interval $(0, \pi)$ in 6 equal parts. [AMIE] [Ans. 1.305]

GAUSS QUADRATURE FORMULA

Carl Frederich Gauss approached the problem of numerical integration in a different way. Instead of finding the area under the given curve, he tried to evaluate the function at some points along with the abscissa. Here the values of abscissa are not equal. Then apply certain weight to the evaluated function.

Thus for Gauss two point formula,

$$\int_a^b f(x)\, dx = \int_{-1}^{1} f(t)\, dt$$

$$= \omega_1 f(t_1) + \omega_2 f(t_2) \quad \ldots (1)$$

The function $f(t)$ is evaluated at t_1 and t_2. ω_1 and ω_2 are the weights given to the two functions.

The basic methodology is explained as given below for **Gauss two point formula.**

GAUSS - TWO POINT FORMULA

First change the interval (a, b) to $(-1, 1)$ by using the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$$

Thus the independent variable 'x' is changed to 't'.

Then we use an interpolation formula which will give the true value of the integral at certain points. Here the interpolation points are t_1 and t_2.

In equation (1), we want to find the four unknown quantities ω_1, ω_2 and t_1, t_2. So we need four algebraic equations to solve it. Let the equation (1) be exact for

$$f(t) = 1$$
$$f(t) = t$$
$$f(t) = t^2 \text{ and } f(t) = t^3$$

Now $\quad f(t) = 1$

$$\Rightarrow \int_{-1}^{1} 1\, dt = 2 = \omega_1 + \omega_2 \quad [\because f(t_1) = f(t_2) = 1] \quad \dots (2)$$

$$f(t) = t$$

$$\Rightarrow \int_{-1}^{1} t\, dt = \left(\frac{t^2}{2}\right)_{-1}^{1} = 0 = \omega_1 t_1 + \omega_2 t_2 \quad \dots (3)$$

$$f(t) = t^2$$

$$\Rightarrow \int_{-1}^{1} t^2\, dt = \left(\frac{t^3}{3}\right)_{-1}^{1} = \frac{2}{3}$$

$$= \omega_1 t_1^2 + \omega_2 t_2^2 \quad \dots (4)$$

$$f(t^2) = \int_{-1}^{1} t^3\, dt$$

$$\Rightarrow \left(\frac{t^4}{4}\right)_{-1}^{1} = 0 = \omega_1 t_1^3 + \omega_2 t_2^3 \quad \dots (5)$$

This set of equations (2), (3), (4) and (5) can be solved as follows.

From (3), we get

$$\omega_1 t_1 = -\omega_2 t_2 \quad \dots (6)$$

From (5), we get

$$\omega_1 t_1^3 = -\omega_2 t_2^3 \quad \ldots (7)$$

From (6) and (7), we get

$$t_1 = -t_2$$

$$\omega_1 = \omega_2 = 1$$

From (4), we get $t_1^2 + t_2^2 = \frac{2}{3}$

$$\Rightarrow \quad t_1 = \frac{1}{\sqrt{3}}$$

$$t_2 = \frac{-1}{\sqrt{3}}$$

From equation (1), we get

$$I = \int_{-1}^{1} f(t)\,dt = \omega_1 f(t_1) + \omega_2 f(t_2)$$

$$\boxed{I = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)} \quad \ldots(A)$$

$[\because \omega_1 = \omega_2 = 1]$

Example 1

Evaluate $\int_1^2 \frac{dx}{x}$ *using Gauss 2 point formula.*

Solution

Transform the variable x to t by the transformation

$$\boxed{x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t}$$

$$= \left(\frac{1+2}{2}\right) + \left(\frac{2-1}{2}\right)t$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2}$$

i.e., $dx = \frac{dt}{2}$

$$\therefore \ I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{2}{3+t} \cdot \frac{dt}{2} = \int_{-1}^{1} \frac{dt}{3+t}$$

Here $f(t) = \frac{1}{3+t}$ ... (A)

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3 + \frac{1}{\sqrt{3}}} = 0.2795$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{3 - \frac{1}{\sqrt{3}}} = 0.41288$$

$$\boxed{I = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right),}$$

where $f(t) = \frac{1}{3+t}$ [By (A)]

$$\boxed{\mathbf{I = 0.6923}}$$

Example 2

Evaluate $\int_1^2 \frac{dx}{1+x^3}$ ***using Gaussian 2 point formula.***

Solution

Transform the variable x to t by

$$\boxed{x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t}$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2} \qquad \ldots (A)$$

$$dx = \frac{dt}{2}$$

$$\therefore \; I = \int_{1}^{3} \frac{dx}{1+x^3} = \int_{-1}^{1} \frac{1}{1+\left(\frac{3+t}{2}\right)^3} \frac{dt}{2}$$

$$= 4 \int_{-1}^{1} \frac{dt}{8+(3+t)^3} = 4\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right],$$

Here $f(t) = \dfrac{1}{8+(3+t)^3}$ [By (A)]

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{8+\left(3+\frac{1}{\sqrt{3}}\right)^3} = 0.0185$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{8+\left(3-\frac{1}{\sqrt{3}}\right)^3} = 0.045$$

$$= 4\,[\,0.0185 + 0.045\,]$$

$$\boxed{I = 0.254}$$

Example 3

Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$ *using Gauss 2 point formula.*

[A.U. Apr./May '05]

Solution

For the interval -1 to 1, the Gauss 2 point formula is

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Here $f(x) = \dfrac{1}{1+x^2}$

$$\therefore \quad f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$\therefore \quad \int_{-1}^{1} f(x)\,dx = \frac{3}{4} + \frac{3}{4} = \frac{6}{4} = 1.5$$

$$\therefore \quad \int_{-1}^{1} \frac{dx}{1+x^2} = 1.5$$

Example 4

Evaluate $\int_{-1}^{1} (3x^2 + 5x^4)\,dx$ *using Gauss 2 point formula.*

Solution

For the interval -1 to 1, the Gauss 2 point formula is

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = 3x^2 + 5x^4$

$$\therefore \quad f\left(\frac{-1}{\sqrt{3}}\right) = 3\left(\frac{-1}{\sqrt{3}}\right)^2 + 5\left(\frac{-1}{\sqrt{3}}\right)^4$$

$$= 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \left(\frac{5}{9}\right)$$

$$= \frac{14}{9} = 1.556$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 3\left(\frac{1}{\sqrt{3}}\right)^2 + 5\left(\frac{1}{\sqrt{3}}\right)^4$$

$$= 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \left(\frac{5}{9}\right)$$

$$= \frac{14}{9} = 1.556$$

$$\int_{-1}^{1} f(x)\,dx = (1.556 + 1.556) = 3.112$$

$$\therefore \int_{-1}^{1} (3x^2 + 5x^4)\,dx = 3.112$$

Example 5

Evaluate $\int_{-2}^{2} e^{-\frac{x}{2}}\,dx$ *using Gauss 2 point formula.*

Solution

Transform the variable x to t by the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

Here $a = -2$, $b = 2$,

$$\therefore \quad x = \left(\frac{-2+2}{2}\right) + \left(\frac{2+2}{2}\right)t$$

$$x = 0 + 2t$$

$$x = 2t$$

$$i.e., \quad dx = 2\,dt$$

$$\therefore \text{I} = \int_{-2}^{2} e^{-\frac{x}{2}}\,dx = \int_{-1}^{1} e^{-\frac{2t}{2}}\,(2dt)$$

$$= 2 \int_{-1}^{1} e^{-t}\, dt$$

Here $\quad f(x) = e^{-t} \qquad \cdots (A)$

$$\therefore \quad f\left(\frac{-1}{\sqrt{3}}\right) = e^{\frac{1}{\sqrt{3}}} = 1.7813$$

$$f\left(\frac{1}{\sqrt{3}}\right) = e^{\frac{-1}{\sqrt{3}}} = 0.5614$$

$$I = 2\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= 2\,[1.7813 + 0.5614]$$

$$= 2\,(2.3427)$$

$$\boxed{I = 4.6854}$$

Example 6

Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$ ***by Gaussian two point formula.***

Solution

Transform the variable x to t by the transformation

$$\boxed{x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t}$$

Here $a = 0,\ b = \frac{\pi}{2}$,

$$\therefore \quad x = \left(\frac{0 + \frac{\pi}{2}}{2}\right) + \left(\frac{\frac{\pi}{2} - 0}{2}\right)t$$

$$x = \frac{\pi}{4} + \frac{\pi}{4} t$$

$$x = \frac{\pi}{4}(1 + t)$$

$$dx = \frac{\pi}{4} dt$$

i.e.,

$$\therefore \; I = \int_0^{\frac{\pi}{2}} \sin x \, dx = \int_{-1}^{1} \sin \frac{\pi}{4}(1 + t) \left(\frac{\pi}{4} dt\right)$$

$$I = \frac{\pi}{4} \int_{-1}^{1} \sin \frac{\pi}{4}(1 + t) \, dt$$

Here

$$f(t) = \sin \frac{\pi}{4}(1 + t) \qquad \ldots \text{(A)}$$

$$\therefore \; f\left(\frac{1}{\sqrt{3}}\right) = \sin \frac{\pi}{4}\left(1 + \frac{1}{\sqrt{3}}\right)$$

3.14/4 0.707108

$$= \sin \frac{\pi}{4}(1 + 0.5773)$$

$$= \sin (0.7854)(1.5773)$$

= 0.013700

$$f\left(\frac{1}{\sqrt{3}}\right) = 0.9454$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \sin \frac{\pi}{4}\left(1 - \frac{1}{\sqrt{3}}\right)$$

$$= \sin (0.7854)(1 - 0.5773)$$

$$= \sin (0.7854)(0.4227) = 0.3259$$

$$I = \frac{\pi}{4}\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{4}[0.9454 + 0.3259]$$

$$= \frac{\pi}{4}(1.2713) = 0.99848$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x \, dx = 0.99848$$

Example 7

Evaluate $\int_0^{\frac{\pi}{2}} \log(1+x)\,dx$ using Gauss two point formula.

Solution

Transform the variable x to t by the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

Here $a = 0, b = \frac{\pi}{2}$,

$$\therefore \quad x = \left(\frac{0+\frac{\pi}{2}}{2}\right) + \left(\frac{\frac{\pi}{2}-0}{2}\right)t$$

$$x = \frac{\pi}{4} + \frac{\pi}{4}t$$

$$x = \frac{\pi}{4}(1+t)$$

i.e.,

$$dx = \frac{\pi}{4}dt$$

$$\therefore \ I = \int_0^{\frac{\pi}{2}} \log(1+x)\,dx$$

$$= \int_{-1}^{1} \log\left[1 + \frac{\pi}{4}(1+t)\right]\left(\frac{\pi}{4}\,dt\right)$$

$$= \frac{\pi}{4}\int_{-1}^{1} \log\left[1 + \frac{\pi}{4}(1+t)\right]dt$$

$$I = \frac{\pi}{4}\int_{-1}^{1} f(t)\,dt$$

Here

$$f(t) = \log\left[1 + \frac{\pi}{4}(1+t)\right] \quad \ldots (A)$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \log\left[1 + \frac{\pi}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\right]$$

$$= \log\left[1 + (0.7854)(1.5773)\right]$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 0.8060$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \log\left[1 + \frac{\pi}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\right]$$

$$= \log\left[1 + (0.7854)(0.4227)\right]$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = 0.2866$$

$$I = \frac{\pi}{4}\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{4}[0.8060 + 0.2866]$$

$$= \frac{\pi}{4}(1.0926)$$

$$I = 0.858$$

$$\therefore \int_{0}^{\frac{\pi}{2}} \log(1+x)\,dx = \mathbf{0.8580}$$

Example 86

Find the value of the following integral using Gaussian quadrature technique $\int_3^5 \frac{4}{(2x^2)}\, dx$.

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$$

Here $a = 3$, $b = 5$

$$\therefore \quad x = \left(\frac{5+3}{2}\right) + \left(\frac{5-3}{2}\right) t$$

$$= 4 + t$$

$$\therefore \quad \int_3^5 \frac{4}{(2x^2)}\, dx = \int_{-1}^{1} f(t)\, dt$$

$$= \int_{-1}^{1} \frac{2}{(t+4)^2}\, dt$$

Here $f(t) = \frac{2}{(t+4)^2}$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\left(\frac{1}{\sqrt{3}} + 4\right)^2} = 0.09546$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{\left(-\frac{1}{\sqrt{3}} + 4\right)^2} = 0.17073$$

$$\therefore \quad \int_3^5 \frac{4}{(2x^2)}\, dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = 0.09546 + 0.1707$$

$$\mathbf{I = 0.26619}$$

Example 9

Evaluate $\int_0^1 \frac{dx}{1+x^2}$, *using Gauss 2 point formula.*

Solution

Gauss 2 – Point formula :

Transform the variable x to t by

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$$

$$= \frac{1}{2} + \frac{t}{2} = \frac{t+1}{2}$$

i.e., $x = \frac{t+1}{2}$ when $x = 0,\ t = -1$

$dx = \frac{dt}{2}$ $\quad x = 1,\ t = 1$

$$I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} \cdot \frac{dt}{2}$$

$$= 2\int_{-1}^{1} \frac{dt}{4+(t+1)^2}$$

$$= 2\left(f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)\right) \quad \text{where } f(t) = \frac{1}{4+(t+1)^2}$$

$$= 2\left(\frac{1}{4+\left(\frac{1}{\sqrt{3}}+1\right)^2} + \frac{1}{4+\left(\frac{-1}{\sqrt{3}}+1\right)^2}\right)$$

$$= 2\left[\frac{1}{4+(1.5773)^2} + \frac{1}{4+(0.4226)^2}\right]$$

$$= 2\,[\,0.154135 + 0.23932\,]$$

I = 0.78691

❑ GAUSSIAN QUADRATURE (3 POINT) FORMULA

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

where the interval (a, b) is changed into $(-1, 1)$ by the transformation,

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

Then
$$\int_{-1}^{1} f(t)\,dt = A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)$$

where
$$A_1 = A_3 = 0.5555$$
$$A_2 = 0.8888$$
$$t_1 = -0.7745$$
$$t_2 = 0$$
$$t_3 = 0.7745$$

Example 1

Evaluate $\int_1^2 \frac{dx}{x}$ *using Gauss 3 - point formula.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$= \frac{3}{2} + \frac{t}{2}$$

i.e.,
$$x = \frac{3+t}{2}$$

$$\therefore I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} f(t)\,dt$$

$$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \quad \cdots (1)$$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \cdots (2)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{3-0.7745} = 0.4493 \\ f(t_2) &= f(0) = \frac{1}{3} = 0.3333 \\ f(t_3) &= f(0.7745) = \frac{1}{3+0.7745} = 0.2649 \end{aligned}\right\} \quad \cdots (3)$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.4493) + 0.8888\,(0.3333) + (0.2649)\,(0.5555)$$

$$\boxed{I = 0.6929}$$

Example 2

Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ *using the three point Gaussian Quadrature.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$$

where $a = 0.2$, $b = 1.5$

$$= \frac{1.7}{2} + \frac{1.3t}{2}$$

i.e., $\quad x = \dfrac{1.7 + 1.3\,t}{2} \Rightarrow dx = \dfrac{1.3\,dt}{2} = 0.65\,dt$

$$\therefore \quad I = \int_{0.2}^{1.5} e^{-x^2}\, dx$$

$$= \int_{-1}^{1} -e^{\left(\frac{1.7 + 1.3\,t}{2}\right)^2} (0.65)\, dt$$

$$= 0.65 \int_{-1}^{1} -e^{\left(\frac{1.7 + 1.3\,t}{2}\right)^2} dt$$

$$I = 0.65[A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)] \quad \text{...(1)}$$

$$\text{where } f(t) = e^{-\left(\frac{1.7 + 1.3\,t}{2}\right)^2} \quad \text{...(2)}$$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\}$$

$$f(t_1) = f(-0.7745) = e^{-\left(\frac{1.7 + 1.3(-0.7745)}{2}\right)^2} \quad \text{...(3)}$$

$$= 0.8868$$

$$f(t_2) = f(0) = e^{-\left(\frac{1.7 + 1.3\,(0)}{2}\right)^2}$$

$$= 0.48555$$

$$f(t_3) = f(0.7745) = e^{-\left(\frac{1.7 + 1.3\,(0.7745)}{2}\right)^2} \quad \text{...(4)}$$

$$= 0.16013$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.8868) + 0.8888\,(0.4855) + (0.5555)(0.16013)$$

$$= 0.4926 + 0.4315 + 0.08895$$

$$\boxed{I = 1.01307}$$

$\times 0.65 = 0.6584$

Example 3

Evaluate $\int_0^1 \frac{1}{1+t}\, dt$ *by Gaussian quadrature formula.*

Solution

Transform the variable from t to x by the transformation

$$t = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)x \qquad \dots (1)$$

Here $a = 0, b = 1$.

$$t = \frac{1}{2} + \frac{x}{2} = \frac{x+1}{2} = dt = \frac{dx}{2}$$

$\therefore$

when $t = 0, x = -1$

$t = 1, x = 1$

$$I = \int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx/2}{1+\left(\frac{x+1}{2}\right)}$$

$$= \int_{-1}^{1} \frac{dx}{2+x+1}$$

$$\boxed{I = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)} \qquad \dots (2)$$

where $f(x) = \frac{1}{x+3}$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \qquad \dots (3)$$

$$\left.\begin{aligned} f(x_1) &= f(-0.7745) = \frac{1}{(-0.7745)+3} = 0.4493 \\ f(x_2) &= f(0) = \frac{1}{3} = 0.3333 \\ f(x_3) &= f(0.7745) = \frac{1}{0.7745+3} = 0.2649 \end{aligned}\right\} \qquad \dots (4)$$

Substituting (3) and (4) in (2), we get

$$I = [0.5555 \times 0.4493 + 0.8888 \times 0.3333 + 0.5555 \times 0.2649]$$

$$= [0.2495 + 0.2962 + 0.14715]$$

$$\boxed{I = 0.69285}$$

Example 4

Evaluate $\int_0^1 \frac{dx}{1+x^2}$*, using Gauss 3 point formula.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$$

$$= \frac{1}{2} + \frac{t}{2} = \frac{t+1}{2}$$

i.e., $x = \frac{t+1}{2}$ when $x = 0, t = -1$

$dx = \frac{dt}{2}$ $x = 1, t = 1$

$$\therefore \quad I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} \frac{dt}{2}$$

$$= 2\int_{-1}^{1} \frac{dt}{4+(t+1)^2}$$

$$\boxed{I = 2\{A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)\}} \quad \ldots (1)$$

where $f(t) = \frac{1}{4+(t+1)^2}$

$$\left.\begin{aligned} A_1 = A_3 &= 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots (2)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{4+(-0.7745+1)^2} \\ &= 0.2468 \\ f(t_2) &= f(0) = \frac{1}{4+1} = 0.2 \\ f(t_3) &= f(0.7745) = \frac{1}{4+(0.7745+1)^2} \\ &= 0.13988 \end{aligned}\right\} \quad \ldots(3)$$

Substituting (2) and (3) in (1), we get

$$I = 2[\,0.5555(0.2468) + 0.8888(0.2) + 0.5555(0.13988)]$$

$$= 2[\,0.39256\,]$$

$$\boxed{I = 0.78512}$$

Example 5

Evaluate $\int_1^2 \frac{dx}{1+x^3}$ *using Gauss 3 point formula.*

Solution

Transform the variable from x to t by the transformation

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2}$$

$$\therefore I = \int_1^2 \frac{dx}{1+x^3} = \int_{-1}^{1} \frac{1}{1+\frac{(3+t)^3}{8}} \cdot \frac{dt}{2}$$

$$= 4\int_{-1}^{1} \frac{1}{8+(3+t)^3}\, dt \quad \ldots (1)$$

$$\therefore I = 4\int_{-1}^{1} \frac{1}{8+(3+t)^3}\, dt$$

$$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \quad \cdots (2)$$

where $f(t) = \dfrac{1}{(3+t)^3 + 8}$

$$\left.\begin{array}{l} A_1 = A_3 = 0.5555 \\ A_2 = 0.8888 \end{array}\right\} \quad \cdots (3)$$

$$\left.\begin{array}{l} f(t_1) = f(-0.7745) = \dfrac{1}{(3-0.7745)^3 + 8} = 0.0525 \\ f(t_2) = f(0) = \dfrac{1}{35} = 0.0285 \\ f(t_3) = f(0.7745) = \dfrac{1}{(3+0.7745)^3 + 8} = 0.0162 \end{array}\right\} \quad \cdots (4)$$

Substituting (3) and (4) in (2), we get

$$I = 4\,[0.5555 \times 0.0525 + 0.8888 \times 0.0285 + 0.5555 \times 0.0162]$$

$$= 4\,[\,0.06349\,]$$

$$\boxed{I = 0.25396}$$

❑ EXERCISES ❑

1. Applying Gauss's quadrature 3 point formula, evaluate $\int_5^{12} \dfrac{dx}{x}$.

[*Ans* : 0.25009]

2. Evaluate $\int_0^1 x\, dx$ by 3 point Gaussian formula.

[*Ans* : 0.4999]

3. Evaluate by Gaussian 3 point formula $\int_0^{10} e^{\frac{1}{1+x^2}}\, dx$.

[*Ans* : 11.986]