The Solution of Numerical Algebraic and Transcendental Equations

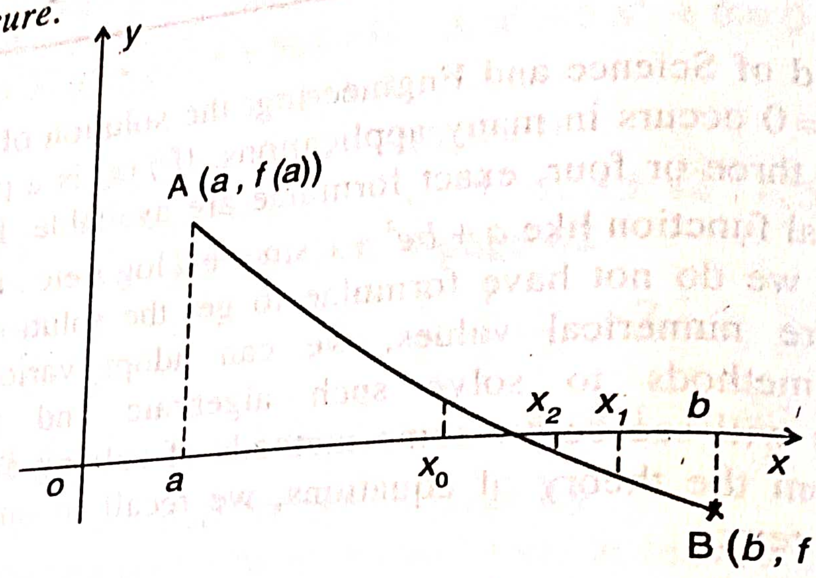
3·1. In the field of Science and Engineering, the solution of equations of the form $f(x)=0$ occurs in many applications. If $f(x)$ is a polynomial of degree two or three or four, exact formulae are available. But, if $f(x)$ is a transcendental function like $a+be^x+c\sin x+d\log x$ etc., the solution is not exact and we do not have formulae to get the solutions. When the coefficients are numerical values, we can adopt various numerical approximate methods to solve such algebraic and transcendental equations. We will see below some methods of solving such numerical equations. From the theory of equations, we recall to our memory the following theorem:

If $f(x)$ is continuous in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x)=0$ will have atleast one real root between a and b.

3.1.1. The Bisection method (or BOLZANO's method) (or Interval halving method)

AIM: Suppose we have an equation of the form $f(x)=0$ whose solution in the range (a, b) is to be searched. We also assume that $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are of opposite signs, atleast one real root between a and b should exist. For convenience, let $f(a)$ be positive and $f(b)$ be negative. Then atleast one root exists between a and b. As a first approximation, we assume that root to be $x_0=\dfrac{a+b}{2}$ (mid point of the ends of the range). Now, find the sign of $f(x_0)$. If $f(x_0)$ is negative, the root lies between a and x_0. If $f(x_0)$ is positive, the root lies between x_0 and b. Any one of this is true. Suppose $f(x_0)$ is positive as shown in the Fig. 3·1, then the root lies between x_0 and

b and take the root as $x_1 = \frac{x_0 + b}{2}$. Now $f(x_1)$ is negative (as in the Fig. 3.1). Hence the root lies between x_0 and x_1 and let the root be (approximate) $x_2 = \frac{x_0 + x_1}{2}$. Now $f(x_2)$ is negative as in the Fig. 3.1, the root lies between x_0 and x_2 and let $x_3 = \frac{x_0 + x_2}{2}$ and so on. In this way taking the mid-point of the range as the approximate root, we form a sequence of approximate roots $x_0', x_1, x_2, \ldots$ whose limit of convergence is the exact root. However, depending on the precision required, we stop the process after some steps. Though simple, the convergence of this method is *slow but sure.*



Note. After n bisections, the length of the subinterval which contains x_n is $\frac{b-a}{2^n}$. If the error is to be made less than a small quantity ε, say,

$$\frac{b-a}{2^n} < \varepsilon. \text{ That is, } 2^n > \frac{b-a}{\varepsilon}$$

The number of iterations n should be greater than $\dfrac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log 2}$.

Example 1. *Find the positive root of $x^3 - x = 1$ correct to four decimal places by bisection method.*

Solution. Let $f(x) = x^3 - x - 1$

Here, $f(0) = -1 = -ve$ and $f(1) = -ve$

$f(2) = 5 = +ve$. Hence a root lies between 1 and 2. We can take the range as (1, 2) and proceed. We can still shorten the range.

$$f(1.5) = 0.8750 = +ve$$

and

$$f(1) = -1 = -ve$$

Hence, the root lies between 1 and 1·5 ...(1)

Let $$x_0 = \frac{1 + 1\cdot5}{2} = 1\cdot2500$$

$$f(x_0) = f(1\cdot25) = -0\cdot29688$$

Hence the root lies between 1·25 and 1·5 ...(2)

Now, $$x_1 = \frac{1\cdot25 + 1\cdot5}{2} = 1\cdot3750$$

$$f(1\cdot3750) = 0\cdot22461 = +ve$$

The root lies between 1·2500 and 1·3750.

Now $$x_2 = \frac{1\cdot2500 + 1\cdot3750}{2} = 1\cdot3125$$

$$f(1\cdot3125) = -0\cdot051514$$

Therefore, root lies between 1·3750 and 1·3125

Now $$x_3 = \frac{1\cdot3125 + 1\cdot3750}{2} = 1\cdot3438$$

$$f(x_3) = f(1\cdot3438) = 0\cdot082832 = +ve$$

The root lies between 1·3125 and 1·3438

Hence $$x_4 = \frac{1\cdot3125 + 1\cdot3438}{2} = 1\cdot3282$$

$$f(1\cdot3282) = 0\cdot014898$$

Therefore the root lies between 1·3125 and 1·3282

$$x_5 = \frac{1}{2}(1\cdot3125 + 1\cdot3282) = 1\cdot3204$$

$$f(1\cdot3204) = -0\cdot0183\cdot40$$

The root lies between 1·3204 and 1·3282

$$x_6 = \frac{1}{2}(1\cdot3204 + 1\cdot3282) = 1\cdot3243$$

$$f(1\cdot3243) = -ve$$

Hence, the root lies between 1·3243 and 1·3282

$$\therefore \quad x_7 = \frac{1}{2}(1\cdot3243 + 1\cdot3282) = 1\cdot3263$$

$$f(1\cdot3263) = +ve$$

∴ The root lies between 1·3243 and 1·3263

$$x_8 = \frac{1}{2}(1\cdot3243 + 1\cdot3263) = 1\cdot3253$$

$$f(1\cdot3253) = +ve$$

The root lies between 1·3243 and 1·3253

$$\therefore \quad x_9 = \frac{1}{2}(1\cdot3243 + 1\cdot3253) = 1\cdot3248$$

$$f(1\cdot3248) = +ve$$

∴ The root lies between 1·3243 and 1·3248

$$x_{10} = \frac{1}{2}(1{\cdot}3243 + 1{\cdot}3248) = 1{\cdot}32455$$

$$f(1{\cdot}32455) = -ve$$

The root lies between 1·3248 and 1·32455

$$x_{11} = \frac{1}{2}(1{\cdot}3248 + 1{\cdot}32455) = 1{\cdot}3247$$

∴ $f(1{\cdot}3247) = -ve$

∴ The root lies between 1·3247 and 1·3248

$$x_{12} = \frac{1}{2}(1{\cdot}3247 + 1{\cdot}3248) = 1{\cdot}32475$$

Hence,

Therefore, the approximate root is 1·32475

(This is not correct to 5 decimal places).

Example 2. *Assuming that a root of $x^3 - 9x + 1 = 0$ lies in interval (2, 4), find that root by bisection method.*

Solution. Let $f(x) = x^3 - 9x + 1$

$$f(2) = -ve \text{ and } f(4) = +ve$$

Therefore, a root lies between 2 and 4

Let $x_0 = \dfrac{2+4}{2} = 3$

Now $f(3) = +ve$; hence the root lies between 2 and 3

$$x_1 = \frac{2+3}{2} = 2{\cdot}5$$

$$f(x_1) = f(2{\cdot}5) = -ve$$

The root lies between 2·5 and 3.

$$x_2 = \frac{2{\cdot}5+3}{2} = 2{\cdot}75$$

$$f(2{\cdot}75) = -ve$$

The root lies between 2·75 and 3

$$x_3 = \frac{1}{2}(2{\cdot}75 + 3) = 2{\cdot}875$$

$$f(x_3) = f(2{\cdot}875) = -ve$$

Therefore, the root lies between 2·875 and 3

$$x_4 = \frac{1}{2}(2{\cdot}875 + 3) = 2{\cdot}9375$$

$$f(2{\cdot}9375) = -ve$$

∴ The root lies between 2·9375 and 3

$$x_5 = \frac{1}{2}(2{\cdot}9375 + 3) = 2{\cdot}9688$$

$$f(2{\cdot}9688) = +ve$$

The root lies between 2·9688 and 2·9375

$$x_6 = \frac{1}{2}(2{\cdot}9375 + 2{\cdot}9688) = 2{\cdot}9532$$

$$f(2{\cdot}9532) = +ve$$

$\therefore$ The root lies between 2·9375 and 2·9532

$$x_7 = \frac{1}{2}(2.9375 + 2.9532) = 2.9454$$

$$f(2.9454) = +ve$$

The root lies between 2·9375 and 2·9454

$$x_8 = \frac{1}{2}(2.9375 + 2.9454) = 2.9415$$

$$f(2.9415) = -ve$$

The root lies between 2·9415 and 2·9454.

$$x_9 = \frac{1}{2}(2.9415 + 2.9454) = 2.9435$$

$$f(2.9435) = +ve$$

The root lies between 2·9415 and 2·9435.

$$x_{10} = 2.9425$$

$$f(2.9425) = -ve$$

The root lies between 2·9425 and 2·9435.

$$x_{11} = 2.9430$$

$$f(2.9430) = +ve$$

$$x_{12} = 2.94275$$

$$x_{13} = 2.942875$$

Approximate root is 2·9429.

Example 3. *Find the positive root of $x - \cos x = 0$ by bisection method.*

Solution. Let $f(x) = x - \cos x$

$$f(0) = -ve, \; f(0.5) = 0.5 - \cos(0.5) = -0.37758$$

$$f(1) = 1 - \cos 1 = 0.45970$$

Hence, the root lies between 0·5 and 1.

$$x_0 = \frac{0.5 + 1}{2} = 0.75$$

$$f(0.75) = 0.75 - \cos(0.75) = 0.018311 = +ve$$

$\therefore$ The root lies between 0·5 and 0·75.

$$x_1 = \frac{0.5 + 0.75}{2} = 0.625$$

$$f(0.625) = 0.625 - \cos(0.625) = -0.18596$$

The root lies between 0·625 and 0·750.

$$x_2 = \frac{1}{2}(0.625 + 0.750) = 0.6875$$

$$f(0.6875) = -0.085335$$

$\therefore$ The root lies between 0·6875 and 0·75.

$$x_3 = \frac{1}{2}(0.6875 + 0.75) = 0.71875$$

$$f(0.71875) = 0.71875 - \cos(0.71875) = -0.03387...$$

The root lies between 0.71875 and 0.75

$$x_4 = \frac{1}{2}(0.71875 + 0.75) = 0.73438$$

$$f(0.73438) = -0.0078664 = -ve$$

∴ The root lies between 0.73438 and 0.75

$$x_5 = 0.742190$$

$$f(0.74219) = 0.0051999 = +ve$$

$$x_6 = \frac{1}{2}(0.73438 + 0.742190) = 0.73829$$

$$f(0.73829) = -0.0013305$$

The root lies between 0.73829 and 0.74219

$$x_7 = \frac{1}{2}(0.73829 + 0.74219) = 0.7402$$

$$f(0.7402) = 0.7402 - \cos(0.7402) = 0.0018663$$

The root lies between 0.73829 and 0.7402

$$x_8 = 0.73925$$

$$f(0.73925) = 0.00027593$$

$$x_9 = 0.7388 \qquad \text{(correct to 4 places)}$$

The root is 0.7388.

Example 4. *Find the positive of $x^4 - x^3 - 2x^2 - 6x - 4 = ($ bisection method.*

Solution. Let $f(x) = x^4 - x^3 - 2x^2 - 6x - 4$

$f(2)$ and $f(3)$ are opposite in sign since $f(2) = -ve$ and $f(3) =$

Therefore the root lies between 2 and 3

$$x_0 = \frac{2+3}{2} = 2.5$$

$f(2.5) = -ve$; hence root lies between 2.5 and 3

$$\therefore \quad x_1 = \frac{2.5+3}{2} = 2.75$$

Proceeding in the same manner, the sequence of mid-points is 2.75, 2.63, 2.69, 2.72, 2.735, 2.728, 2.7315, 2.7298, 2.7307, 2.7... 2.7313, 2.7314, 2.7315,...

$$f(2.7315) = -0.0232$$

The root of the equation is 2.7315 approximately.

Example 5. *Using bisection method, find the negative root $x^3 - 4x + 9 = 0$ by bisection method.*

Solution. Let

$$f(x) = x^3 - 4x + 9$$

$$f(-x) = -x^3 + 4x + 9$$

The negative root of $f(x) = 0$ is the positi...

$\therefore$ We will find the positive root of $f(-x)=0$, firstly,

i.e., $$\phi(x)=x^3-4x-9=0$$

$$\phi(2)=-ve \quad \text{and} \quad \phi(3)=+ve$$

$\therefore$ The root lies between 2 and 3.

Hence $$x_0=\frac{2+3}{2}=2{\cdot}5$$

$$\phi(2{\cdot}5)=(2{\cdot}5)^3-4\,(2{\cdot}5)-9=-ve$$

Therefore, the root lies between 2·5 and 3.

Hence, $$x_1=\tfrac{1}{2}(2{\cdot}5+3)=2{\cdot}75$$

$$\phi(2{\cdot}75)=+ve$$

$\therefore$ The root lies between 2·5 and 2·75

$$x_2=\tfrac{1}{2}(2{\cdot}5+2{\cdot}75)=2{\cdot}625$$

$$\phi(2{\cdot}625)=(2{\cdot}625)^3-4(2{\cdot}625)-9=-1{\cdot}4121=-ve$$

The root lies between 2·625 and 2·75.

$$x_3=\tfrac{1}{2}(2{\cdot}625+2{\cdot}75)=2{\cdot}6875$$

$$\phi(2{\cdot}6875)=-ve$$

$\therefore$ The root lies between 2·6875 and 2·75.

$$x_4=\tfrac{1}{2}(2{\cdot}6875+2{\cdot}75)=2{\cdot}71875$$

$$\phi(2{\cdot}71875)=+ve$$

$\therefore$ The root lies between 2·6875 and 2·71875.

$\therefore$ $$x_5=\tfrac{1}{2}(2{\cdot}6875+2{\cdot}71875)=2{\cdot}703125$$

$$\phi(2{\cdot}703125)=(2{\cdot}703125)^3-4(2{\cdot}703125)-9=-ve$$

$\therefore$ The root lies between 2·703125 and 2·71875.

$$x_6=\tfrac{1}{2}(2{\cdot}703125+2{\cdot}71875)=2{\cdot}710938$$

Proceeding in the same way,

$$x_7=2{\cdot}707031,\ x_8=2{\cdot}705078,\ x_9=2{\cdot}706054,$$

$$x_{10}=2{\cdot}70654,\ x_{11}=2{\cdot}706297,\ x_{12}=2{\cdot}706418,$$

$$x_{13}=2{\cdot}70648,\ x_{14}=2{\cdot}70651 \text{ etc.}$$

We can conclude the root to be 2·7065 for $\phi(x)=0$

Hence the negative root of the given equation is $-2{\cdot}7065$.

3·2. Iteration method (or Method of successive approximations)

Suppose we want the approximate roots of the equation

$$f(x)=0 \qquad \text{...(1)}$$

Now, write the equation (1) in the form

$$x=\phi(x) \qquad \text{...(2)}$$

Assume x_0 to be the starting approximate value to the actual root α of $x=\phi(x)$. Setting $x=x_0$ in the right hand side of (2), we get the first approximation

$$x_1=\phi(x_0)$$

Again setting $x=x_1$ on the R.H.S. of (2), we get successive approximations.

$$x_2=\phi(x_1)$$
$$x_3=\phi(x_2)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$x_n=\phi(x_{n-1})$$

The sequence of approximate roots $x_1, x_2, \ldots, x_n$, if it converges to α is taken as the root of the equation $f(x)=0$

Note. The convergence of the sequence is not guaranteed always unless the choice of x_0 is properly chosen.

3·2·1. The condition for the convergence of the method

Theorem. Let $f(x)=0$ ...(1)

be the given equation whose actual root is α. The equation $f(x)=0$ be written as $x=\phi(x)$. Let I be the interval containing the root $x=\alpha$. If $|\phi'(x)|<1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to α, if the initial starting value x_0 is chosen in I.

Proof. Since α is an actual root of $x=\phi(x)$, we have

$$\alpha=\phi(\alpha) \quad \ldots(2)$$

Further

$$x_1=\phi(x_0)$$
$$x_2=\phi(x_1)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$x_n=\phi(x_{n-1}) \quad \ldots(3)$$

from which the sequence $x_0, x_1, x_2, \ldots, x_n$ of approximations is got. Hence,

$$x_n-\alpha=\phi(x_{n-1})-\phi(\alpha) \quad \ldots(4)$$

By mean value theorem of differential calculus,

$\phi(x_{n-1})-\phi(\alpha)=(x_{n-1}-\alpha)\,\phi'(\theta)$ where $x_{n-1}<\theta<\alpha$

Using in (4),

$$x_n-\alpha=(x_{n-1}-\alpha)\,\phi'(\theta) \quad \ldots(5)$$

Let $|\phi'(x)|\le k$ for all x in the interval I which contains

$$x_0, x_1, x_2, \ldots, x_n, \alpha$$

Hence, (5) reduces to,

$$|x_n - \alpha| \le |x_{n-1} - \alpha| k$$

Similarly,

$$|x_{n-1} - \alpha| \le |x_{n-2} - \alpha| k$$

$$|x_{n-2} - \alpha| \le |x_{n-3} - \alpha| k$$

....

....

$$|x_1 - \alpha| \le |x_0 - \alpha| k$$

Multiplying vertically and cancelling the factors,

$$|x_n - \alpha| \le k^n |x_0 - \alpha| \qquad ...(6)$$

If $\quad k < 1,\ k^n \to 0$ as $n \to \infty$

Hence $\quad |x_n - \alpha| \to 0$ as $n \to \infty$

i.e.,

$$\underset{n \to \infty}{\text{Lt}}\ x_n = \alpha$$

Therefore, the sequence of approximations $x_0, x_1, x_2, \ldots, x_n \ldots$ converges to the exact root α if

$|\phi'(x)| < k < 1$ for all values of x in I. The sequence will converge rapidly if $|\phi'(x)|$ is very small.

If $|\phi'(x)| > 1$, $|x_n - \alpha|$ will become very great and the sequence will not converge.

Note 1. Since $|x_n - \alpha| \le k |x_{n-1} - \alpha|$ where k is a constant, the convergence is linear and the convergence is of order *one*.

The sufficient condition for the convergence is $|\phi'(x)| < 1$ for all x in I.

3·2·2. Order of convergence of an iterative process

Let $x_0, x_1, x_2, \ldots, x_n, \ldots$ be the successive approximations of the root α of $f(x) = 0$. Let e_i be the error in the root x_i, $i = 1, 2, 3, \ldots$

If α is the exact root,

$$e_i = x_i - \alpha \quad \text{and} \quad e_{i+1} = x_{i+1} - \alpha$$

If $p \ge 1$ can be found out such that $|e_{i+1}| \le |e_i|^p . k$ where k is a positive constant for every i, then p is called the *order of convergence*.

If $p = 1$, the convergence is linear and if $p = 2$, it is quadratic.

Example 1. *Solve $e^x - 3x = 0$ by the method of iteration.*

Solution. Let $\quad f(x) = e^x - 3x = 0$

$$f(0) = 1 = +ve\ ; \quad f(1) = e - 3 = -ve$$

$\therefore$ a root lies between 0 and 1.

Let $\qquad x = \frac{1}{3} e^x = \phi(x)$

$$\phi'(x) = \frac{1}{3}e^x \text{ and } \phi'(0) = 1/3, \phi'(1) < 1$$

In the interval $(0, 1)$, $|\phi'(x)| < 1$

Select $x_0 = 0.6$, $x_1 = \frac{1}{3}e^{x_0} = \frac{1}{3}e^{0.6} = 0.60737$

$$x_2 = \frac{1}{3}e^{0.60737} = 0.61187, \quad x_3 = \frac{1}{3}e^{0.61187} = 0.61452$$

$$x_4 = \frac{1}{3}e^{0.61452} = 0.61626, \quad x_5 = \frac{1}{3}e^{0.61626} = 0.61733$$

$$x_6 = \frac{1}{3}e^{0.61733} = 0.61799, \quad x_7 = \frac{1}{3}e^{0.61799} = 0.61840$$

$$x_8 = \frac{1}{3}e^{0.61840} = 0.61865, \quad x_9 = \frac{1}{3}e^{0.61865} = 0.61881$$

$$x_{10} = \frac{1}{3}e^{0.61881} = 0.61891, \quad x_{11} = \frac{1}{3}e^{0.61891} = 0.61897$$

$$x_{12} = \frac{1}{3}e^{0.61897} = 0.61900, \quad x_{13} = \frac{1}{3}e^{0.61900} = 0.61902$$

We can take 0·6190 as the correct value of the root of the equation.

Example 2. *Find a real root of the equation $\cos x = 3x - 1$ correct to 4 decimal places by iteration method.*

Solution. Let $f(x) = \cos x - 3x + 1 = 0$

$$f(0) = 2 = +ve\,; \quad f\left(\frac{\pi}{2}\right) = 1 - 3\left(\frac{\pi}{2}\right) = -ve$$

Therefore, a root lies between 0 and $\pi/2$.

The given equation may be written as

$$x = \frac{1}{3}(1 + \cos x) = \phi(x)$$

$$\phi'(x) = -\frac{1}{3}\sin x$$

$$|\phi'(x)| = \left|\frac{1}{3}\sin x\right| < 1 \text{ for all } x \text{ and in particular in } (0, \pi/2),$$

i.e., $(0, 1.5708)$.

Hence, the iteration method may be applied.

Let us take $x_0 = 0.6$

$$x_1 = \frac{1}{3}[1 + \cos(0.6)] = 0.60845$$

$$x_2 = \frac{1}{3}[1 + \cos(0.60845)] = 0.60684$$

$$x_3 = \frac{1}{3}[1 + \cos(0.60684)] = 0.60715$$

$$x_4 = \frac{1}{3}[1 + \cos(0.60715)] = 0.60709$$

$$x_5 = \frac{1}{3}[1 + \cos(0.60709)] = 0.60710$$

$$x_6 = \frac{1}{3}[1 + \cos(0.60710)] = 0.60710$$

Due to repetition of x_5 and x_6, we stop our work here. Hence the root is 0·6071 correct to 4 decimal places.

Example 3. *Solve the equation $x^3 + x^2 - 1 = 0$ for the positive root by iteration method.*

Solution. Let $f(x) = x^3 + x^2 - 1 = 0$

$$f(0) = -1 = -ve\ ;\ f(1) = 1 = +ve$$

The root lies between 0 and 1.

We write the given equation as $x^2(x+1) = 1$

i.e.,
$$x = \frac{1}{\sqrt{x+1}} = \phi(x)$$

$$\phi'(x) = -\frac{1}{2}\frac{1}{(x+1)^{3/2}}$$

$$|\phi'(0)| = \frac{1}{2} < 1 \quad \text{and} \quad |\phi'(1)| < 1$$

That is $|\phi'(x)| < 1$ for all x in (0, 1).

Hence, the iterative method can be applied.

Take $x_0 = 0.75$ as starting value

$$x_1 = \frac{1}{\sqrt{1 + x_0}} = \frac{1}{\sqrt{1.75}} = 0.75593, \quad x_2 = \frac{1}{\sqrt{1.75593}} = 0.75465$$

$$x_3 = \frac{1}{\sqrt{1.75465}} = 0.75493, \quad x_4 = \frac{1}{\sqrt{1.75493}} = 0.75487$$

$$x_5 = \frac{1}{\sqrt{1.75487}} = 0.75488, \quad x_6 = \frac{1}{\sqrt{1.75488}} = 0.75488$$

Hence the root is 0·75488.

Example 4. *Solve for x from $\cos x - xe^x = 0$ by iteration method.*

Solution. Let $f(x) = \cos x - xe^x$

$$f(0) = 1 = +ve\ ;\ f(1) = \cos 1 - e^1 = -2.1780 = -ve$$

Therefore, the root lies between 0 and 1; *i.e.,* in (0, 1)

From $f(x)=0$, we get $x=\dfrac{\cos x}{e^x}=\phi(x)$

$$\phi'(x)=\frac{e^x(-\sin x)-\cos x.e^x}{e^{2x}}$$

$$=\frac{-(\sin x+\cos x)}{e^x}=\frac{-\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}{e^x}$$

$$|\phi'(x)|<\frac{\sqrt{2}}{e^x}$$

$$|\phi'(0\cdot5)|<\frac{\sqrt{2}}{e^{0\cdot5}}=0\cdot85776<1$$

$$|\phi'(1)|<\frac{\sqrt{2}}{e}=0\cdot52026<1$$

$|\phi'(x)|<1$ for x in the range $(0\cdot5,\ 1)$

$f(0\cdot5)=0\cdot0532=+ve$; $f(1)=-ve$

The root lies between 0·5 and 1.

Take $x_0=0\cdot52$ as starting value.

$$x_1=\phi(x_0)=\frac{\cos(0\cdot52)}{e^{0\cdot52}}=0\cdot51594$$

$$x_2=\phi(x_1)=\frac{\cos(0\cdot51594)}{e^{0\cdot51594}}=0\cdot51924$$

$$x_3=\frac{\cos(0\cdot51924)}{e^{0\cdot51924}}=0\cdot51655$$

$$x_4=\frac{\cos(0\cdot51655)}{e^{0\cdot51655}}=0\cdot51874$$

$x_5=0\cdot51696$ $\quad$ $x_6=0\cdot51841$

$x_7=0\cdot51723$ $\quad$ $x_8=0\cdot51819$

$x_9=0\cdot51741$ $\quad$ $x_{10}=0\cdot51804$

$x_{11}=0\cdot51753$ $\quad$ $x_{12}=0\cdot51794$

$x_{13}=0\cdot51761$ $\quad$ $x_{14}=0\cdot51788$

$x_{15}=0\cdot51766$ $\quad$ $x_{16}=0\cdot51784$

$x_{17}=0\cdot51769$ $\quad$ $x_{18}=0\cdot51781$

$x_{19}=0\cdot51771$

The root is 0·5177 correct to 4 decimals.

Example 5. *Solve* $x^3=2x+5$ *for the positive root by iteration method.*

Solution. Let $f(x) = x^3 - 2x - 5 = 0$

$f(2) = -1 = -ve$; $f(3) = 16 = +ve$

The root lies between 2 and 3 and closer to 2. $f(x) = 0$ can be written as

$x^3 = 2x + 5$; *i.e.*, $x = (2x+5)^{\frac{1}{3}} = \phi(x)$

$$\phi'(x) = \frac{2}{3} \cdot \frac{1}{(2x+5)^{2/3}}$$

$|\phi'(x)| < 1$ for all x in (2, 3).

Take $x_0 = 2{\cdot}0$

$x_1 = (2x_0 + 5)^{1/3} = 9^{1/3} = 2{\cdot}0801$

$x_2 = (9{\cdot}1602)^{1/3} = 2{\cdot}0924$; $x_3 = (9{\cdot}1848)^{1/3} = 2{\cdot}0942$

$x_4 = (9{\cdot}1884)^{1/3} = 2{\cdot}0945$; $x_5 = (9{\cdot}1890)^{1/3} = 2{\cdot}0945$

Therefore the root is 2·0945.

Example 6. *Find a positive root of $3x - \sqrt{1 + \sin x} = 0$ by iteration method.*

Solution. Writing the given equation as

$$x = \frac{1}{3}\sqrt{1 + \sin x} = \phi(x), \quad \phi'(x) = \frac{\cos x}{6\sqrt{1 + \sin x}}$$

The root of given equation lies in (0, 1) since $f(0) = -ve$ and $f(1) = +ve$

In (0, 1), $|\phi'(x)| < 1$ for all x

So, we can use iteration method.

Taking $x_0 = 0{\cdot}4$, $x_1 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}4)} = 0{\cdot}39291$

$$x_2 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}39291)} = 0{\cdot}39199$$

$$x_3 = \frac{1}{3}\sqrt{1 + \sin(0{\cdot}39199)} = 0{\cdot}39187$$

$$x_4 = 0{\cdot}39185$$

$$x_5 = 0{\cdot}39185$$

The root is 0·39185.

3·3. Regula Falsi method (or the method of false position)

Consider the equation $f(x) = 0$ and let $f(a)$ and $f(b)$ be of opposite signs. Also, let $a < b$. The curve $y = f(x)$ will meet the x-axis at some point between $A(a, f(a))$ and $B(b, f(b))$. The equation of the chord joining the two points $A(a, f(a))$ and $B(b, f(b))$ is $\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$. Th

Hence, $x_4 = \dfrac{0.851184 \times f(1) - 1 \times f(0.851184)}{f(1) - f(0.851184)} = 0.852452$

Now the root lies between 0.852452 and 1

In the same way, $x_5 = 0.85261$, $x_6 = 0.85261$

Hence the required root is 0.85261.

EXERCISE 3.1

1. Find a positive root of the following equations by bisection method :

   (i) $x^3 - 4x - 9 = 0$ (ii) $x^3 + 1.2x^2 = 4x + 4.8$

   (iii) $e^x = 3x$ (iv) $3x = \sqrt{1 + \sin x}$

   (v) $x^3 + 3x - 1 = 0$ (vi) $3x = \cos x + 1$

   (vii) $x^3 + x^2 - 1 = 0$ (viii) $2x = 3 + \cos x$

2. Find a positive root of the above equations by iteration method and Regula Falsi method.

3. Solve the following by iteration method :

   (i) $\sin x = \dfrac{x+1}{x-1}$ (ii) $3x - \cos x - 2 = 0$

   (iii) $3x = 6 + \log_{10} x$ (iv) $x^3 + x + 1 = 0$

   (v) $2x - \log_{10} x = 7$ (vi) $x^3 + 2x^2 + 10x = 20$

   (vii) $2 \sin x = x$ (viii) $3x + \sin x = e^x$

   (ix) $x^3 + x^2 = 100$ (x) $\cos x = 3x - 1$

4. Solve the following for a positive root by False position method :

   (i) $xe^x = 3$ (ii) $4x = e^x$

   (iii) $x \log_{10} x = 1.2$ (iv) $\tan x + \tanh x = 0$

   (v) $e^{-x} = \sin x$ (vi) $x^3 - 5x - 7 = 0$

   (vii) $x^3 + 2x^2 + 10x - 20 = 0$ (viii) $2x - \log_{10} x = 7$

   (ix) $xe^x = \cos x$ (x) $x^3 - 5x + 1 = 0$

   (xi) $e^x = 3x$ (xii) $x^2 - \log_e x = 12$

   (xiii) $3x - \cos x = 1$ (xiv) $2x - 3 \sin x = 5$

   (xv) $2x = \cos x + 3$

3·4. Newton-Raphson method (or Newton's method)

Given an approximate value of a root of an equation, a better and closer approximation to the root can be found by using an iterative process called Newton's method or Newton-Raphson method.

Let α_0 be an *approximate value* of a root of the equation $f(x) = 0$

Let α be the exact root nearer to α_0

Then $\alpha = \alpha_0 + h$ where h is very small, positive or negative.

$\therefore$ $f(\alpha) = f(\alpha_0 + h) = 0$ since α is the exact root of $f(x) = 0$

By Taylor expansion,

$$f(\alpha) = f(\alpha_0 + h) = f(\alpha_0) + h f'(\alpha_0) + \frac{h^2}{2!} f''(\alpha_0) + \cdots = 0$$

i.e., If h is small, neglecting $h^2, h^3, \ldots$ etc, we get

$$f(\alpha_0) + hf'(\alpha_0) \approx 0$$

$$\therefore \qquad h \approx -\frac{f(\alpha_0)}{f'(\alpha_0)} \quad \text{if} \quad f'(\alpha_0) \neq 0$$

$$\therefore \qquad \alpha = \alpha_0 + h = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} \text{ approximately}$$

Let this value be α_1

$$\therefore \qquad \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

α_1 is a better approximate root than α_0

Starting with this α_1, we get

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} \text{ which is still better.}$$

Continuing like this, we iterate this process until $|\alpha_{r+1} - \alpha_r|$ is less than the quantity desired.

$$\therefore \qquad \alpha_{r+1} = \alpha_r - \frac{f(\alpha_r)}{f'(\alpha_r)}, \quad r = 0, 1, 2, \ldots$$

This is the iterative formula of Newton-Raphson method.

3·4·1. Geometrical meaning of Newton's method

Let $x = \alpha$ be an exact root of $f(x) = 0$. The curve cuts the x-axis at P whose x-coordinate is α. The x coordinates of the points where $y = f(x)$ cuts the x-axis are the real roots of $f(x) = 0$.

Now, let α_0 be an approximate root of $f(x) = 0$. The ordinate at

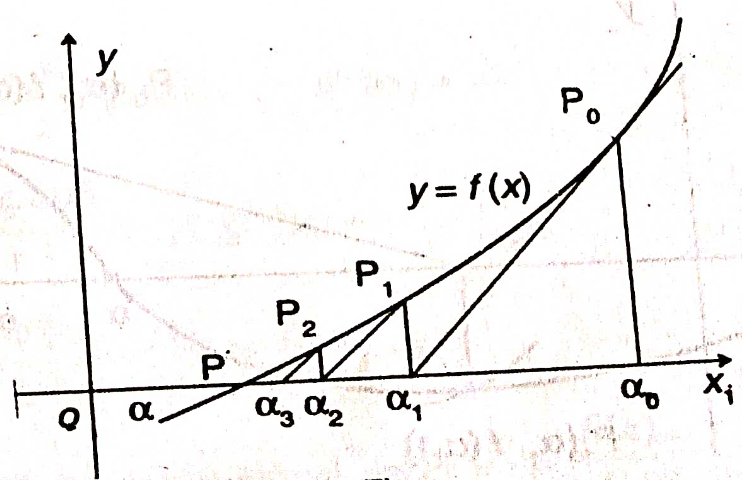


Fig.

$x = \alpha_0$ meets the curve at $P_0\,(\alpha_0, f(\alpha_0))$. The slope of the tangent at P_0 to the curve is $f'(\alpha_0)$. The equation of the tangent at P_0 to the curve is

$$y - f(\alpha_0) = f'(\alpha_0)(x - \alpha_0)$$

This cuts the x-axis at $x = \alpha_1$. To get the point, solve the equation of the tangent at P_0 with $y = 0$.

$$\therefore \quad -f(\alpha_0) = f'(\alpha_0)\,(x - \alpha_0)$$

$$x - \alpha_0 = \frac{-f(\alpha_0)}{f'(\alpha_0)}$$

$$\therefore \quad x = \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

This is exactly the formula got by Newton's method also.

This α_1 is nearer to α than α_0

Now $P_1(\alpha_1, f(\alpha_1))$ is on the curve.

The tangent at P_1 to the curve meets the x-axis at $x = \alpha_2$ where

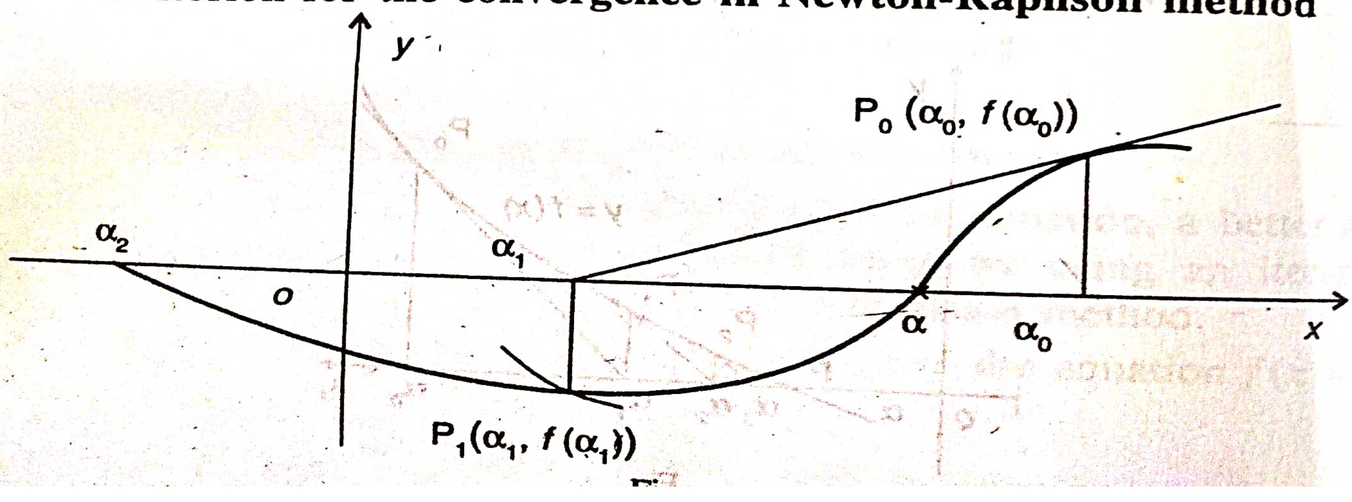
$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)}$$

Now $P_2\,(\alpha_2, f(\alpha_2))$ is on the curve and the tangent at P_2 meets the x-axis at $x = \alpha_3$ where $\alpha_3 = \alpha_2 - \dfrac{f(\alpha_2)}{f'(\alpha_2)}$. Thus we get a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ and every time we get a better approximation. The limit of this sequence is α.

Note 1. The method is also called *method of tangents.*

2. The starting approximate value α_0 must be nearer to the exact value α. Then only the sequence $\alpha_1, \alpha_2, \alpha_3 \ldots$ will converge (approaches to the value α). If the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ does not converge, this method of Newton is of no use. For example, the following figure indicates the diverging nature of the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$

3·4·2. Criterion for the convergence in Newton-Raphson method



Here, in Newton's method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This is really an iteration method where

$$x_{i+1} = \phi(x_i) \text{ and } \phi(x_i) = x_i - \frac{f(x_i)}{f'(x_i)}$$

Hence the equation is

$$x = \phi(x) \text{ where } \phi(x) = x - \frac{f(x)}{f'(x)}$$

The sequence $x_1, x_2, x_3, \ldots$ converges to the exact value if $|\phi'(x)| < 1$

i.e., if $\left| 1 - \frac{[f'(x)]^2 - f(x) f''(x)}{[f'(x)]^2} \right| < 1$

i.e., if $\left| \frac{f(x) f''(x)}{[f'(x)]^2} \right| < 1$

i.e., if $|f(x) f''(x)| < [f'(x)]^2$ ...(1)

This is the criterion for the convergence. The interval containing α should be selected in which (1) is satisfied.

3·4·3. Order of convergence of Newton's method

Let α be the root of $f(x) = 0$

Let e_i be the error at the ith stage of iteration.

i.e., $e_i = x_i - \alpha$ where x_i is the approximate root at the ith iteration.

If $f(x) = 0$ is expressed as $x = \phi(x)$, then $x_{i+1} = \phi(x_i)$

$$\begin{aligned} x_{i+1} &= \phi(x_i) \\ &= \phi(\alpha + e_i) \\ &= \phi(\alpha) + \frac{e_i}{1!} \phi'(\alpha) + \frac{e_i^2}{2!} \phi''(\alpha) + \cdots \\ &= \alpha + \frac{e_i}{1!} \phi'(\alpha) + \frac{e_i^2}{2!} \phi''(\alpha) + \cdots \end{aligned}$$

$$x_{i+1} - \alpha = \frac{e_i \phi'(\alpha)}{1!} + \frac{e_i^2}{2!} \phi''(\alpha) + \cdots$$

i.e., $$e_{i+1} = \frac{e_i \phi'(\alpha)}{1!} + \frac{e_i^2}{2!} \phi''(\alpha) + \cdots \quad \ldots(1)$$

In Newton-Raphson method, $\phi(x) = x - \frac{f(x)}{f'(x)}$

$$\phi'(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$$

$\phi'(\alpha) = 0$ since $f(\alpha) = 0$

$$\phi''(x) = \frac{[f'(x)]^2 \{f(x) f'''(x) + f'(x) f''(x)\} - 2f(x) [f''(x)]^2 f'(x)}{[f'(x)]^4}$$

$\phi''(\alpha) = \dfrac{f''(\alpha)}{f'(\alpha)}$ since $f(\alpha) = 0$

∴ (1) becomes,

$$e_{i+1} = \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}$$ omitting higher power of e_i

∴ The convergence is quadratic and is of order 2.

Note 1. The choice of α_0 is very important for the convergence.

2. When $f'(x)$ is very large, the correct values of root can be found out with minimum number of iterations.
3. If $f(a)$ and $f(b)$ are of opposite signs, a root of $f(x) = 0$ lies between a and b. This idea can be used to fix an approximate root.
4. The error at any stage is proportional to the square of the error in the previous stage.

Example 1: *Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to five decimal places.*

Solution. Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(1) = 2 - 3 - 6 = -7 = -ve$ and $f(2) = 16 - 6 - 6 = 4 = +ve$

∴ a root lies between 1 and 2

By Descarte's rule of sign, we can prove that there is only one positive root.

Take $\alpha_0 = 2$

$$\therefore \quad \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} = \alpha_0 - \frac{2\alpha_0^3 - 3\alpha_0 - 6}{6\alpha_0^2 - 3} = \frac{4\alpha_0^3 + 6}{6\alpha_0^2 - 3}$$

$$\alpha_{i+1} = \frac{4\alpha_i^3 + 6}{6\alpha_i^2 - 3}$$

$$\alpha_1 = \frac{4(2)^3 + 6}{6(2)^2 - 3} = \frac{38}{21} = 1{\cdot}809524$$

$$\alpha_2 = \frac{4(1{\cdot}809524)^3 + 6}{6(1{\cdot}809524)^2 - 3} = \frac{29{\cdot}700256}{16{\cdot}646263} = 1{\cdot}784200$$

$$\alpha_3 = \frac{4(1{\cdot}784200)^3 + 6}{6(1{\cdot}784200)^2 - 3} = \frac{28{\cdot}719072}{16{\cdot}100218} = 1{\cdot}783769$$

$$\alpha_4 = \frac{4(1.783769)^3 + 6}{6(1.783769)^2 - 3} = \frac{28.702612}{16.090991} = 1.783769$$

The better approximate root is 1.783769

Example 2. *Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$ correct to 5 decimal places.*

Solution. Let $f(x) = x^3 - 6x + 4$; $f(0) = 4 = +ve$; $f(1) = -1 = -ve$

$\therefore$ a root lies between 0 and 1

This root is nearer to 1. Take $\alpha_0 = 1$

$$f'(x) = 3x^2 - 6$$

$$x - \frac{f(x)}{f'(x)} = x - \frac{3x^3 - 6x + 4}{3x^2 - 6} = \frac{2x^3 - 4}{3x^2 - 6}$$

$$\therefore \quad \alpha_1 = \frac{2\alpha_0^3 - 4}{3\alpha_0^2 - 6} = \frac{2-4}{3-6} = \frac{2}{3} = 0.66666666$$

$$\alpha_2 = \frac{2\left(\frac{2}{3}\right)^3 - 4}{3\left(\frac{2}{3}\right)^2 - 6} = 0.73015873$$

$$\alpha_3 = \frac{2(0.73015873)^3 - 4}{3(0.73015873)^2 - 6} = \frac{3.22145837}{4.40060469} = 0.73204903$$

$$\alpha_4 = \frac{2(0.73204903)^3 - 4}{3(0.73204903)^2 - 6} = \frac{3.21539602}{4.39231265} = 0.73205081$$

$\therefore$ The root is 0.73205 correct to 5 decimal places.

Example 3. *Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.* [BR. Ap. '93]

Solution. Let $f(x) = 3x - \cos x - 1$; $f(0) = -2 = -ve$

$$f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = +ve$$

$\therefore$ a root lies between 0 and 1

$$f'(x) = 3 + \sin x$$

$$x - \frac{f(x)}{f'(x)} = x - \frac{3x - \cos x - 1}{3 + \sin x} = \frac{x \sin x + \cos x + 1}{3 + \sin x}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$\therefore$ take $\alpha_0 = 0.5$

$$\therefore \quad \alpha_1 = \frac{0.5 \sin(0.5) + \cos(0.5) + 1}{3 + \sin(0.5)}$$

$$= \frac{2.11729533}{3.479425539} = 0.608518649$$

$$\alpha_2 = \frac{0.608518649 \times \sin(0.608518649) + \cos(0.608518649) + \ldots}{3 + \sin(0.608518649)}$$

$$= \frac{2.168335703}{3.57165265} = 0.607101878$$

$$\alpha_3 = \frac{0.607101878 \times \sin(0.607101878) + \cos(0.607101878) + \ldots}{3 + \sin(0.607101878)}$$

$$= \frac{2.16765013}{3.57048962} = 0.607101648$$

$\therefore$ The root is 0·607102 correct to six decimals.

Note. If we have started with $\alpha_0 = 0.6$, the convergence is faster.

Example 4. *Find the positive root of $x = \cos x$ using Newton's method.*

Solution. Let $f(x) = x - \cos x$

$$f(0) = -1 = -ve\ ;\ f(1) = 1 - \cos 1 = 0.459697$$

$\therefore$ a root lies between 0 and 1 and it is closer to 1. Therefore, take $\alpha_0 = 0.7$.

$$f'(x) = 1 + \sin x$$

$$x - \frac{f(x)}{f'(x)} = x - \frac{x - \cos x}{1 + \sin x} = \frac{x \sin x + \cos x}{1 + \sin x}$$

$$x_{i+1} = \frac{x_i \sin x_i + \cos x_i}{1 + \sin x_i}$$

$$\alpha_1 = \frac{0.7 \sin(0.7) + \cos(0.7)}{1 + \sin(0.7)} = \frac{1.21579457}{1.64421769} = 0.739436499$$

$$\alpha_2 = \frac{0.739436499 \times \sin(0.739436499) + \cos(0.739436499)}{1 + \sin(0.739436499)}$$

$$= \frac{1.23713372}{1.67387168} = 0.739085162$$

$$\alpha_3 = \frac{1.23694179}{1.67361205} = 0.739085136$$

$\therefore$ Correct value of the root is 0·7390851.

Example 5. *Find an iterative formula to find $\sqrt{N}$ (where N is positive number) and hence find $\sqrt{5}$.*

Solution. $x = \sqrt{N}$ $\therefore$ $x^2 - N = 0$. Let $f(x) = x^2 - N$; $f'(x) = 2x$

$$\alpha_{i+1} = \alpha_i - \frac{\alpha_i^2 - N}{2\alpha_i} = \alpha_i - \frac{\alpha_i}{2} + \frac{N}{2\alpha_i} = \frac{1}{2}\left(\alpha_i + \frac{N}{\alpha_i}\right)$$

$\therefore$ $\alpha_{i+1} = \frac{1}{2}\left(\alpha_i + \frac{N}{\alpha_i}\right)$ is the iterative formula to find $\sqrt{N}$.

To find $\sqrt{5}$, put $N = 5$

Also $x = \sqrt{5}$ lies between 2 and 3. Take $\alpha_0 = 2$

$$\alpha_1 = \frac{1}{2}\left(\alpha_0 + \frac{5}{\alpha_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$$

$$\alpha_2 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.23611111$$

$$\alpha_3 = \frac{1}{2}\left(2.23611111 + \frac{5}{2.23611111}\right) = 2.23606798$$

Similarly, $\alpha_4 = \frac{1}{2}\left(\alpha_3 + \frac{5}{\alpha_3}\right) = 2.23606798$

Hence the approximate value of $\sqrt{5}$ is 2.23606798.

Example 6. *Find an iterative formula to find the reciprocal of a given number N and hence find the value of* $\frac{1}{19}$

Solution. Let $x = \frac{1}{N}$ $\therefore$ $N = \frac{1}{x}$

Let $f(x) = \frac{1}{x} - N = 0$; $f'(x) = -\frac{1}{x^2}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\frac{1}{x_i} - N}{\left(-\frac{1}{x_i^2}\right)}$$

$$= x_i + x_i^2\left(\frac{1}{x_i} - N\right) = 2x_i - Nx_i^2 = x_i(2 - Nx_i)$$

$\therefore$ $x_{i+1} = x_i(2 - Nx_i)$ is the iterative formula.

To find $\frac{1}{19}$, take $N = 19$

Further $\frac{1}{20} = 0.05$; $\therefore$ take $\alpha_0 = 0.05$

$$x_1 = 0.05(2 - 19 \times 0.05) = 0.0525$$

$$x_2 = 0.0525(2 - 19 \times 0.0525) = 0.05263125$$

$$x_3 = 0.05263125(2 - 19 \times 0.05263125)$$

$$= 0.0526315789$$

Similarly $x_4 = 0.0526315789$

Hence the value of $\frac{1}{19}$ is 0.0526315789.

Example 7. *Find the root of $4x - e^x = 0$ that lies between 2 an*

Solution. Let $f(x) = 4x - e^x$

$f(2) = 8 - e^2 = 0.6109 = +ve$

$f(3) = 12 - e^3 = -8.0855 = -ve$

$\therefore$ a root lies between 2 and 3 and it is closer to 2.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{4x_i - e^{x_i}}{4 - e^{x_i}} = \frac{e^{x_i}(1 - x_i)}{4 - e^{x_i}}$$

Take $\alpha_0 = 2$

$$\alpha_1 = \frac{e^2(1-2)}{4 - e^2} = 2.18026963$$

$$\alpha_2 = \frac{10.4438422}{4.87019074} = 2.14444213$$

$$\alpha_3 = \frac{9.77041972}{4.53727722} = 2.15336627$$

$$\alpha_4 = \frac{9.93487335}{4.61380604} = 2.15329237$$

$\therefore$ The root is approximately 2.1533 correct to 4 decimal places.

3.4.4. Generalised Newton's method

If α is a root of $f(x) = 0$ with multiplicity p, then the iteration formula will be

$$x_{n+1} = x_n - p\frac{f(x_n)}{f'(x_n)}$$

This means that $\frac{1}{p}f'(x_n)$ is the slope of the line through (x_n, y_n) and intersecting the axis of x at $(x_{n+1}, 0)$.

Since α is a root of $f(x) = 0$ with multiplicity p, it implies that α is also a root of $f'(x) = 0$ with multiplicity $(p - 1)$ and it is a root of $f''(x) = 0$ with multiplicity $(p - 2)$ and so on. Therefore

$$x_0 - p\frac{f(x_0)}{f'(x_0)}, \quad x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)}, \quad x_0 - (p-2)\frac{f''(x_0)}{f'''(x_0)}$$

will all have the same value if the initial approximation x_0 is chosen close to the actual root.

Example 8. *Find the double root of $x^3 - 5.4x^2 + 9.24x - 5.096 = 0$ given that it is nearer to 1.5.*

Solution. Let $f(x) = x^3 - 5.4x^2 + 9.24x - 5.096$

$f'(x) = 3x^2 - 10.8x + 9.24$

$$x - \frac{pf(x)}{f'(x)} = x - \frac{2f(x)}{f'(x)}$$

$$= x - \frac{2(x^3 - 5.4x^2 + 9.24x - 5.096)}{3x^2 - 10.8x + 9.24}$$

$$= \frac{x^3 - 9.24x + 10.192}{3x^2 - 10.8x + 9.24}$$

$$\therefore \quad \alpha_{i+1} = \frac{\alpha_i^3 - 9.24\,\alpha_i + 10.192}{3\alpha_i^2 - 10.8\,\alpha_i + 9.24}$$

Take $\alpha_0 = 1.5$

$\alpha_1 = 1.3952,$ $\quad\alpha_2 = 1.3966$

$\alpha_3 = 1.4024,$ $\quad\alpha_4 = 1.4211$ etc.

The root is approximately 1.4 correct to one decimal place.

EXERCISE 3.2

Using Newton-Raphson method, solve the following:

1. Find the positive root of $x^3 - x - 1 = 0$
2. Find a positive root of $f(x) = \cos x - xe^x = 0$
3. Find a real root of $x^3 + 2x^2 + 50x + 7 = 0$
4. Find the value of $\frac{1}{31}$ using Newton-Raphson method.
5. Find a root of $x \log_{10} x = 4.772393$
6. Find the positive root of $x - 2\sin x = 0$
7. Find a real root of $x^3 + x + 1 = 0$
8. Find the cube root of 24 correct to 3 decimal places.
9. Find a real root of $x^3 - x - 2 = 0$
10. Find a positive root of $x^4 - x - 9 = 0$
11. Find a positive root of $2(x - 3) = \log_{10} x$
12. Find the value of $\sqrt{35}$
13. Find the cube root of 24
14. Solve for a positive root of $x^4 - x = 10$
15. Solve for a positive root of $x \tan x = 1.28$
16. Solve for a positive root of $xe^x = 1$
17. Find a positive root of $x^3 - 5x + 3 = 0$
18. Evaluate $\sqrt{12}$ by Newton's method.
19. Find a +ve root of $\sqrt[4]{24}$ and $\sqrt[3]{17}$
20. Find a positive root of $\cos x = x^2$

21. Solve for a +ve root of $2x - 3 \sin x = 5$
22. Find a positive root of $e^{0.4x} - 0.4x = 9$
23. Find the smallest positive root of $x^3 - 2x + 0.5 = 0$
24. Find a negative root of $\sin x = 1 + x^3$
25. Find a positive root of $xe^x = \cos x$
26. Find the negative root of $x^2 + 4 \sin x = 0$
27. Solve for positive root by Newton's method of $2x - \log_{10} x = 7$
28. Using generalised Newton-Raphson method, find the double root of $x^3 - x^2 - x + 1 = 0$ choosing $x_0 = 0.8$
29. Find a root of $x^3 - 3x - 5 = 0$
30. Find a root of the equation $x \sin x + \cos x = 0$
31. Using Newton's method, show that the iteration formula for finding
(*i*) the *p*th root of *a* is

$$x_{n+1} = \frac{(p-1)\, x_n^{\,p} + a}{p x_n^{\,p-1}}$$

(*ii*) the reciprocal of the *p*th root of *a* is

$$x_{n+1} = \frac{x_n\, (p + 1 - a x_n^{\,p})}{p}$$

Hence find the values of $(10)^{1/3}$ and $\dfrac{1}{\sqrt{\sin (0.5)}}$

3.5. Horner's method

This numerical method is employed to determine both the commensurable and the incommensurable real roots of a numerical polynomial equation. Firstly, we find the integral part of the root and then by every iteration, we find each decimal place value in succession.

Suppose a positive root of $f(x) = 0$ lies between a and $a + 1$. (where a is an integer)

Let that root be $a.a_1a_2a_3a_4...$

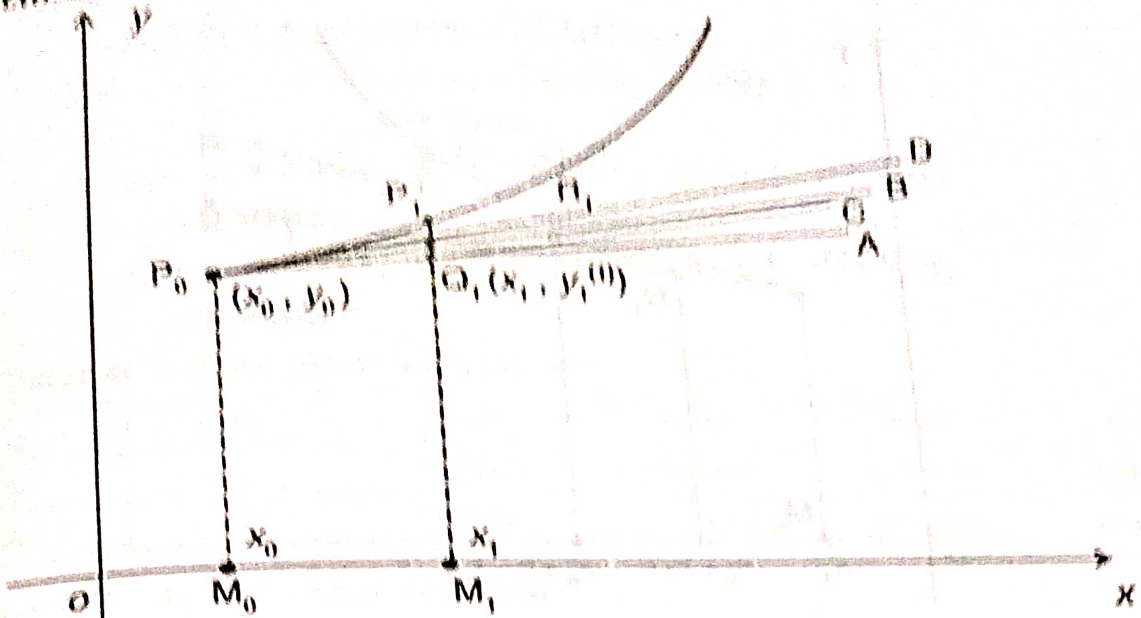
First, diminish the roots of $f(x) = 0$ by the integral part a and let $\phi_1(x) = 0$ possess the root $0.a_1a_2a_3a_4...$

Secondly, multiply the roots of $\phi_1(x) = 0$ by 10 and let $\phi_2(x) = 0$ possess $a_1.a_2a_3a_4...$ as a root.

Thirdly, find the value of a_1 and then diminish the roots by a_1 and let $\phi_3(x) = 0$ possess a root $0.a_2a_3a_4...$

Now repeating the process we find $a_2, a_3, a_4, ...$ each time.

Example 1. *Find the positive root of $x^3 + 3x - 1 = 0$, correct to two decimal places, by Horner's method.*



$$y_1 = y_0 + \frac{1}{2} h \left[f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0)) \right] \quad ...(3)$$

Writing generally,

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right] \quad ...(4)$$

Equation (4) gives the formula for y_{n+1}. This is improved Euler's method.

Note 1. The difference between Euler's method and improved Euler's method is that in the latter we take the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ instead of the slope at (x_0, y_0) in the former method.

11·11. Modified Euler method

In the previous improved Euler method, *we averaged the slopes*, whereas in modified Euler method, we will average the *points*.

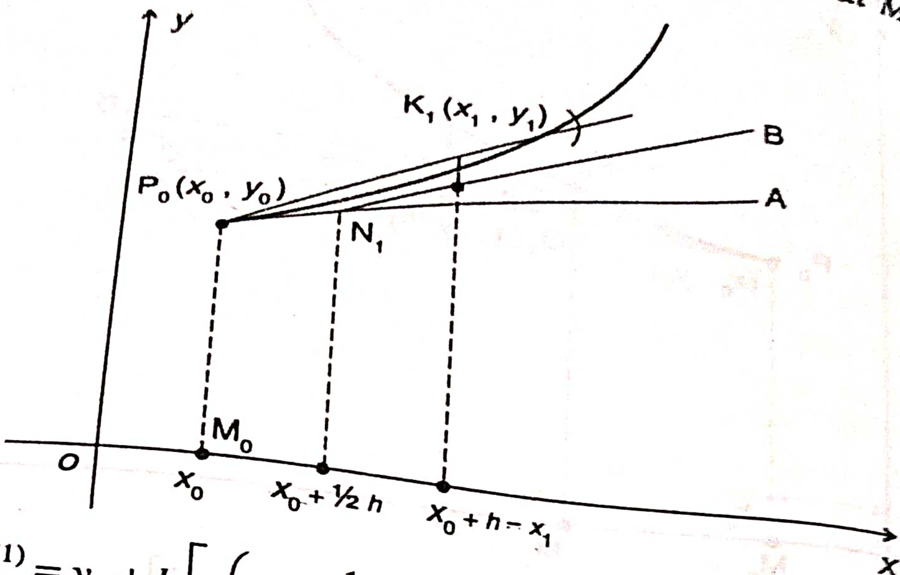
Let $P_0 (x_0, y_0)$ be the point on the solution curve.

Let P_0A be the tangent at (x_0, y_0) to the curve. Now let this tangent meet the ordinate at $x = x_0 + \frac{1}{2} h$ at N_1

$$\text{y-coordinate of } N_1 = y_0 + \frac{1}{2} h f(x_0, y_0) \quad ...(1)$$

Calculate the slope at N_1 *i.e.*, $f\left(x_0 + \frac{1}{2} h, y_0 + \frac{1}{2} h f(x_0, y_0)\right)$

Now draw the line through $P (x_0, y_0)$ with this slope as the slope. Let this line meet $x = x_1$ at $K_1 (x_1, y_1^{(1)})$. This $y_1^{(1)}$ is taken as the approximate value of y at $x = x_1$.



$\therefore \quad y_1^{(1)} = y_0 + h\left[f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$

In general,

$$\mathbf{y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]}$$

or

$$\mathbf{y(x+h) = y(x) + h\left[f\left(x + \frac{1}{2}h,\ y + \frac{1}{2}hf(x, y)\right)\right]} \quad ...(3)$$

Equations (2) or (3) is called **modified Euler's formula.**

Note 2. Hence the Euler predictor is

$$y_{n+1} = y_n + h\,y_n'$$

and the corrector is

$$y_{n+1} = y_n + \frac{h}{2}\left(y_n' + y'_{n+1}\right)$$

in the *Improved Euler method:*

Note 3. There is a lot of confusion among the authors: Some take the improved Euler method as the modified Euler method and the modified Euler method is not mentioned at all. You can see this in some books.

Example 1. *Given $y' = -y$ and $y(0) = 1$, determine the values of y at $x = (0{\cdot}01)\ (0{\cdot}01)\ (0{\cdot}04)$ by Euler method.*

Solution. $y' = -y$ and $y(0) = 1$; $f(x, y) = -y$.

Here, $x_0 = 0$, $y_0 = 1$, $x_1 = 0{\cdot}01$, $x_2 = 0{\cdot}02$, $x_3 = 0{\cdot}03$, $x_4 = 0{\cdot}04$.

We have to find y_1, y_2, y_3, y_4. Take $h = 0{\cdot}01$.

By Euler algorithm,

$$y_{n+1} = y_n + h\,y_n' = y_n + hf(x_n, y_n) \quad ...(1)$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0{\cdot}01)(-1) = 1 - 0{\cdot}01 = 0{\cdot}99$$

$$y_{n+1} = y_n + \frac{1}{2} h [f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h [f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))] \quad ...(1)$$

$$= 0 + \frac{0.2}{2} [y_0 + e^{x_0} + y_0 + h (y_0 + e^{x_0}) + e^{x_0 + h}]$$

$$= (0.1) [0 + 1 + 0 + 0.2 (0 + 1) + e^{0.2}]$$

$$y(0.2) = (0.1) [1 + 0.2 + 1.2214] = 0.24214$$

$$y_2 = y_1 + \frac{1}{2} h [f(x_1, y_1) + f(x_1 + h, y_1 + h f(x_1, y_1))] \quad ...(2)$$

Here $f(x_1, y_1) = y_1 + e^{x_1} = 0.24214 + e^{0.2} = 1.46354$

$$y_1 + h f(x_1, y_1) = 0.24214 + (0.2)(1.46354) = 0.53485$$

$$f(x_1 + h, y_1 + h f(x_1, y_1)) = f(0.4, 0.53485)$$

$$= 0.53485 + e^{0.4}$$

$$= 2.02667$$

using (2),

$$y_2 = y(0.4) = 0.24214 + (0.1) [1.46354 + 2.02667]$$

$$= 0.59116$$

$$y(0.4) = \mathbf{0.59116}$$

Example 4. *Compute y at x = 0·25 by Modified Euler method given $y' = 2xy$, $y(0) = 1$.* *(BR. Nov. 1995)*

Solution. Here, $f(x, y) = 2xy$: $x_0 = 0$, $y_0 = 1$.

Take $h = 0.25$, $x_1 = 0.25$

By Modified Euler method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2} h f(x_n, y_n)\right)\right] \quad ...(1)$$

$$\therefore \quad y_1 = y_0 + h\left[f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2} h f(x_0, y_0)\right)\right]$$

$$f(x_0, y_0) = f(0, 1) = 2(0)(1) = 0.$$

$$\therefore \quad y_1 = 1 + (0.25) [f(0.125, 1)]$$

$$= 1 + (0.25) [2 \times 0.125 \times 1]$$

$$y(0.25) = \mathbf{1.0625}$$

By solving $\frac{dy}{dx} = 2xy$, we get $y = e^{x^2}$ using $y(0) = 1$,

$$y(0.25) = e^{(0.25)^2} = 1.0645$$

Exact value of $y(0.25) = \mathbf{1.0645}$

Error is only 0·002.

Note: To improve the result we can take $h = 0.125$ and get $y(0.125)$ first then get $y(0.25)$. Of course, labour is more.

Example 5. *Solve the equation* $\frac{dy}{dx} = 1 - y$*, given* $y(0) = 0$ *using* ***Modified Euler's method*** *and tabulate the solutions at* $x = 0.1, 0.2,$ *and* 0.3*. Compare your results with the* ***exact solutions.***

Also, get the solutions by ***Improved Euler method.*** (*Anna Ap. 2005*)

(*B.R. Nov. 1991*)

Solution. Here, $x_0 = 0, y_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

$y' = 1 - y \quad \therefore \quad f(x, y) = 1 - y; \quad f(x_0, y_0) = 1 - y_0 = 1$

by *Modified Euler method,*

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right) \quad ...(1)$$

$$\therefore \quad y_1 = y_0 + hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right) \quad ...(2)$$

$$x_0 + \frac{1}{2}h = \frac{0.1}{2} = 0.05$$

$$y_0 + \frac{1}{2}hf(x_0, y_0) = 0 + \frac{0.1}{2}[1] = 0.05$$

using (2),

$$\therefore \quad y_1 = 0 + 0.1\,[f(0.05, 0.05)] = (0.1)(1 - 0.05)$$

$$y_1 = y\,(0.1) = \mathbf{0.095}$$

$$\therefore \quad f(x_1, y_1) = 1 - y_1 = 0.905;$$

$$y_2 = y_1 + hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hf(x_1, y_1)\right)$$

$$= 0.095 + (0.1)\,[f(0.15, 0.14025)]$$

$$= 0.095 + (0.1)\,[1 - 0.14025]$$

$$y\,(0.2) = \mathbf{0.18098}$$

$$y_3 = y_2 + hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}hf(x_2, y_2)\right) \quad ...(3)$$

$$x_2 + \frac{1}{2}h = 0.25$$

$$y_2 + \frac{1}{2}hf(x_2, y_2) = 0.18098 + (0.05)\,[1 - 0.18098]$$

$$= 0.22193$$

using (3), we get

$$y\,(0.3) = y_3 = 0.18098 + (0.1)\,[1 - 0.22193] = \mathbf{0.258787.}$$

Exact solution: $\frac{dy}{dx} = 1 - y$ gives $\frac{dy}{1 - y} = dx$

$$\therefore \quad -\log(1 - y) = x + c$$

$$\log(1 - y) = -x - c$$

$\therefore \quad 1 - y = e^{-x}.A$

At $x = 0, y = 0 \quad \therefore \quad A = 1 \quad \therefore \quad y = 1 - e^{-x}$

using this exact solution, ...(4)

$$y(0 \cdot 1) = 1 - e^{-0 \cdot 1} = 0 \cdot 09516258$$

$$y(0 \cdot 2) = 1 - e^{-0 \cdot 2} = 0 \cdot 181269247$$

$$y(0 \cdot 3) = 1 - e^{-0 \cdot 3} = 0 \cdot 259181779$$

By Improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right] \quad \ldots(5)$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h \left[f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0)) \right] \quad \ldots(6)$$

$$f(x_0, y_0) = 1 - y = 1 - 0 = 1$$

$$f(x_1, y_0 + h f(x_0, y_0)) = f(0 \cdot 1, 0 \cdot 1) = 1 - 0 \cdot 1 = 0 \cdot 9$$

using in (6),

$$y_1 = y(0 \cdot 1) = 0 + \frac{0 \cdot 1}{2} [1 + 0 \cdot 9] = \frac{0 \cdot 19}{2} = \mathbf{0 \cdot 095}$$

$$y_2 = y_1 + \frac{1}{2} h \left[f(x_1, y_1) + f(x_2, y_1 + h f(x_1, y_1)) \right] \quad \ldots(7)$$

$$f(x_1, y_1) = 1 - y_1 = 1 - 0 \cdot 095 = 0 \cdot 905$$

$$f(x_2, y_1 + h f(x_1, y_1)) = f(0 \cdot 2, 0 \cdot 095 + (0 \cdot 1)(0 \cdot 905)) = 0 \cdot 8145$$

using in (7), we get

$$y_2 = y(0 \cdot 2) = 0 \cdot 095 + \frac{0 \cdot 1}{2} [0 \cdot 905 + 0 \cdot 8145]$$

$$y(0 \cdot 2) = \mathbf{0 \cdot 18098}$$

$$y_3 = y_2 + \frac{1}{2} h \left[f(x_2, y_2) + f(x_3, y_2 + h f(x_2, y_2)) \right] \quad \ldots(8)$$

$$f(x_2, y_2) = 1 - y_2 = 1 - 0 \cdot 18098 = 0 \cdot 81902$$

$$y_2 + h f(x_2, y_2) = 0 \cdot 18098 + (0 \cdot 1)(0 \cdot 81902) = 0 \cdot 26288$$

using in (8),

$$y_3 = y(0 \cdot 3) = 0 \cdot 18098 + \frac{0 \cdot 1}{2} [0 \cdot 81902 + 1 - 0 \cdot 26288]$$

$$y(0 \cdot 3) = \mathbf{0 \cdot 258787}$$

The values are tabulated.

x	*Modified Euler*	*Improved Euler*	*Exact solution*
0·1	0·095	0·095	0·09516
0·2	0·18098	0·18098	0·18127
0·3	0·258787	0·258787	0·25918

$$= f(0{\cdot}2,\ 1{\cdot}18732) = 1{\cdot}18732 - \frac{2 \times 0{\cdot}2}{1{\cdot}18732} = 0{\cdot}8504268$$

using in (3),

$$y_2 = 1{\cdot}095909 + \frac{0{\cdot}1}{2}[0{\cdot}913412 + 0{\cdot}850427]$$

$$= \mathbf{1{\cdot}1841009}$$

x	0	0·1	0·2
y	1	1·095907	1·1841009

Example 8. *Using Modified Euler method, find y (0·2), y (0·1) given*

$$\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1.$$ (MS. Ap. '92)

Solution. Here, $x_0 = 0, y_0 = 1, h = 0{\cdot}1, x_1 = 0{\cdot}1, f(x, y) = x^2 + y^2$

By Modified Euler method,

$$y_1 = y_0 + hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}hf(x_0, y_0)\right) \quad ...(1)$$

$$y_0 + \frac{1}{2}hf(x_0, y_0) = y_0 + \frac{1}{2}h\left(x_0^2 + y_0^2\right)$$

$$= 1 + \frac{0{\cdot}1}{2}(0 + 1) = 1{\cdot}05$$

using in (1)

$$y_1 = 1 + (0{\cdot}1)\,[f(0{\cdot}05,\ 1{\cdot}05)]$$

$$y(0{\cdot}1) = 1 + (0{\cdot}1)\,[(0{\cdot}05)^2 + (1{\cdot}05)^2] = \mathbf{1{\cdot}1105}$$

$$y_2 = y_1 + hf\left(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}hf(x_1, y_1)\right)$$

$$f(x_1, y_1) = f(0{\cdot}1,\ 1{\cdot}1105) = (0{\cdot}1)^2 + (1{\cdot}1105)^2 = 1{\cdot}2432$$

$$y_1 + \frac{1}{2}hf(x_1, y_1) = 1{\cdot}1105 + (0{\cdot}05)(1{\cdot}24321) = 1{\cdot}17266$$

$$\therefore\ y_2 = 1{\cdot}1105 + (0{\cdot}1)\,[f(0{\cdot}15,\ 1{\cdot}172660)]$$

$$= 1{\cdot}1105 + (0{\cdot}1)\,[(0{\cdot}15)^2 + (1{\cdot}17266)^2]$$

$$y(0{\cdot}2) = \mathbf{1{\cdot}25026}.$$

2. Find $y(0{\cdot}6), y(0{\cdot}8), y(1)$ given $\frac{dy}{dx} = x + y, y(0) = 0$ taking $h = 0{\cdot}2$ by improved Euler method.
3. Using Improved Euler method find

$$y(0{\cdot}2), y(0{\cdot}4) \text{ given } \frac{dy}{dx} = y + x^2, y(0) = 1.$$

4. Use Euler's method to find $y(0{\cdot}4)$ given $y' = xy, y(0) = 1$.
5. Use Improved Euler method to find $y(0{\cdot}1)$ given $y' = \frac{y-x}{y+x}, y(0) = 1$.
6. Use Modified Euler method and obtain $y(0{\cdot}2)$ given $\frac{dy}{dx} = \log(x+y), y(0) = 1, h = 0{\cdot}2$.
7. Using Modified Euler method, get $y(0{\cdot}2), y(0{\cdot}4), y(0{\cdot}6)$ given $\frac{dy}{dx} = y - x^2, y(0) = 1$.
8. Using Euler's Improved method, find $y(0{\cdot}2), y(0{\cdot}4)$ given

$$\frac{dy}{dx} = x + |\sqrt{y}|, \ y(0) = 1.$$

9. Find $y(0{\cdot}1)$ given $y' = x^2 + y, y(0) = 1$ using Improved Euler method.

Using Euler's method do the problems (10-11):

10. Find $y(1{\cdot}5)$ taking $h = 0{\cdot}5$ given

$$y' = y - 1, y(0) = 1{\cdot}1.$$

11. If $y' = 1 + y^2, y(0) = 1, h = 0{\cdot}1$, find $y(0{\cdot}4)$.
12. Use Euler's improved method to calculate $y(0{\cdot}5)$, taking $h = 0{\cdot}1$, and $y' = y + \sin x, y(0) = 2$.
13. Find $y(1{\cdot}6)$ if $y' = x \log y - y \log x, y(1) = 1$ if $h = 0{\cdot}1$.
14. Find by Improved Euler to get $y(1{\cdot}2), y(1{\cdot}4)$ given $\frac{dy}{dx} = \frac{2y}{x} + x^3$ if $y(1) = 0{\cdot}5$.
15. Use Improved Euler and Modified Euler method, to get $y(1{\cdot}6)$ if

$$\frac{dy}{dx} = y^2 - \frac{y}{x}, \text{ if } y(1) = 1.$$

16. Solve $y' = 3x^2 + y$ given $y(0) = 4$, if $h = 0{\cdot}25$ to obtain $y(0{\cdot}25), y(0{\cdot}5)$.
17. Given $y' = \frac{y}{x} - \frac{5}{2}x^2 y^3; y(1) = \frac{1}{\sqrt{2}}$ find $y(2)$ if $h = 0{\cdot}125$.
18. Find $y(0{\cdot}2)$ by Improved Euler method, given $y' = -xy^2, y(0) = 2$ if $h = 0{\cdot}1$.

11·12. Runge-Kutta Method

The use of the previous methods to solve the differential equation numerically is restricted due to either slow convergence or due to labour involved, especially in Taylor-series method. But, in Runge-Kutta methods, the derivatives of higher order are not required and we require only the given function values at different points. Since the derivation of fourth order Runge-Kutta method is tedious, we will derive Runge-Kutta method of second order.

11·13. Second order Runge-Kutta method (for first order O.D.E.)

AIM. To solve $\frac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$. ...(1)

Proof. By Taylor series, we have,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + O(h^3) \quad ...(2)$$

Differentiating the equation (1) w.r.t. x,

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + y' f_y = f_x + ff_y \quad ...(3)$$

Using the values of y' and y'' got from (1) and (3), in (2), we get

$$y(x+h) - y(x) = hf + \frac{1}{2} h^2 [f_x + ff_y] + O(h^3)$$

$$\therefore \quad \Delta y = hf + \frac{1}{2} h^2 (f_x + ff_y) + O(h^3) \quad ...(4)$$

Let $\Delta_1 y = k_1 = f(x, y). \Delta x = hf(x, y)$ ...(5)

$\Delta_2 y = k_2 = hf(x + mh, y + mk_1)$ ...(6)

and le $\Delta y = ak_1 + bk_2$ ...(7)

where a, b and m are constants to be determined to get the better accuracy of Δy.

Expand k_2 and Δy in powers of h.

Expanding k_2, by Taylor series for two variables, we have

$$k_2 = hf(x + mh, y + mk_1)$$

$$= h\left[f(x, y) + \left(mh\frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right) f + \frac{\left(mh\frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right)^2}{2!} f + \cdots \right]$$

$$= h\left[f + mhf_x + mhff_y + \frac{\left(mh\frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right)^2 f}{2!} + \cdots \right] \quad ...(8)$$

since $k_1 = hf$

$$= hf + mh^2 (f_x + ff_y) + \cdots \text{ higher powers of } h \quad ...(9)$$

Substituting k_1, k_2 in (7),

$$\Delta y = ahf + b\left[hf + mh^2 (f_x + ff_y) + O(h^3) \right]$$

$$= (a+b) hf + bmh^2 (f_x + ff_y) + O(h^3) \quad ...(10)$$

Equating Δy from (4) and (10), we get

$= hf + mh^2 (f_x + ff_y) + \cdots$ higher powers of h ...(9)

Substituting k_1, k_2 in (7),

$$\Delta y = ahf + b\left[hf + mh^2 (f_x + ff_y) + O(h^3)\right]$$

$$= (a+b)\,hf + bmh^2 (f_x + ff_y) + O(h^3) \quad ...(10)$$

Equating Δy from (4) and (10), we get

$$a + b = 1 \quad \text{and} \quad bm = \frac{1}{2} \quad ...(11)$$

Now we have only two equations given by (1) to solve for three unknowns a, b, m.

From $a + b = 1, a = 1 - b$ and also $m = \frac{1}{2b}$ using (7),

$$\Delta y = (1-b)\,k_1 + bk_2$$

where $k_1 = hf(x, y)$

$$k_2 = hf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

Now $\Delta y = y(x+h) - y(x)$

$$\therefore y(x+h) = y(x) + (1-b)\,hf + bhf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

i.e.,

$$y_{n+1} = y_n + (1-b)\,hf(x_n, y_n) + bhf\left(x_n + \frac{h}{2b}, y_n + \frac{h}{2b} f(x_n, y_n)\right) + O(h^3)$$

From this general second order Runge-Kutta formula, setting $a = 0, b = 1, m = \frac{1}{2}$, we get the second order Runge-Kutta algorithm as

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

and $\Delta y = K_2$ where $h = \Delta x$.

Second order R.K. algorithm

Since the derivations of third and fourth order Runge-Kutta algorithms are tedious, we state them below for use.

The third order Runge-Kutta method algorithm is given below:

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

$$k_3 = hf(x + h, y + 2k_2 - k_1)$$

and $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$

Third order R.K. algorithm

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2} k_2\right) = (0{\cdot}1) f(0{\cdot}15, 1{\cdot}1763848)$$

$$= (0{\cdot}1)(0{\cdot}15 + 1{\cdot}1763848) = 0{\cdot}13263848$$

$$k_4 = h f(x_1 + h, y_1 + k_3) = (0{\cdot}1) f(0{\cdot}2, 1{\cdot}24298048)$$

$$= 0{\cdot}144298048$$

$$y(0{\cdot}2) = y(0{\cdot}1) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1{\cdot}110342 + \frac{1}{6}(0{\cdot}794781608)$$

$$y(0{\cdot}2) = \mathbf{1{\cdot}2428055}$$

Correct to four decimal places, $y(0{\cdot}2) = \mathbf{1{\cdot}2428}$.

Example 2. *Obtain the values of y at $x = 0{\cdot}1, 0{\cdot}2$ using R.K. method of (i) second order (ii) third order and (iii) fourth order for the differential equation $y' = -y$, given $y(0) = 1$.* (MKU 1971)

Solution. Here, $f(x, y) = -y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0{\cdot}1$, $x_2 = 0{\cdot}2$.

(i) Second order:

$$k_1 = h f(x_0, y_0) = (0{\cdot}1)(-y_0) = -0{\cdot}1$$

$$k_2 = h f\left(x_0 + \frac{1}{2} h, y_0 + \frac{1}{2} k_1\right) = (0{\cdot}1) f(0{\cdot}05, 0{\cdot}95)$$

$$= -0{\cdot}1 \times 0{\cdot}95 = -0{\cdot}095 = \Delta y$$

$$y_1 = y_0 + \Delta y = 1 - 0{\cdot}095 = 0{\cdot}905$$

$$y_1 = y(0{\cdot}1) = \mathbf{0{\cdot}905}$$

Again starting from $(0{\cdot}1, 0{\cdot}905)$ replacing (x_0, y_0) by (x_1, y_1) we get

$$k_1 = (0{\cdot}1) f(x_1, y_1) = (0{\cdot}1)(-0{\cdot}905) = -0{\cdot}0905$$

$$k_2 = h f\left(x_1 + \frac{1}{2} h, y_1 + \frac{1}{2} k_1\right)$$

$$= (0{\cdot}1)\,[f(0{\cdot}15, 0{\cdot}85975)] = (0{\cdot}1)(-0{\cdot}85975) = -0{\cdot}085975$$

$$\Delta y = k_2$$

$$\therefore\ y_2 = y(0{\cdot}2) = y_1 + \Delta y = \mathbf{0{\cdot}819025}$$

(ii) Third order:

$$k_1 = h f(x_0, y_0) = -0{\cdot}1$$

$$k_2 = h f\left(x_0 + \frac{1}{2} h, y_0 + \frac{1}{2} k_1\right) = -0{\cdot}095$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$$

$$= (0{\cdot}1) f(0{\cdot}1, 0{\cdot}9) = (0{\cdot}1)(-0{\cdot}9) = -0{\cdot}09$$

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$y(0{\cdot}1) = y_1 = y_0 + \Delta y = 1 - 0{\cdot}09 = \mathbf{0{\cdot}91}$$

Again taking (x_1, y_1) as (x_0, y_0) repeat the process.

$\therefore \quad k_1 = hf(x_1, y_1) = (0{\cdot}1)(-0{\cdot}91) = -0{\cdot}091$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0{\cdot}1)f(0{\cdot}15, 0{\cdot}865) = (0{\cdot}1)(-0{\cdot}865) = -0{\cdot}0865$$

$$k_3 = hf(x_1 + h, y_1 + 2k_2 - k_1)$$

$$= (0{\cdot}1)f(0{\cdot}2, 0{\cdot}828) = -0{\cdot}0828$$

$$y_2 = y_1 + \Delta y = 0{\cdot}91 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= 0{\cdot}91 + \frac{1}{6}(-0{\cdot}091 - 0{\cdot}3460 - 0{\cdot}0828)$$

$$y(0{\cdot}2) = \mathbf{0{\cdot}823366}$$

(*iii*) Fourth order:

$$k_1 = hf(x_0, y_0) = (0{\cdot}1)f(0, 1) = -0{\cdot}1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0{\cdot}1)f(0{\cdot}05, 0{\cdot}95) = -0{\cdot}095$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0{\cdot}1)f(0{\cdot}05, 0{\cdot}9525)$$

$$= -0{\cdot}09525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0{\cdot}1)f(0{\cdot}1, 0{\cdot}90475)$$

$$= -\mathbf{0{\cdot}090475}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \Delta y = 1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y(0{\cdot}1) = \mathbf{0{\cdot}9048375}$$

Again start from this (x_1, y_1) and replace (x_0, y_0) and repeat

$$k_1 = hf(x_1, y_1) = (0{\cdot}1)(-y_1) = -0{\cdot}09048375$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$= (0{\cdot}1)f(0{\cdot}15, 0{\cdot}8595956) = -0$ cdot 08595956

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$= (0{\cdot}1)f(0{\cdot}15, 0{\cdot}8618577) = -0$ cdot 08618577

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0{\cdot}1)f(0{\cdot}2, 0{\cdot}8186517) = -0{\cdot}08186517$$

$$\Delta y = \frac{1}{6}(-0{\cdot}09048375 - 2 \times 0{\cdot}08595956$$

$$- 2 \times 0{\cdot}08618577 - 0{\cdot}08186517)$$

$k_4 = (0{\cdot}1) f(0{\cdot}1, 2{\cdot}084238, 0{\cdot}5878)$;
$= (0{\cdot}1)(0{\cdot}1 + 0{\cdot}5878)$
$= 0{\cdot}06878$

$l_4 = (0{\cdot}1)(0{\cdot}1 - (2{\cdot}084238)^2)$
$= -0{\cdot}4244$

$$y_1 = 2 + \frac{1}{6}[0{\cdot}1 + 2(0{\cdot}085 + 0{\cdot}084238) + 0{\cdot}06878] = 2{\cdot}0845$$

$$z_1 = 1 + \frac{1}{6}[-0{\cdot}4 - (0{\cdot}41525 + 0{\cdot}4122) \times 2 - 0{\cdot}4244]$$
$$= 0{\cdot}5868$$

$y(0{\cdot}1) = 2{\cdot}0845$ and $z(0{\cdot}1) = 0{\cdot}5868$.

11·15. Runge-Kutta method for second order differential equation

AIM. *To solve $y'' = f(x, y, y')$, given $y(x_0) = y_0$, $y'(x_0) = y_0'$.*

Now, set $y' = z$ and $y'' = z'$

Hence, differential equation reduces to

$$\frac{dy}{dx} = y' = z$$

and $$\frac{dz}{dx} = z' = y'' = f(x, y, y') = f(x, y, z)$$

$$\therefore \left.\begin{aligned} \frac{dy}{dx} &= z \\ \text{and} \quad \frac{dz}{dx} &= f(x, y, z) \end{aligned}\right\}$$

are simultaneous equations where $f_1(x, y, z) = z$
$f_2(x, y, z) = f(x, y, z)$ given.
Also $y(0)$ and $z(0)$ are given.

Starting from these equations, we can use the previous article and solve the problem.

Example 7. *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, find the value of $y(0{\cdot}1)$ by using Runge-Kutta method of fourth order.*

Solution. $y'' = -xy' - y$, $y(0) = 1$, $y'(0) = 0$, $h = 0{\cdot}1$, $y_0 = 1$,
$x_0 = 0$, $y_1 = y(0{\cdot}1)$

Setting $y' = z$

The equation becomes,

$$y'' = z' = -xz - y$$

$$\therefore \frac{dy}{dx} = z = f_1(x, y, z) \quad \text{...(1)}$$

$$\frac{dz}{dx} = -xz - y = f_2(x, y, z) \quad \text{...(2)}$$

given $y_0 = 1$, $z_0 = y_0' = 0$.

By algorithm,

$$k_1 = hf_1(x_0, y_0, z_0) = (0{\cdot}1) f_1(0, 1, 0) = (0{\cdot}1)(0) = 0$$

$$l_1 = hf_2(x_0, y_0, z_0) = (0{\cdot}1) f_2(0, 1, 0) = (-1)(0{\cdot}1) = -0{\cdot}1$$

$$k_2 = hf_1\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)$$

$$=(0\cdot1)f_1(0\cdot05, 1, -0\cdot05) = (0\cdot1)(-0\cdot05) = -0\cdot005$$

$$l_2 = (0\cdot1)f_2(0\cdot05, 1, -0\cdot05) = (0\cdot1)[+(0\cdot05)(0\cdot05) - 1]$$

$$= -0\cdot09975$$

$$k_3 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)$$

$$=(0\cdot1)f_1(0\cdot05, 0\cdot9975, -0\cdot0499)$$

$$=(0\cdot1)(-0\cdot0499) = -0\cdot00499$$

$$l_3 = hf_2(0\cdot05, 0\cdot9975, -0\cdot0499)$$

$$= -(0\cdot1)[(0\cdot05)(-0\cdot04991) + 0\cdot9975]$$

$$= -0\cdot09950$$

$$k_4 = hf_1(x_0+h, y_0+k_3, z_0+l_3)$$

$$=(0\cdot1)f_1(0\cdot1, 0\cdot99511, -0\cdot0995)$$

$$=(0\cdot1)(-0\cdot0995) = -0\cdot00995$$

$$l_4 = hf_2(0\cdot1, 0\cdot99511, -0\cdot0995)$$

$$=(0\cdot1)[-\{(0\cdot1)(-0\cdot0995) + 0\cdot99511\}]$$

$$= -0\cdot0985$$

$$\therefore \quad y_1 = y_0 + \Delta y = 1 + \frac{1}{6}[0 + 2(-0\cdot005) + 2(-0\cdot00499) - 0\cdot00995]$$

$$= 0\cdot9950$$

$y(0\cdot1) = 0\cdot9950$.

EXERCISE 11.4

Evaluate using Runge-Kutta methods. Unless otherwise mentioned, use fourth order R.K. method.

1. Find $y(0\cdot2)$ given $\frac{dy}{dx} = y - x$, $y(0) = 2$ taking $h = 0\cdot1$.
2. Evaluate $y(1\cdot4)$ given $\frac{dy}{dx} = x + y$, $y(1\cdot2) = 2$.
3. Obtain the value of y at $x = 0\cdot2$ if y satisfies
   $\frac{dy}{dx} - x^2y = x$; $y(0) = 1$ taking $h = 0\cdot1$.
4. Solve $\frac{dy}{dx} = xy$ for $x = 1\cdot4$, taking $y(1) = 2$, $h = 0\cdot2$.
5. Solve: $y' = \frac{y-x}{y+x}$ given $y(0) = 1$, to obtain $y(0\cdot2)$.
6. Solve the initial value problem

$\frac{du}{dt} = -2tu^2, u(0) = 1$ with $h = 0.2$ on the interval $(0, 0.6)$ by using fourth order R.K. method. (Nov. 1991)

7. Evaluate for $y(0.1), y(0.2), y(0.3)$ given $y' = \frac{1}{2}(1 + x)y^2, y(0) = 1$.

8. Solve: $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, y(1) = 1$ for $y(1.1)$ taking $h = 0.05$.

9. Find $y(0.5), y(1), y(1.5), y(2)$ taking $h = 0.5$ given $y' = \frac{1}{x + y}, y(0) = 1$.

10. Evaluate $y(1.2)$ and $y(1.4)$ given $y' = \frac{2xy + e^x}{x^2 + xe^x}, y(1) = 0$. (MS. Ap. 1990)

11. Find y for $x = 0.2\,(0.2)\,0.6$ given $\frac{dy}{dx} = 1 + y^2, y(0) = 0$.

12. Find $y(0.2)$ given $\frac{dy}{dx} = -xy, y(0) = 1$, taking $h = 0.2$ by R.K. method of 4th order.

13. Find $y(0.1), y(0.2)$ given $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ by (i) second order, third order and fourth order R.K. method.

14. Determine y at $x = 0.2\,(0.2)\,(0.6)$ by R.K method given $\frac{dy}{dx} = \frac{1}{1 + x}$, given $y(0) = 0$.

15. Find $y(0.2)$ given
$y' = 3x + \frac{1}{2}y, y(0) = 1$ by using Runge-Kutta method of 4th order.

16. Solve $y' = xy + 1$ as $x = 0.2, 0.4, 0.6$ given $y(0) = 2$, taking $h = 0.2$.

17. Given $y' = x^3 + \frac{1}{2}y, y(1) = 2$, find $y(1.1), y(1.2)$.

18. Solve $10y' = x^2 + y^2$, given $y(0) = 1$ for $x = 0.1\,(0.1)\,(0.3)$.

19. Solve $8y' = x + y^2$ given $y(0) = 0.5$ for $x = 0.1\,(0.1)\,(0.4)$.

20. Solve the system: $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy$ for $x = 0.3\,(0.3)\,(0.9)$ taking $x = 0$, $y = 0, z = 1$. (MKU 1979)

21. Solve: $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y$, given $y(0) = 0, z(0) = 1$ for $x = 0.0$ to 0.2 taking $h = 0.1$.

22. Solve $\frac{dy}{dx} = -xz, \frac{dz}{dx} = y^2$, given $y(0) = 1, z(0) = 1$ for $x = 0\,(0.2)\,(0.4)$.

23. Evaluate $y(1.1), z(1.1)$ given $\frac{dy}{dx} = xyz, \frac{dz}{dx} = \frac{xy}{z}, y(1) = 1/3, z(1) = 1$.

24. Using R.K. method determine $x(0.1), y(0.1)$ given $\frac{dx}{dt} = xy + t,\; x(0) = 1$