4

Matrices

4.1 DEFINITION

Let us consider a set of simultaneous equations,

$$x + 2y + 3z + 5t = 0$$
$$4x + 2y + 5z + 7t = 0$$
$$3x + 4y + 2z + 6t = 0.$$

Now we write down the coefficients of x, y, z, t of the above equations and enclose them within brackets and then we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and columns and bounded by the brackets, is called a matrix.

It has got 3 rows and 4 columns and in all $3 \times 4 = 12$ elements. It is termed as 3×4 matrix, to be read as [3 by 4 matrix]. In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, a_{ij} lies in the ith row and jth column.

4.2 VARIOUS TYPES OF MATRICES

(1) **Row Matrix.** If a matrix has only one row and any number of columns, it is called a Row matrix, *e.g.*,

$$[2 \ 7 \ 3 \ 9]$$

(2) **Column Matrix.** A matrix, having one column and any number of rows, is called Column matrix, e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(3) **Null Matrix or Zero Matrix.** Any matrix, in which all the elements are zeros, called a Zero matrix or Null matrix *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(*d*) **Square Matrix.** A matrix, in which the number of rows is equal to the number columns, is called a square matrix *e.g.*,

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

(*e*) **Diagonal Matrix.** A square matrix is called a diagonal matrix, if all its non-diag elements are zero *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(*f*) **Unit or Identity Matrix.** A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(*g*) **Symmetric Matrix.** A square matrix will be called symmetric, if for all values of i and j, *i.e.* $a_{ij} = a_{ji}$ or $A' = A$ *e.g.*,

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(*h*) **Skew Symmetric Matrix.** A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ij}$ for all values of i and j, or $A' = -A$

(2) All diagonal elements are zero, *e.g.*,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(*i*) **Triangular Matrix.** A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix*. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix *e.g.*,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix   Lower triangular matrix

(*j*) **Transpose of a Matrix.** If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by A' or A^T *e.g.*,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(*k*) **Orthogonal Matrix.** A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A' is an identity matrix *e.g.*,

$$A \cdot A' = I$$

if $|A| = 1$, matrix A is proper.

(*l*) **Conjugate of a Matrix**

Let
$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Conjugate of matrix of A is $\overline{A}$

$$\therefore \quad \overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

(*m*) **Matrix A^{θ}.** Transpose of the conjugate of a matrix A is denoted by A^{θ}.

Let

$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & +i & 3+2i \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

or

$$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(*n*) **Unitary Matrix.** A square matrix A is said to be unitary if

$$A^{\theta} A = I$$

e.g. $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$, $A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$, $A \cdot A^{\theta} = I$

(*o*) **Hermitian Matrix.** A square matrix $A = (a_{ij})$ is called Hermiatian matrix, if every *i-j*th element of A is equal to conjugate complex *j-i*th element of A.

In other words $a_{ij} = \bar{a}_{ji}$

e.g.

$$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$ *i.e.* conjugate transpose of A

or $A = (\bar{A})'$.

(*p*) **Skew Hermitian Matrix.** A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every *i-j*th element of A is equal to negative conjugate complex of *j-i*th element of A.

In other words $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form

$$a_{ii} = -\bar{a}_{ii} \quad \text{or} \quad a_{ii} + \bar{a}_{ii} = 0$$

If $a_{ii} = a + ib$ then $\bar{a}_{ii} = a - ib$

$$(a+ib) + (a-ib) = 0 \quad \text{or} \quad 2a = 0 \text{ or } a = 0$$

So a_{ii} is pure imaginary or $a_{ii} = 0$.

Hence all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g.

$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that

$$A^{\theta} = -A$$

$$(\bar{A})' = -A$$

7. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

8. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha, & \cos\alpha \end{bmatrix}$, then show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, where n is positive integer.

9. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

10. If $f(x) = x^3 - 20x + 8$, find $f(A)$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **Ans.** 0

11. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

12. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ show that $A^3 = A^{-1}$.

13. Verify whether the following matrix is orthogonal. $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

14. Verify that $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix.

15. Show that $\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is an orthogonal matrix.

16. Show that $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

17. If A and B are square matrices of the same order, explain in general
(*i*) $(A+B)^2 \neq A^2 + 2AB + B^2$ (*ii*) $(A-B)^2 \neq A^2 - 2AB + B^2$ (*iii*) $(A+B)(A-B) \neq A^2 - B^2$

18. Let A and B be any two matrices such that $AB = 0$ and A is non-singular.
Then (*a*) $B = 0$; (*b*) B is also non-singular; (*c*) $B = A$; (*d*) B is singular.
(*A.M.I.E.T.E., Winter 1996*) **Ans.** (

4.10 ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be $|A|$.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ Then $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

The matrix formed by the co-factors of the elements in $|A|$ is

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

where $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2, \quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1$

$$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1, \quad B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2$$

$$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1, \quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1$$

$$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2, \quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1$$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Then the transpose of the matrix of co-factors

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

is called the adjoint of the matrix A and is written as adj A.

4.11 PROPERTY OF ADJOINT MATRIX

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant A.

If A be a square matrix, then $(\text{Adjoint} A) \cdot A = A \cdot (\text{Adjoint} A) = |A| \cdot I$

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ and adj. $A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$$A \cdot (\text{adj. } A) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 A_1 + a_2 A_2 + a_3 A_3 & a_1 B_2 + a_2 B_2 + a_3 B_3 & a_1 C_1 + a_2 C_2 + a_3 C_3 \\ b_1 A_1 + b_2 A_2 + b_3 A_3 & b_1 B_1 + b_2 B_2 + b_3 B_3 & b_1 C_1 + b_2 C_2 + b_3 C_3 \\ c_1 A_1 + c_2 A_2 + c_3 A_3 & c_1 B_1 + c_2 B_2 + c_3 B_3 & c_1 C_1 + c_2 C_2 + c_3 C_3 \end{bmatrix}$$

$$\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I$$

4.12 INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that

$$AB = BA = I \qquad (I = \text{unit matrix})$$

hen B is called the inverse of A *i.e.* $B = A^{-1}$ and A is the inverse of B.

Condition for a square matrix A to possess an inverse is that matrix A is non-singular, ., $|A| \neq 0$

If A is a square matrix and B be its inverse, then $AB = I$

Taking determinant of both sides

$$|AB| = |I| \quad \text{or} \quad |A||B| = I$$

From this relation it is clear that $|A| \neq 0$

i.e. the matrix A is non-singular.

To find the inverse matrix by the help of adjoint matrix

We know that $A \cdot (Adj.\ A) = |A| I$

or $A \cdot (Adj) \cdot \frac{A}{|A|} = I$ Provided $|A| \neq 0$

and $A \cdot A^{-1} = I \quad \therefore \quad A^{-1} = \frac{1}{|A|}(Adj \cdot A)$

Example 18. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *find* A^{-1}.

Solution. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = 3(-3+4) + 3(2-0) + 4(-2-0) = 3+6-8 = 1$$

The co-factors of elements of various rows of $|A|$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

Therefore the matrix formed by the co-factors of $|A|$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}, \quad Adj.\ A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} Adj.\ A = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$ **Ans.**

Example 19. *If* $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, *prove that* $A^{-1} = A'$, A' *being the transpose of A.*

(*A.M.I.E., Winter 2000*)

Solution. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}, \quad A' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$$AA' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } AA' = I$$

$\therefore \quad A' = A^{-1}$ **Proved**

Example 20. *If a matrix A satisfies a relation $A^2 + A - I = 0$ prove that A^{-1} exists and that $A^{-1} = I + A$, I being an identity matrix.*

Solution. Here $A^2 + A - I = 0$ or $A^2 + AI = I$ or $A(A + I) = I$

$\therefore$ $|A||A+I| = |I|$

$\therefore$ $|A| \neq 0$ and so A^{-1} exists.

Again $A^2 + A - I = 0$ or $A^2 + A = I$ ...(1)

Multiplying (1) by A^{-1} we get

$$A^{-1}(A^2 + A) = A^{-1}I \quad \text{or} \quad A + I = A^{-1}$$

or $A^{-1} = I + A$ **Proved.**

Example 21. *If A and B are non-singular matrices of the same order then,* (A.M.I.E., Winter 1998, Summer 1996)

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

Hence prove that $(A^{-1})^m = (A^m)^{-1}$ for any positive integer m.

Proof. We know that

$$(AB) \cdot (B^{-1}A^{-1}) = [(AB)B^{-1}] \cdot A^{-1} = [A(BB^{-1})] \cdot A^{-1}$$
$$= [AI] \cdot A^{-1} = A \cdot A^{-1} = I$$

Also $B^{-1}A^{-1} \cdot (AB) = B^{-1}[A^{-1} \cdot (AB)] = B^{-1}[(A^{-1}A) \cdot B]$
$$= B^{-1}[I \cdot B] = B^{-1} \cdot B = I$$

By definition of the inverse of a matrix then $B^{-1}A^{-1}$ is inverse of AB.

or $B^{-1}A^{-1} = (AB)^{-1}$ **Proved.**

$$(A^m)^{-1} = [A \cdot A^{m-1}]^{-1} = (A^{m-1})^{-1} A^{-1}$$
$$= (A \cdot A^{m-2})^{-1} \cdot A^{-1} = [(A^{m-2})^{-1} \cdot A^{-1}] \cdot A^{-1} = (A^{m-2})^{-1}(A^{-1})^2$$
$$= [A \cdot A^{m-3}]^{-1}(A^{-1})^2 = [(A^{m-3})^{-1} \cdot A^{-1}](A^{-1})^2 = (A^{m-3})^{-1}(A^{-1})^3$$
$$= A^{-1}(A^{-1})^{m-1} = (A^{-1})^m$$ **Proved.**

Example 22. *Prove that the inverse of a matrix is unique.*

Proof. We suppose that B and C are two inverse matrices of a given matrix say A.

Then $AB = BA = I$ $\because$ B is inverse of A.

and $AC = CA = I$ $\because$ C is inverse of A.

But $C \cdot (AB) = (CA) \cdot B$ (Associative law)

or $C \cdot I = I \cdot B$ or $C = B$

Hence the inverse of matrix A is unique. **Proved.**

Example 23. *Find A satisfying the Matrix equation.*

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Solution. $$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Both sides of the equation are pre-multiplied by the inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ *i.e.* $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}$$

$$A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}$$

Again both sides are post-multiplied by the inverse of $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ *i.e.* $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$$A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

$$A = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$ **Ans.**

Example 24. *Given* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

find C such that $BCA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution. $BCA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

or $$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = -\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

or $$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = -\begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix}$$

$$C = -\begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}^{-1}$$

$$C = \begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$$ **Ans.**

Exercise 4.3

Find the adjoint and inverse of the following matrices: (1–5)

1. $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$ **Ans.** $-\frac{1}{3}\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ **Ans.** $\frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

5. $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, show that $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, show that $(AB)^{-1} = B^{-1}A^{-1}$.

8. Given the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ compute det (A), A^{-1} and the matrix B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$

   Also compute BA. Is $AB = BA$?

   Ans. $5, \frac{1}{5}\begin{bmatrix} 9 & -2 & -4 \\ 1 & 2 & -1 \\ -12 & 1 & 7 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $AB \neq BA$

9. Find the condition of k such that the matrix

   $$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & k & 6 \\ -1 & 5 & 1 \end{bmatrix}$$ has an inverse. Obtain A^{-1} for $k = 1$.

   (*A.M.I.E.T.E., Summer 1997*)

   Ans. $k \neq -\frac{3}{5}$, $A^{-1} = \frac{1}{8}\begin{bmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{bmatrix}$

10. Prove that $(A^{-1})^T = (A^T)^{-1}$.

11. Let I be the unit matrix of order n and adj. $(2I) = 2^k I$. Then k equals

    (*a*) 1 (*b*) 2 (*c*) $n - 1$ (*d*) n. **Ans.** (c)

4.13 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Let the equations be

$$a_1 x + a_2 y + a_3 z = d_1$$
$$b_1 x + b_2 y + b_3 z = d_2$$
$$c_1 x + c_2 y + c_3 z = d_3$$

We write the above equations in the matrix form

$$\begin{bmatrix} a_1 x + a_2 y + a_3 z \\ b_1 x + b_2 y + b_3 z \\ c_1 x + c_2 y + c_3 z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B \qquad \ldots(1)$$

Find the inverse of the following matrices by partitioning:

4. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{10}\begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ **Ans.** $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 & 2 & 7 \\ 2 & 3 & 3 & 2 \\ 52 & 7 & 3 & 9 \\ 2 & 3 & 2 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{2}\begin{bmatrix} -1 & 11 & 7 & -26 \\ -1 & -7 & -3 & 16 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 2 \end{bmatrix}$

4.25 CHARACTERISTIC ROOTS OR EIGEN VALUES

(*a*) For a given square matrix A, $A - \lambda I$ matrix is called the *characteristic matrix*, where λ is scalar and I is the unit matrix.

Let
$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

characteristic matrix

(*b*) **Characteristic Polynomial**

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

For example $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)(6 - 5\lambda + \lambda^2 - 2) - 2(2 - \lambda - 1) + 1(2 - 3 + \lambda)$$
$$= \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

(*c*) **Characteristic Equation**

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A *e.g.*

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

(*d*) **Characteristic Roots or Eigenvalues**

The roots of characteristic equation $|A - \lambda I| = 0$ are called characteristic roots of matrix A. *e.g.*

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$
$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0 \quad \therefore \quad \lambda = 1, 1, 5$$

Characteristic roots are 1, 1, 5.

Some Important Properties of Eigenvalues

(1) Any square matrix A and its transpose A' have the same eigenvalues.

Note. The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.

(2) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

(3) The product of the eigenvalues of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, then the eigenvalues of

(*i*) kA are $k\lambda_1, k\lambda_2, \ldots\ldots, k\lambda_n$ (*ii*) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots, \lambda_n^m$

(*iii*) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots, \frac{1}{\lambda_n}$.

Example 59. *Find the characteristic roots of the matrix* (A.M.I.E.T.E., Winter 1997)

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution. The characteristic equation of the given matrix is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

or $$(6-\lambda)(9-6\lambda+\lambda^2-1)+2(-6+2\lambda+2)+2(2-6+2\lambda) = 0$$

or $$-\lambda^3+12\lambda^2-36\lambda+32 = 0$$

By trial, $\lambda = 2$ is a root of this equation.

or $$(\lambda-2)(\lambda^2-10\lambda+16) = 0 \quad \text{or} \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

or $\lambda = 2, 2, 8$ are the characteristic roots or Eigen values. **Ans.**

Example 60. *The matrix A is defined as*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Find the eigenvalues of $3A^3+5A^2-6A+2I$.

Solution. $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = 0$$

or $$(1-\lambda)(3-\lambda)(-2-\lambda) = 0 \quad \text{or} \quad \lambda = 1, 3, -2$$

Eigenvalues of $A^3 = 1, 27, -8$; Eigenvalues of $A^2 = 1, 9, 4$

Eigenvalues of $A = 1, 3, -2$; Eigenvalues of $I = 1, 1, 1$

$\therefore$ Eigenvalues of $3A^3+5A^2-6A+2I$

First eigenvalue $= 3(1)^2+5(1)^2-6(1)+2 = 4$

Second eigenvalue $= 3(27)+5(9)-6(3)+2(1) = 110$

Third eigenvalue $= 3(-8)+5(4)-6(-2)+2(1) = 10$

Required eigenvalues are 4, 110, 10. **Ans.**

Example 61. *If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, find the eigen values of the matrix* $(A - \lambda I)^2$.

Solution. $(A - \lambda I)^2 = A^2 - 2\lambda AI + \lambda^2 I^2 = A^2 - 2\lambda A + \lambda^2 I$

Eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2 \ldots \lambda_n^2$

Eigenvalues of $2\lambda A$ are $2\lambda\lambda_1, 2\lambda\lambda_2, 2\lambda\lambda_3 \ldots. 2\lambda\lambda_n$.

Eigenvalues of $\lambda^2 I$ are λ^2,

$\therefore$ Eigenvalues of $A^2 - 2\lambda A + \lambda^2 I$ are

$$\lambda_1^2 - 2\lambda\lambda_1 + \lambda^2, \lambda_2^2 - 2\lambda\lambda_2 + \lambda^2, \lambda_3^2 - 2\lambda\lambda_3 + \lambda^2 \ldots.$$

or
$$(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, (\lambda_3 - \lambda)^2, \ldots.. (\lambda_n - \lambda)^2$$ **Ans.**

Example 62. *Prove that the following matrices have the same characteristic equation.*

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

Solution. Characteristic equation of first matrix is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0.$$

$$(a - \lambda)[\lambda^2 - \lambda(b + c) + bc - a^2] - b(b^2 - ac - b\lambda) + c(ab - c^2 + c\lambda) = 0$$

or
$$-\lambda^3 + \lambda^2(a + b + c) + \lambda(-ab - ac - bc + a^2 + b^2 + c^2)$$
$$+ (abc - a^3 - b^3 + abc + abc - c^3) = 0$$

or
$$\lambda^3 - \lambda^2(a + b + c) - \lambda(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$+ (a^3 + b^3 + c^3 - 3abc) = 0.$$

The symmetry of the result shows that characteristic equation for the other two matrices will also be same. **Proved**

Example 63. *Prove that a matrix A and its transpose A′ have the same characteristic roots.*

Solution. Characteristic equation of matrix A is

$$|A - \lambda I| = 0. \quad \ldots(1)$$

Characteristic equation of matrix A' is

$$|A' - \lambda I| = 0 \quad \ldots(2)$$

Clearly both (1) and (2) are same, as we know that

$$|A| = |A'|$$

i.e., a determinant remains unchanged when rows be changed into columns and columns into rows.

Proved

Example 64. *If A and P be square matrices of the same type and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.*

(*A.M.I.E.T.E., Summer 1998, Winter 1996*)

Solution. Let us put $B = P^{-1}AP$ and we will show that characteristic equations for both A and B are the same and hence they have the same characteristic roots.

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$$

$\therefore \quad |B - \lambda I| = |P^{-1}(A - \lambda I) P| = |P^{-1}| |A - \lambda I| |P|$

$= |A - \lambda I| |P^{-1}| |P| = |A - \lambda I| |P^{-1} P|$

$= |A - \lambda I| |I| = |A - \lambda I|$ as $|I| = 1$.

Thus the matrices A and B have the same characteristic equations and hence same characteristic roots. **Proved**

Example 65. *If A and B be two square invertible matrices, then prove that AB and BA have the same characteristic roots.*

Solution. Now $\quad AB = IAB = B^{-1} B (AB) = B^{-1} (BA) B \quad$ ...(1)

But by Ex. 64, matrices BA and $B^{-1} (BA) B$ have same characteristic roots or matrices BA and AB by (1) have same characteristic roots. **Proved**

Example 66. *If A and B be n rowed square matrices and if A be invertible, show that the matrices $A^{-1} B$ and BA^{-1} have the same characteristic roots.*

Solution. $\quad A^{-1} B = A^{-1} BI = A^{-1} B (A^{-1} A) = A^{-1} (BA^{-1}) A. \quad$ ...(1)

But by Ex. 64, matrices BA^{-1} and $A^{-1} (BA^{-1}) A$ have same characteristic roots or matrices BA^{-1} and $A^{-1} B$ by (1) have same characteristic roots. **Proved**

Example 67. *Show that 0 is a characteristic root of a matrix, if and only if, the matrix is singular.*

Solution. Characteristic equation of matrix A is given by

$$|A - \lambda I| = 0.$$

If $\lambda = 0$ then from above it follows that $|A| = 0$ *i.e.* Matrix A is singular.

Again if Matrix A is singular *i.e.*, $|A| = 0$ then

$$|A - \lambda I| = 0 \Rightarrow |A| - \lambda |I| = 0, 0 - \lambda \cdot 1 = 0 \Rightarrow \lambda = 0.$$ **Proved**

Example 68. *Show that characteristic roots of a triangular matrix are just the diagonal elements of the matrix.*

Solution. Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Characteristic equation is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} a_{11} - \lambda & 0 & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 & 0 \\ a_{31} & a_{32} & a_{33} - \lambda & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} - \lambda \end{vmatrix} = 0$$

On expansion it gives

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)(a_{44} - \lambda) = 0$$

$\therefore \quad \lambda = a_{11}, a_{22}, a_{33}, a_{44}$

which are diagonal elements of matrix A. **Proved**

Example 69. *The characteristic roots of a Hermitian matrix are all real.*

Solution. We know that matrix A is Hermitian if

$$A^{\theta} = A \text{ i.e., where } A^{\theta} = (\overline{A'}) \text{ or } (\overline{A})'.$$

Also $(\lambda A)^\theta = \bar{\lambda} A^\theta$ and $(AB)^\theta = B^\theta A^\theta$.

If λ is a characteristic root of matrix A then $AX = \lambda X$. ...(1)

$\therefore$ $(AX)^\theta = (\lambda X)^\theta$ or $X^\theta A^\theta = \bar{\lambda} X^\theta$.

But A is Hermitian $\therefore$ $A^\theta = A$.

$\therefore$ $X^\theta A = \bar{\lambda} X^\theta$ $\therefore$ $X^\theta AX = \bar{\lambda} X^\theta X$. ...(2)

Again from (1) $IX^\theta AX = X^\theta \lambda X = \lambda X^\theta X$. ...(3)

Hence from (2) and (3) we conclude that $\bar{\lambda} = \lambda$ showing that λ is real.

Deduction 1. *From above we conclude that characteristic roots of real symmetric matrix are all real, as in this case, real symmetric matrix will be Hermitian.*

For symmetric, we know that $A' = A$. $(\bar{A}') = \bar{A}$.

or $A^\theta = A$ $\therefore$ $\bar{A} = A$ as A is real. Rest as above.

Deduction 2. *Characteristic of a skew Hermitian matrix is either zero or a pure imaginary number.*

If A is skew Hermitian then iA is Hermitian.

Also λ be a characteristic root of A then $AX = \lambda X$.

$\therefore$ $(i \cdot A) X = (i\lambda) X$.

Above shows that $i\lambda$ is characteristic root of matrix iA, which is Hermitian and hence $i\lambda$ should be real, which will be possible if λ is either pure imaginary or zero.

Example 70. *The modulus of each characteristic root of a unitary matrix is unity.*

Solution. If A be a unitary matrix, then we know $A^\theta A = I$.

If λ be characteristic root of A, then $AX = \lambda X$. ...(1)

$$(AX)^\theta = (\lambda X)^\theta \text{ or } X^\theta A^\theta = \bar{\lambda} X^\theta. \quad ...(2)$$

$$X^\theta A^\theta AX = \lambda X^\theta \lambda X = \bar{\lambda} X \lambda X^\theta X.$$

But $A^\theta A = I$.

$$X^\theta IX = \bar{\lambda}\lambda X^\theta X \text{ or } X^\theta X = \bar{\lambda}\lambda X^\theta X$$

$\therefore$ $(1 - \lambda\bar{\lambda}) X^\theta X = 0$.

But $X \neq 0 \therefore X^\theta X \neq 0$ and hence $1 - \lambda\bar{\lambda} = 0$ or $1 = \bar{\lambda} \cdot \lambda$ or $|\lambda|^2 = 1$.

i.e. characteristic roots are unimodular. .

Example 71. *If λ is an eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is also eigenvalue.*

(*A.M.I.E.T.E., Winter 1995*)

[**Hint:** $AA' = I$ if λ is the eigen value of A, then $\lambda^2 = I$, $\lambda = \frac{1}{\lambda}$]

4.26 CAYLEY-HAMILTON THEOREM

Statement. Every square matrix satisfies its own characteristic equation.

If $|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n)$ be the characteristic polynomial of $n \times n$ matrix $A = (a_{ij})$, then the matrix equation

$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots + a_n I = 0$ is satisfied by $X = A$ *i.e.*,

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

8. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

satisfies its characteristic equation. Hence find A^{-1}.

(A.M.I.E.T.E., Winter 1995) **Ans.** $\frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

9. Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Hence evaluate A^{-1}. **Ans.** $\frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$

10. Use Cayley Hamilton Theorem to find the inverse of **Ans.** $A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

11. If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix

$\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$ then $\lambda_1 + \lambda_2 + \lambda_3$ is equal to

(*i*) −16 (*ii*) 2 (*iii*) −6 (*iv*) −14 (A.M.I.E.T.E., Winter 1997) **Ans.** (*ii*)

12. The matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ is given. The eigenvalues of $4A^{-1} + 3A + 2I$ are

(A) 6, 15; (B) 9, 12 (C) 9, 15; (D) 7, 15 (A.M.I.E.T.E., Winter 1996) **Ans.** (C)

13. The matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

Find the eigenvalues of $3A^3 + 5A^2 + 6A + I$. **Ans.** 15, − 15, − 53

4.27 CHARACTERISTIC VECTORS OR EIGENVECTORS

Let A be a $n \times n$ square matrix and Y and X are two non-zero column vectors such that

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ x_2 \\ \vdots \\ X_n \end{bmatrix}$$

$Y = AX \Rightarrow A$ transformsvector X to vector Y.

Two vectors X and Y have the same direction. Here we have to determine those vectors X whose images Y are given by

$$Y = \lambda X.$$

Corresponding to each characteristic root λ we have a corresponding non-zero vector X which satisfies the equation $|A - \lambda I| X = 0$. The non-zero vector X is called characteristic vector or Eigenvectors.

19.28 PROPERTIES OF EIGENVECTORS

1. The eigenvector X of a matrix A is not unique.

2. If $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct eigenvalues of an $n \times n$ matrix then corresponding eigen vectors $X_1, X_2, \ldots X_n$ form a linearly independent set.

3. If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.

4. Two eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.

5. Eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.

Normalised form of vectors. To find normalised form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we divide each element by $\sqrt{a^2+b^2+c^2}$.

For example, normalised form of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

4.29 NON SYMMETRIC MATRICES WITH NON REPEATED EIGEN VALUES

Example 76. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Solution.

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0 \quad i.e., \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\text{Let } \lambda = 1, \quad 1 - 6 + 11 - 6 = 0$$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -6 & +11 & -6 \\ & & -1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 1, 2, 3$$

To find eigenvectors for the corresponding eigenvalues we will consider the matrix equation

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1$$

Eigen vector corresponding to eigenvalue $\lambda = 1$

By putting $\lambda = 1$, the matrix equation (1) will become

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -z = 0 \\ x + y + z = 0, \\ k + y + 0 = 0 \end{matrix} \quad \begin{matrix} \text{Let } x = k \\ y = -k \end{matrix}$$

Unit - I

Cramer rule.

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = -1/\Delta$$

1) Solve by cramer rule by the equation

$$5x_1 + 3y_1 - 4z_1 - 4 = 0$$

$$3x_2 + 2y_2 + 6z_2 - 6 = 0$$

$$-8x_3 + 6y_3 + 2z_3 - 2 = 0$$

Ans: [illegible] =

$$\begin{bmatrix} 5 & 3 & -4 & -4 \\ 3 & 2 & 6 & -6 \\ -8 & 6 & 2 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & -4 \\ 3 & 2 & 6 \\ -8 & 6 & 2 \end{bmatrix}$$

$$= 5(4-36) - 3(6+48) - 4(18+16)$$

$$= 5(-32) - 3(54) - 4(34)$$

$$= -160 - 162 - 136$$

$$\boxed{\Delta = -458}$$

$$\Delta_1 = \begin{bmatrix} 3 & -4 & -4 \\ 2 & 6 & -6 \\ 6 & 2 & -2 \end{bmatrix}$$

$$= 3(-12+12) + 4(-4+36) - 4(4-36)$$

$$= 3(0) + 4(32) - 4(-32)$$

$$\Delta_1 = 128 + 128$$

$$\boxed{\Delta_1 = 256}$$

$$\Delta_2 = \begin{bmatrix} 5 & -4 & -4 \\ 3 & 6 & -6 \\ -8 & 2 & -2 \end{bmatrix}$$

$= 5(-12+12)+4(-6-48)-4(6+48)$

$= 5(0)+4(-54)-4(54)$

$= -216-216$

$\boxed{\Delta_2 = -432}$

$$\Delta_3 = \begin{bmatrix} 5 & 3 & -4 \\ 3 & 2 & -6 \\ -8 & 6 & -2 \end{bmatrix}$$

$= 5(-4+36)-3(-6-48)-4(18+16)$

$= 5(32)-3(-54)-4(34)$

$= 200+126-136$

$= 326-136$

$\boxed{\Delta_3 = 190}$

$= 160+162-136$

$= 322-136 = 186$

$\therefore \Delta = -458, \Delta_1 = 256, \Delta_2 = -432 ; \Delta_3 = 186$

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = -1/4$$

$$\frac{x}{256} = -y/432 = z/186 = 1/-458 \rightarrow (1).$$

$$\frac{x}{256} = 1/458 \Rightarrow x = 256/458 = 0.5589$$

$$\frac{y}{432} = 1/458 \Rightarrow y = 432/458 = 0.9432$$

$$\frac{z}{186} = 1/458 \Rightarrow z = 186/488 = 0.4069.$$

x, y, z value sub in eqn ①.

$$\frac{0.5589}{256} = \frac{0.9432}{432} = \frac{0.4069}{186} = \frac{1}{458}$$

$$2.1832 = 2.183 = 2.1876 = 2.1834$$

2) Solve by cramer rule by the equation.

$$6x_1 + 2y_1 - 2z_1 - 4 = 0$$

$$6x_2 + 4y_2 + 2z_2 - 6 = 0$$

$$-2x_3 - 2y_3 + 8z_3 - 18 = 0$$

Ans:

அணி $= \begin{bmatrix} 6 & 2 & -2 & -4 \\ 6 & 4 & 2 & -6 \\ -2 & -2 & 8 & -18 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & 2 & -2 \\ 6 & 4 & 2 \\ -2 & -2 & 8 \end{bmatrix}$$

$$= 6(32+4) - 2(48+4) - 2(-12+8)$$

$$= 216 - 104 + (-8)$$

$$\boxed{\Delta = 120}$$

$$\Delta_1 = \begin{bmatrix} 2 & -2 & -4 \\ 4 & 2 & -6 \\ -2 & 8 & -18 \end{bmatrix}$$

$$= 2(-36+48) + 2(-72+12) - 4(32-4)$$

$$= 2(12) + 2(-60) - 4(28)$$

$$= 24 - 120 - 112$$

$$= 24 - 232$$

$$\boxed{\Delta_1 = -208}$$

$$\Delta_2 = \begin{bmatrix} 6 & -2 & -4 \\ 6 & 2 & -6 \\ -2 & 8 & -18 \end{bmatrix}$$

$= 6(-36+48) + 2(-108-12) - 4(48+4)$

$= 6(12) + 2(-120) - 4(52)$

$= 72 - 240 - 208$

$= 72 - 448$

$\boxed{\Delta_2 = -376}$

$$\Delta_3 = \begin{bmatrix} 6 & 2 & -4 \\ 6 & 4 & -6 \\ -2 & -2 & -18 \end{bmatrix}$$

$= 6(-72+12) - 2(-108-12) - 4(12+8)$

$= 6(-60) - 2(-120) - 4(20)$

$= -360 + 240 - 80$

$= -440 + 240$

$\boxed{\Delta_3 = -200}$

$\Delta = 24, \quad \Delta_1 = -208, \quad \Delta_2 = -376, \quad \Delta_3 = -200$

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta}$$

$$\frac{x}{-208} = \frac{-y}{-376} = \frac{z}{-200} = \frac{-1}{24} \rightarrow (1)$$

$$\frac{x}{-208} = \frac{-1}{24} \Rightarrow x = \frac{208}{24} = 8.6666$$

$$\frac{y}{376} = \frac{-1}{24} \Rightarrow y = \frac{-376}{24} = -15.6666$$

$$\frac{z}{-200} = \frac{-1}{24} \Rightarrow z = -\frac{200}{24} = 8.3333$$

x, y, z value sub in eqn (1).

$$\frac{8.6666}{-208} = \frac{-15.6666}{376} = \frac{8.3333}{-200} = -\frac{1}{24}$$

$-0.041 = -0.041 = -0.041 = -0.041$

3) Solve by Cramer rule by the equation

$$x+y+z=1$$
$$3x+5y+6z=4$$
$$9x+2y-3z=17$$

Ans: $x+y+z-1=0$

$$3x+5y+6z-4=0$$
$$9x+2y-32-17=0$$

Aug $= \begin{bmatrix} 1 & 1 & 1 & -1 \\ 3 & 5 & 6 & -4 \\ 9 & 2 & -3 & -17 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 2 & -3 \end{bmatrix}$

$= 1(-15-12) - 1(-9-54) + 1(6-45)$

$= 1(-27) - 1(-63) + 1(-39)$

$= -27 + 63 - 39$

$\boxed{\Delta = -3}$

$\Delta_1 = \begin{bmatrix} 1 & 1 & -1 \\ 5 & 6 & -4 \\ 2 & -3 & -17 \end{bmatrix}$

$= 1(-102-12) - 1(-85+8) - 1(-15-12)$

$= 1(-114) - (-77) - 1(-27)$

$= -114 + 77 + 27$

$= -144 + 104$

$\boxed{\Delta_1 = -10}$

$\Delta_2 = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 6 & -4 \\ 9 & -3 & -17 \end{bmatrix}$

$= 1(-102-12) - 1(-51+36) - 1(-9-54)$

$= 1(-114) - 1(-15) - 1(-63)$

$= -114 + 15 + 63$

$= -114 + 78$

$\boxed{\Delta_2 = -36}$

$$\Delta_3 = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 5 & -4 \\ 9 & 2 & -17 \end{bmatrix}$$

$= 1(-85+8) - 1(-51+36) - 1(6-45)$

$= 1(-77) - 1(-15) - 1(-39)$

$= -77 + 15 + 39$

$= -77 + 54$

$\boxed{\Delta_3 = -23}$

$\Delta = -3, \Delta_1 = -10, \Delta_2 = -36, \Delta_3 = -23$

$$\frac{x}{-10} = \frac{-y}{-36} = \frac{z}{-23} = \frac{-1}{-3} \rightarrow (1)$$

$$\frac{x}{-10} = \frac{1}{3} \Rightarrow x = \frac{-10}{3} = -3.3333$$

$$\frac{y}{36} = \frac{1}{3} \Rightarrow y = \frac{36}{3} = 12$$

$$\frac{z}{-23} = \frac{1}{3} \Rightarrow z = \frac{-23}{3} = 7.6666.$$

Sub x, y, z values in eqn (1).

$$\frac{+3.3333}{+10} = \frac{12}{36} = \frac{+7.6666}{+23}.$$

$0.3333 = 0.3333 = 0.3333 = 0.3333$ //

8 Matrices

8.1 Eigen Values, Eigen-Vectors : Characteristic equation of a matrix

Let $A = [a_{ij}]$ be a square matrix of order n. Suppose there is an n-dimensional nonzero column vector X, such that the action of **A** on **X** (i.e., the matrix product **AX**, gives a vector which is just a multiple of **X**, that is

$$AX = \lambda X$$

where λ is a scalar. In other words, the transformation represented by the matrix **A** just multiplies the vector **X** by a scalar λ. The vector **X** is then called an *eigen vector* of the matrix **A**. λ is called an *Eigen value* of **A** corresponding to the eigenvector **X**. The problem of finding the eigenvectors and the eigenvalues of a matrix is called the *eigenvalue problem.*

Definition of Eigenvector. A nonzero vector **X** *is called an eigenvector of a matrix* **A** *if there is a number* λ *such that* $AX = \lambda X$.

Here λ is called an eigen value of **A** corresponding to the eigenvector **X** and vice versa.

We have, $AX = \lambda X = \lambda I X$, **I** being a unit matrix.

or

$$(A - \lambda I) X = 0.$$

This is the matrix form of an eigenvalue problem.

Since $X \neq 0$, the matrix $(A - \lambda I)$ is singular, so that

$$|A - \lambda I| = 0 \quad \text{... (1)}$$

Equation (1) is called the *characteristic equation* of **A**. The eigenvalues are just the roots of the equation obtained by expanding the determinant in Eq. (1). The n-roots $\lambda_1, \lambda_2, \ldots \lambda_n$ of the characteristic equation are not necessarily all different.

Example 1. *Determine the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Sol. The characteristic equation of **A** is $|A - \lambda I| = 0$ *i.e.,*

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \text{ i.e., } (\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$

$$\therefore \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$$

These are the eigen values of **A**.

To determine eigen vectors let us consider the eigen values one by one.

(*i*) When $\lambda_1 = 2$ the eigen vector X_1 is given by $(A - 2I) X_1 = 0$

i.e.,
$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of coefficient matrix being 2, the equation will have only 3 − 2 i.e., 1 linearly independent solution.

These are equivalent to $x_1 + x_2 + 4x_3 = 0$

$$6x_3 = 0$$
$$3x_3 = 0$$

The last two give $x_3 = 0$. Then first one gives $x_1 + x_2 = 0$.

Take $x_1 = 1$, then $x_2 = -1$ and $x_3 = 0$.

Hence $$X_1 = C_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$ C_1 being a scalar.

(*ii*) When $\lambda_2 = 3$, the eigen vector X_2 is given by $(A - 3I)\, X_2 = 0$

i.e.,
$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These are equivalent to $x_2 + 4x_3 = 0$

$$-x_2 + 6x_3 = 0$$
$$2x_3 = 0$$

giving $x_3 = 0, x_2 = 0$ and x_1 is arbitrary, say $x_1 = 1$.

Then $$X_2 = C_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$ C_2 being a scalar.

(*iii*) When $\lambda_3 = 5$, the eigen vector X_3 is given by $(A - 5I)\, X_3 = 0$

i.e.,
$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These are equivalent to $-2x_1 + x_2 + 4x_3 = 0$

$$-3x_2 + 6x_3 = 0$$

giving $x_2 = 2x_3 = \frac{2}{3}x_1$, i.e., $2x_1 = 3x_2 = 6x_3$

Take $x_3 = 1$, so that $x_2 = 2$ and $x_1 = 3$

Hence $$X_3 = C_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$ C_3 being a scalar.

Example 2. *Find the eigen values and normalised eigen vectors of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

The characteristic equation of **A** is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

i.e., $(1-\lambda)\,[(1-\lambda)^2 - 1] = 0$

i.e., $(1-\lambda)\,(\lambda^2 - 2\lambda) = 0$

i.e., $\lambda\,(1-\lambda)\,(\lambda - 2) = 0$

i.e., $\lambda = 0,\ 1,\ 2\,.$

Thus the *eigen values* of the matrix **A** are 0, 1, 2.

Eigen value equation is

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = 0 \qquad \text{... (1)}$$

For $\lambda = 0$, Eq. (1) reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalent to the following equations

$$x_1 = 0$$
$$x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$

Solving these equations, we get

$$x_1 = 0,\ x_2 = -x_3 = k \text{ (say)}$$

$\therefore$
$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \{x_1, x_2, x_3\} = \{\,0, k, -k\,\}$$

If the eigen vectors be normalised to unity, then $|\mathbf{X_1}| = 1$

or $$\sqrt{[0^2 + k^2 + (-k)^2]} = 1 \text{ or } k = \frac{1}{\sqrt{(2)}}$$

$\therefore$ Normalised eigen vector

$$\mathbf{X_1} = \left\{0, \frac{1}{\sqrt{(2)}}, \frac{-1}{\sqrt{(2)}}\right\}$$

For $\lambda = 1$, Eq. (1) reduces to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalent to the following equations

$$x_2 = 0$$
$$x_3 = 0$$

$\therefore \mathbf{X_2} = \{1, 0, 0\}$ in normalised form.

For $\lambda = 2$, Eq. (1) reduces to

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to the following equations

$$-x_1 = 0$$
$$-x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$

Solving these equations, we get

$$x_1 = 0, x_2 = x_3$$

Within the arbitrary scale factor, the *eigen vector* corresponding to $\lambda = 2$ is given by

$$\mathbf{X_3} = \{x_1, x_2, x_3\} = \{0, k, k\}$$

For $|\mathbf{X_3}|$ normalised to unity

$$\sqrt{0^2 + k^2 + k^2} = 1 \quad i.e., \quad k = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \mathbf{X_3} = \left\{0, \frac{1}{\sqrt{(2)}}, \frac{1}{\sqrt{2}}\right\} \text{ in normalised form.}$$

Thus the *normalised eigen vectors of the given matrix* **A** *corresponding to the eigen values* 0 , 1, 2 *are*

$$\left\{0, \frac{1}{\sqrt{(2)}}, -\frac{1}{\sqrt{2}}\right\}, \{1, 0, 0\}, \left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\} \quad \text{respectively}$$

8.2. Cayley-Hamilton Theorem

Statement. *Every square matrix satisfies its own characteristic equation*

OR

If $$|\mathbf{A} - \lambda \mathbf{I}| = a_0 + a_1 \lambda + a_2\lambda^2 + \ldots + a_n\lambda^n = 0$$

be the characteristic equation of a square matrix **A**, then

$$a_0 \mathbf{I} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \ldots + a_n \mathbf{A}^n = 0$$

Proof. The characteristic polynomial is

$$|\mathbf{A} - \lambda \mathbf{I}| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \quad \ldots(1)$$

Each element of characteristic matrix $(\mathbf{A} - \lambda\mathbf{I})$ is an ordinary polynomial of degree n (at most). Therefore the co-factor of every element of $|\mathbf{A} - \lambda\mathbf{I}|$ is an ordinary polynomial of degree $n - 1$ (at most). Consequently each element of

$$\mathbf{B} = \text{adj}\,(\mathbf{A} - \lambda\mathbf{I}) \quad \ldots(2)$$

is an ordinary polynomial of degree $(n - 1)$ (at most).

$$\therefore \quad \mathbf{B} = \text{adj}\,(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_0 + \mathbf{B}_1\lambda + \mathbf{B}_2\lambda^2 + \cdots + \mathbf{B}_{n-1}\lambda^{n-1} \quad \ldots(3)$$

Here $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2, \ldots \mathbf{B}_{n-1}$ are all square matrices of the same order n whose elements are polynomials in the elements of **A**.

Now, $$(\mathbf{A} - \lambda\mathbf{I})\,\text{adj}\,(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\,\mathbf{I}$$

Using Eqs. (3) and (1), we get

$$(\mathbf{A} - \lambda\mathbf{I})\,[\mathbf{B}_0 + \mathbf{B}_1\lambda + \mathbf{B}_2\lambda^2 + \ldots + \mathbf{B}_{n-1}\lambda^{n-1}]$$

$$= (a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n)\,\mathbf{I} \quad \ldots(4)$$

Comparing the coefficients of like powers of λ on both the sides, we get

$$\mathbf{A}\mathbf{B}_0 = a_0\mathbf{I}$$

$$\mathbf{A}\mathbf{B}_1 - \mathbf{B}_0 = a_1\mathbf{I}$$

$$\mathbf{A}\mathbf{B}_2 - \mathbf{B}_1 = a_2\mathbf{I}$$

....

....

$$\mathbf{A}\mathbf{B}_{n-1} - \mathbf{B}_{n-2} = a_{n-1}\mathbf{I}$$

$$-\mathbf{B}_{n-1} = a_n\mathbf{I}$$

Premultiplying these by $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3 \ldots \mathbf{A}^n$ in order and adding,

$$a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n = 0$$

This is *Cayley-Hamilton theorem.*

Corollary. *To determine* $\mathbf{A}^{-1}$ *by using Cayley-Hamilton theorem.*

Let **A** be a non-singular matrix of order n so that $|\mathbf{A}| \neq 0$.

According to Cayley-Hamilton theorem

$$a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n = 0 \quad ...(1)$$

The characteristic polynomial is

$$|\mathbf{A} - \lambda\mathbf{I}| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots a_n\lambda^n \quad ...(2)$$

For $\lambda = 0$, Eq. (2) gives $|\mathbf{A}| = a_0$. $\therefore a_0 \neq 0$

Now dividing Eq. (1) by a_0, we get

$$\mathbf{I} = -\left(\frac{a_1}{a_0}\mathbf{A} + \frac{a_2}{a_0}\mathbf{A}^2 + \cdots + \frac{a_n}{a_0}\mathbf{A}^n\right) \quad ...(3)$$

Pre-multiplying Eq. (3) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1} = -\left(\frac{a_1}{a_0}\mathbf{I} + \frac{a_2}{a_0}\mathbf{A} + \cdots + \frac{a_n}{a_0}\mathbf{A}^{n-1}\right) \quad ...(4)$$

Example 1. *Find the characteristic equation of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix}$$

and verify the Cayley-Hamilton theorem for it. Hence or otherwise find $\mathbf{A}^{-1}$.

Sol. $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

$$\therefore |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 18\lambda + 30$$

Hence the characteristic equation is

$$-\lambda^3 + \lambda^2 + 18\lambda + 30 = 0$$

Now, in order to verify Cayley-Hamilton theorem, we have to show that

$$-\mathbf{A}^3 + \mathbf{A}^2 + 18\mathbf{A} + 30\,\mathbf{I} = 0$$

Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\qquad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}$$

$\therefore\; -\mathbf{A}^3 + \mathbf{A}^2 + 18\mathbf{A} + 30\mathbf{I}$

$$= -\begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} + 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} + 30\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence Cayley - Hamilton theorem is verified.

To find $\mathbf{A}^{-1}$: $30\,\mathbf{I} - \mathbf{A}^3 + \mathbf{A}^2 + 18\,\mathbf{A} = 0$

$$\mathbf{I} = \frac{1}{30}(\mathbf{A}^3 - \mathbf{A}^2 - 18\,\mathbf{A})$$

Premultiplying the above equation by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1} = \frac{1}{30}(\mathbf{A}^2 - \mathbf{A} - 18\,\mathbf{I})$$

$$\mathbf{A}^{-1} = \frac{1}{30}\begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - \frac{1}{30}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \frac{18}{30}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{5}{30} & \frac{1}{30} & \frac{11}{30} \\ \frac{10}{30} & -\frac{8}{30} & \frac{2}{30} \\ \frac{5}{30} & \frac{5}{30} & -\frac{5}{30} \end{bmatrix}$$

Example 2. *Illustrate the Cayley-Hamilton theorem for the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= -5 + 5\lambda + \lambda^2 - \lambda^3$$

Hence the characteristic equation is ,

$$-5 + 5\lambda + \lambda^2 - \lambda^3 = 0$$

We have to show that $-5\mathbf{I} + 5\mathbf{A} + \mathbf{A}^2 - \mathbf{A}^3 = 0$

Now, $-5\mathbf{I} + 5\mathbf{A} + \mathbf{A}^2 - \mathbf{A}^3$

$$= \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} + \begin{pmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -5 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

Example 3. *Find the characteristic equation of the matrix*

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by **A**. *Hence find the inverse of* **A** .

sol. The characteristic equation of **A** is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

i. e. , $$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

i. e., $$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \qquad ...(1)$$

We have to show that

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore$ $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ $$A^3 - 6A^2 + 9A - 4I = 0$$

To find A^{-1}: $$I = \frac{1}{4}(A^3 - 6A^2 + 9A)$$

Premultiplying the above equation by A^{-1}, we get

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

8.3 . Theorems on eigen values and eigen vectors

Theorem 1. *The eigen values of a Hermitian matrix are all real.*

Proof. For a hermitian matrix **A**

$$A^{\dagger} = A$$

Here, $A^{\dagger}$ is the transposed conjugate of **A** .

Let λ be an eigen value of a Hermitian matrix **A**. Then by definition, there exists a vector $\mathbf{X} \neq 0$, such that

$$\mathbf{AX} = \lambda\,\mathbf{X}. \quad \text{...(1)}$$

Premultiplying Eq. (1) by $\mathbf{X}\dagger$, we get

$$\mathbf{X}\dagger\,\mathbf{AX} = \mathbf{X}\dagger\,\lambda\mathbf{X} = \lambda\mathbf{X}\dagger\mathbf{X} \quad (2)$$

Taking transposed conjugate of both sides in (2), we get

$$(\mathbf{X}\dagger\,\mathbf{AX})\dagger = (\lambda\,\mathbf{X}\dagger\mathbf{X})\,\dagger$$

or $$\mathbf{X}\dagger\,\mathbf{A}\dagger\,(\mathbf{X}\dagger)\dagger = \lambda * \mathbf{X}\dagger(\mathbf{X}\dagger)\dagger$$

or $$\mathbf{X}\dagger\mathbf{A}\dagger\mathbf{X} = \lambda^*\mathbf{X}\dagger\mathbf{X}$$

or $$\mathbf{X}\,\dagger\mathbf{AX} = \lambda^*\mathbf{X}\dagger\mathbf{X} \quad (\because \mathbf{A}\dagger = \mathbf{A})$$

$$\mathbf{X}\dagger\lambda\mathbf{X} = \lambda^*\mathbf{X}\dagger\mathbf{X} \quad (\text{using Eq. 1})$$

or $$(\lambda - \lambda^*)\,\mathbf{X}\dagger\mathbf{X} = 0 \quad \text{...(3)}$$

As **X** is an eigen vector $\mathbf{X} \neq 0$; $\therefore \mathbf{X}\dagger\mathbf{X} \neq 0$

Then Eq. (3) gives,

$$\lambda - \lambda^* = 0$$

or $$\lambda = \lambda^*$$

This means that the conjugate of λ is equal to itself. This is only possible when λ is real.

Thus *the eigen-values of a Hermitian matrix are all real.*

Theorem 2. *The eigen values of a real symmetric matrix are all real.*

Proof. For real symmetric matrix **A**,

$$\mathbf{A}^* = \mathbf{A} \quad \text{and} \quad \mathbf{A}^T = \mathbf{A}$$

$$\therefore \quad (\mathbf{A}^T)^* = \mathbf{A} \qquad i.e.,\ \mathbf{A}\dagger = \mathbf{A}$$

Thus real symmetric matrix is a hermitian matrix.

By Th. 1, the eigen values of a hermitian matrix are all real.

Consequently, the *eigen values of a real symmetric matrix are all real.*

Theorem 3. *The eigen values of skew-hermitian matrix are either zero* or *purely imaginary.*

Proof. For a skew-hermitian matrix **A**,

$$\mathbf{A}\dagger = -\mathbf{A} \quad \text{...(1)}$$

Let **X** be an eigen vector of **A** corresponding to eigen value λ. Then

$$\mathbf{AX} = \lambda\mathbf{X} \quad \text{...(2)}$$

or $$(i\mathbf{A})\,\mathbf{X} = (i\lambda)\,\mathbf{X} \quad \text{...(3)}$$

Now, $$(i\mathbf{A})\dagger = i^*\mathbf{A}\dagger = -i\,(-\mathbf{A}) \quad \text{using (1)}$$

$$\therefore \quad (i\mathbf{A})\dagger = i\mathbf{A}$$

This proves that $i\mathbf{A}$ is *hermitian matrix.*

According to Eq. (3), $i\lambda$ is the eigen value of hermitian matrix $i\mathbf{A}$ corresponding to the eigen vector **X**. Therefore, by theorem 1,

$i\lambda$ is a real number. It follows that λ is either zero or purely imaginary number.

Theorem 4. *The eigen values of a real skew-symmetric matrix are either zero or purely imaginary.*

Proof. For a real skew- symmetric matrix **A**,

$$\mathbf{A}^* = \mathbf{A} \text{ and } \mathbf{A}^T = -\mathbf{A} \Rightarrow (\mathbf{A}^T)^* = -\mathbf{A}^* = -\mathbf{A}$$

$$\Rightarrow \quad \mathbf{A}^\dagger = -\mathbf{A} \Rightarrow \mathbf{A} \text{ is skew-Hermitian.}$$

$\Rightarrow$ By Th. 3, the result follows.

Theorem 5. *The modulus of each eigen value of a unitary matrix is unity.*

Proof. For a unitary matrix **A**,

$$\mathbf{A}^\dagger \mathbf{A} = \mathbf{I} \quad ...(1)$$

Let X be an eigen vector of A corresponding to eigen value λ. Then

$$\mathbf{AX} = \lambda\mathbf{X} \quad ...(2)$$

Taking transposed conjugate of (2), we get

$$(\mathbf{AX})^\dagger = (\lambda\mathbf{X})^\dagger$$

or $$\mathbf{X}^\dagger\mathbf{A}^\dagger = \lambda^*\mathbf{X}^\dagger \quad ...(3)$$

Post-multiplying (3) by (2), we get

$$(\mathbf{X}^\dagger\mathbf{A}^\dagger)(\mathbf{AX}) = (\lambda^* \mathbf{X}^\dagger)\cdot(\lambda\mathbf{X})$$

or $$\mathbf{X}^\dagger(\mathbf{A}^\dagger\mathbf{A})\mathbf{X} = \lambda\lambda^* \mathbf{X}^\dagger\mathbf{X}$$

or $$\mathbf{X}^\dagger\mathbf{I}\mathbf{X} = \lambda\lambda^*\mathbf{X}^\dagger\mathbf{X} \quad \text{using (1)}$$

or $$\mathbf{X}^\dagger\mathbf{X} = \lambda\lambda^*\mathbf{X}^\dagger\mathbf{X}$$

or $$\mathbf{X}^\dagger\mathbf{X}(1 - \lambda\lambda^*) = 0 \quad ...(4)$$

As $$\mathbf{X} \neq 0, \quad \mathbf{X}^\dagger\mathbf{X} \neq 0$$

Eq. (4) gives, $1 - \lambda\lambda^* = 0$ *i. e.,* $\lambda\lambda^* = |\lambda^2| = 1$

$$\therefore \quad |\lambda| = 1$$

i. e., the modulus of λ is unity.

Theorem 6. *The eigen values of an orthogonal matrix are unimodular (i. e., of unit modulus)*

Proof. For an orthogonal real matrix **A**, we have

$$\mathbf{A}^* = \mathbf{A} \quad ...(1)$$

$$\mathbf{A}^T\mathbf{A} = \mathbf{I} \quad ...(2)$$

Taking complex conjugate of Eq. (2), we get

$$(\mathbf{A}^T\mathbf{A})^* = \mathbf{I}^*$$

or $$(\mathbf{A}^T)^*\mathbf{A}^* = \mathbf{I} \qquad (\because \; I^* = I)$$

or $$\mathbf{A}^\dagger\mathbf{A} = \mathbf{I} \quad \text{using (1)}$$

$\Rightarrow$ **A** is unitary. By Th. 5, the eigen values of a unitary matrix are unimodular. Hence the *eigen values of a real orthogonal matrix are unimodular.*

Theorem 7. *Any two eigen vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal.*

Proof. Let X_1, X_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of a *hermitian matrix* **A**. Then

$$A\dagger = A \quad ...(1)$$

$$AX_1 = \lambda_1 X_1 \quad ...(2)$$

$$AX_2 = \lambda_2 X_2 \quad ...(3)$$

From Th. 1, λ_1 and λ_2 are real numbers.

$$\therefore \quad \lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^* \quad ...(4)$$

Premultiplying (2) and (3) by $X_2\dagger$ and $X_1\dagger$ respectively, we get

$$X_2\dagger AX_1 = \lambda_1 X_2\dagger X_1 \quad ...(5)$$

$$X_1\dagger AX_2 = \lambda_2 X_1\dagger X_2 \quad ..(6)$$

Taking transposed conjugate of (5), we get

$$(X_2\dagger AX_1)\dagger = (\lambda_1 X_2\dagger X_1)\dagger$$

i.e., $$X_1\dagger A\dagger (X_2\dagger)\dagger = \lambda_1^* X_1\dagger (X_2\dagger)\dagger$$

i. e., $$X_1\dagger A\dagger X_2 = \lambda_1 X_1\dagger X_2 \quad \text{using (4)}$$

i. e., $$X_1\dagger AX_2 = \lambda_1 X_1\dagger X_2 \quad \text{using (1)} ...(7)$$

Comparing (6) and (7), we get

$$\lambda_1 X_1\dagger X_2 = \lambda_2 X_1\dagger X_2$$

i. e., $$(\lambda_1 - \lambda_2) X_1\dagger X_2 = 0$$

Since $\lambda_1 - \lambda_2 \neq 0$, otherwise the roots will not be distinct, the only possibility is that $X_1\dagger X_2 = 0$.

It follows that X_1, X_2 are orthogonal.

Theorem 8. *Any two eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.*

Proof. Let A be a real symmetric matrix. Then

$$A^* = A \quad ...(1)$$

$$A^T = A \quad ...(2)$$

Taking transposed conjugate of (2), we get

$$(A^T)^* = A^*$$

or $$A\dagger = A \quad \text{using (1)}$$

⇒A is Hermitian.

By Th. 7, the result follows.

Theorem 9. *Any two eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.*

Proof. Let A be a unitary matrix.

Then $$A\dagger A = I \quad ...(1)$$

Let $\mathbf{X}_1, \mathbf{X}_2$ be two eigen vectors corresponding to *two* distinct eigen values λ_1 and λ_2 of unitary matrix **A**. Then

$$\mathbf{AX}_1 = \lambda_1 \mathbf{X}_1 \quad ...(2)$$

$$\mathbf{AX}_2 = \lambda_2 \mathbf{X}_2 \quad ...(3)$$

Taking transposed conjugate of (2), we get

$$(\mathbf{AX}_1)\dagger = (\lambda_1 \mathbf{X}_1)\dagger$$

or

$$\mathbf{X}_1\dagger\mathbf{A}\dagger = \lambda_1{}^* \mathbf{X}_1\dagger \quad ...(4)$$

Post-multiplying (4) by (3), we get

$$(\mathbf{X}_1\dagger\mathbf{A}\dagger)(\mathbf{AX}_2) = (\lambda_1{}^*\mathbf{X}_1\dagger)(\lambda_2\mathbf{X}_2)$$

or

$$\mathbf{X}_1\dagger(\mathbf{A}\dagger\mathbf{A})\mathbf{X}_2 = \lambda_1{}^*\lambda_2\mathbf{X}_1\dagger\mathbf{X}_2$$

or

$$\mathbf{X}_1\dagger\mathbf{I}\mathbf{X}_2 = \lambda_1{}^*\lambda_2\mathbf{X}_1\dagger\mathbf{X}_2 \quad \text{using (1)}$$

or

$$\mathbf{X}_1\dagger\mathbf{X}_2 = \lambda_1{}^*\lambda_2\mathbf{X}_1\dagger\mathbf{X}_2 \quad (\because \ \mathbf{X}_2 = \mathbf{I}\mathbf{X}_2)$$

or

$$(1 - \lambda_1{}^*\lambda_2)\,\mathbf{X}_1\dagger\mathbf{X}_2 = 0 \quad ...(5)$$

But **A** being a unitary matrix, the modulus of each of its eigen values is unity, *i.e.*,

$$\lambda_1 * \lambda_1 = 1$$

$$\therefore \quad (1 - \lambda_1{}^*\lambda_2) = (\lambda_1{}^*\lambda_1 - \lambda_1{}^*\lambda_2) = \lambda_1{}^*(\lambda_1 - \lambda_2) \neq 0$$

(since $\lambda_1 \neq \lambda_2$) ...(6)

From (5) and (6) it follows that $\mathbf{X}_1\dagger\mathbf{X}_2 = 0$

i. e., $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are orthogonal.*

Theorem 10. *Eigen values of a matrix are invariant under a similarity transformation.*

Proof. Consider two similar matrices A and B, related through similarity transformation such that

$$B = P^{-1}AP$$

The characteristic equation of the matrix B is $|B - \lambda I| = 0$. Here, λ is an eigen value of B.

i. e.,

$$|P^{-1}AP - \lambda I| = 0$$

or

$$|P^{-1}AP - P^{-1}\lambda IP| = 0$$

or

$$|P^{-1}(A - \lambda I)P| = 0$$

or

$$|P^{-1}||A - \lambda I||P| = 0$$

or

$$|P^{-1}P||A - \lambda I| = 0$$

or

$$|A - \lambda I| = 0$$

This relation shows that λ is an eigen value of A also.

This proves the theorem.

Theorem 11. *The eigen values of a diagonal matrix are precisely the elements in the diagonal.*

Proof. Let $A = \text{diag.} [a_{11}, a_{22}, \cdots a_{nn}]$

Then $(A - \lambda I) = \text{diag.} [a_{11} - \lambda, a_{22} - \lambda, \cdots, a_{nn} - \lambda]$

The characteristic equation $| A - \lambda I | = 0$ gives

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

$$\therefore \qquad \lambda = a_{11}, a_{22}, \cdots a_{nn}$$

$a_{11}, a_{22} \cdots a_{nn}$ are the diagonal elements of A. Therefore *the eigen values λ of a diagonal matrix are the elements in the diagonal.*

8.4 Diagonalization of matrices.

Let $\lambda_1, \lambda_2 \cdots \lambda_n$ be n distinct eigen values of a matrix A and $X_1, X_2, \ldots X_n$ be the n corresponding eigen vectors.

Let $\mathbf{X}_i$ be the column vector given by

$$\mathbf{X}_i = \begin{bmatrix} X_{1i} \\ X_{2i} \\ X_{ni} \end{bmatrix} \qquad \text{...(1)}$$

Consider a matrix **P** whose column vectors are n eigen vectors such that

$$\mathbf{P} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix} = [X_{ij}] \qquad \text{...(2)}$$

Suppose that **D** is a diagonal matrix such that

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{diag}[\lambda_1, \lambda_2, \ldots \lambda_n] \qquad \text{...(3)}$$

$$\text{Then } \mathbf{PD} = \begin{bmatrix} \lambda_1 \mathbf{X}_{11} & \lambda_2 \mathbf{X}_{12} \cdots\cdots & \lambda_n \mathbf{X}_{1n} \\ \lambda_1 \mathbf{X}_{21} & \lambda_2 \mathbf{X}_{22} \cdots\cdots & \lambda_n \mathbf{X}_{2n} \\ \lambda_1 \mathbf{X}_{n1} & \lambda_2 \mathbf{X}_{n2} \cdots\cdots & \lambda_n \mathbf{X}_{nn} \end{bmatrix} = [\lambda_j \mathbf{X}_{ij}] \qquad \text{...(4)}$$

(no summation over j)

$= (\mathbf{AX}_1, \mathbf{AX}_2, \ldots \mathbf{AX}_n)$ (expressing matrix as vectors)

$= \mathbf{A}(\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n)$

$= \mathbf{AP}$.. ...(5)

If **P** be a non-singular matrix, then premultiplying (5) by $\mathbf{P}^{-1}$, we get

$$\mathbf{D}=P^{-1}\mathbf{A}P \quad ...(6)$$

Thus, premultiplying **A** by $\mathbf{P}^{-1}$ and postmultiplying by *P*, we get the diagonal matrix whose diagonal elements are the eigen values. *This process is called the diagonalization of the matrix* **A**.

Example. Let $A=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}$

The characteristic equation is

$$|A-\lambda\mathbf{I}|=\begin{vmatrix}3-\lambda & 4\\ 4 & -3-\lambda\end{vmatrix}=0$$

i. e., $-9+\lambda^2-16=0$ or $\lambda=\pm 5$

i.e., $\lambda_1=-5$ and $\lambda_2=5$.

The corresponding eigenvectors are easily obtained :

$$\mathbf{x}_1=\begin{bmatrix}-1\\ 2\end{bmatrix} \text{ and } \mathbf{x}_2=\begin{bmatrix}2\\ 1\end{bmatrix}$$

Let $P=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}$

Then $P^{-1}=-\frac{1}{5}\begin{bmatrix}1 & -2\\ -2 & -1\end{bmatrix}$

Hence, $P^{-1}AP=-\frac{1}{5}\begin{bmatrix}1 & -2\\ -2 & -1\end{bmatrix}\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}\begin{bmatrix}-1 & +2\\ +2 & +1\end{bmatrix}$

$$=-\frac{1}{5}\begin{bmatrix}1 & -2\\ -2 & -1\end{bmatrix}\begin{bmatrix}5 & 10\\ -10 & 5\end{bmatrix}$$

$$=-\frac{1}{5}\begin{bmatrix}25 & 0\\ 0 & -25\end{bmatrix}=\begin{bmatrix}-5 & 0\\ 0 & 5\end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

Practical Method of Diagonalization. To reduce a given square matrix *A* to diagonal form, we first write the characteristic equation for the matrix and evaluate the characteristic roots $\lambda_1, \lambda_2 \ldots \lambda_n$. Then the required diagonal matrix *D* of *A* will be

$$D=\begin{bmatrix}\lambda_1 & 0 & 0 & \ldots & 0\\ 0 & \lambda_2 & 0 & \ldots & 0\\ 0 & 0 & \lambda_3 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & \lambda_n\end{bmatrix}$$

Example 1. *Diagonalise the following matrix*

$$A=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

sol. Let
$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

i. e.,
$$(1-\lambda)(1 - 2\lambda\cos\theta + \lambda^2) = 0$$

Characteristic roots are 1, $\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

i. e., $1, \cos\theta \pm i\sin\theta$

i. e., $1, e^{\pm i\theta}$

$\therefore$ $\lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}, \lambda_3 = 1$ (say)

$\therefore$ The required diagonal matrix is

$$D = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.5 Matrices in Physics

(*i*) **The Rotation Matrix.** Let (x, y, z) be a cartesian coordinate system in a three-dimensional space. Let $\mathbf{u} = \{u_1, u_2\}$ be a vector in the *xy*-plane.

Consider a rotation of the coordinate system about the z- axis through an angle θ in the anticlockwise sense. Let us denote the new coordinate axes by x', y', z.

If the same vector **u** has components u_1', u_2' relative to the new system, then

$$u_1' = u_1\cos\theta + u_2\sin\theta$$
$$u_2' = -u_1\sin\theta + u_2\cos\theta \quad \ldots(1)$$

This could be written in the matrix notation as

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \ldots(2)$$

If we denote the rotation operator by $R_z(\theta)$, we could write the transformation of the vector $\boldsymbol{u}$ to $\boldsymbol{u}'$ symbolically by

$$\mathbf{u} \to \mathbf{u}' = R_z(\theta)\,\mathbf{u} \quad \ldots(3)$$

The operator $R_z(\theta)$ which causes the transformation of two dimensional vector $\boldsymbol{u}$ to $\boldsymbol{u}'$ is represented by two *dimensional matrix*

$$R_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \ldots(4)$$

and is called *two dimensional rotation matrix*. It is an *orthogonal matrix.*