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Transient Currents

12.1. Growth of current in a circuit containing a resistance and inductance

Consider a circuit having an inductance L and a resistance R connected in series to a cell of steady emf E (Fig. 12.1). When the key K is pressed, there is a gradual growth of current in the circuit from zero to maximum value I_0. Let I be the instantaneous current at any instant.

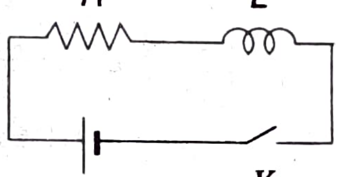

Fig. 12.1

Then, the induced back emf $\varepsilon = -L\dfrac{dI}{dt}$

$$\therefore \qquad E = RI + L\frac{dI}{dt} \qquad \text{... (1)}$$

When the current reaches the maximum value I_0, the back emf,

$$L\frac{dI}{dt} = 0.$$

Hence
$$E = RI_0 \qquad \text{... (2)}$$

Substituting this value for E in Eq. (1),

$$RI_0 = RI + L\frac{dI}{dt} \quad \text{or} \quad R(I_0 - I) = L\frac{dI}{dt}$$

or
$$\frac{dI}{I_0 - I} = \frac{R}{L}dt$$

Integrating,
$$-\log(I_0 - I) = \frac{R}{L}t + C \qquad \text{... (3)}$$

where C is the constant of integration.

When $t = 0$, $I = 0$, $\quad \therefore -\log_e I_0 = C$

Substituting this value of C in Eq. (3),

$$-\log(I_0 - I) = \frac{R}{L}t - \log_e I_0 \quad \text{or} \quad \log_e(I_0 - I) - \log_e I_0 = -\frac{R}{L}t$$

$$\log_e \frac{(I_0 - I)}{I_0} = -\frac{R}{L}t$$

$$\frac{I_0 - I}{I_0} = e^{-(R/L)t} \quad \text{or} \quad 1 - \frac{I}{I_0} = e^{-(R/L)t}$$

$$\therefore \quad I = I_0\left(1 - e^{-(R/L)t}\right) \quad \dots (4)$$

Eq. (4) gives the value of the instantaneous current in the LR circuit. The quantity (L/R) is called the *time constant* of the circuit.

$$\text{If } \frac{L}{R} = t, \quad I = I_0(1 - e^{-1}) = I_0\left(1 - \frac{1}{e}\right) = 0.632 I_0$$

Thus, *the time constant L/R of a L–R circuit is the time taken by the current to grow from zero to 0.632 times the steady maximum value of current in the circuit.*

Similarly, when $t = 2L/R$, $3L/R$ . . . , the value of current will be 0.8647, 0.9502,... of the final maximum current.

When $t = 0$, $I = 0$ and when $t \to \infty$, $I = I_0$

Greater the value of L/R, longer is the time taken by the current I to reach its maximum value (Fig. 12.2).

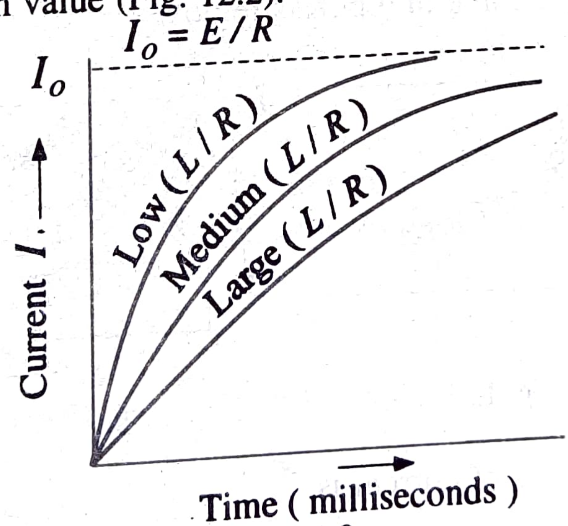

Fig. 12.2

12.2. Decay of Current in a Circuit Containing L and R

When the circuit is broken, an induced emf, equal to $-L\frac{dI}{dt}$ is again produced in the inductance L and it slows down the rate of decay of the current. The current in the circuit decays from maximum value I_0 to zero. During the decay, let I be the current at time t. In this case $E = 0$. The emf equation for the decay of current is

$$0 = RI + L\frac{dI}{dt} \quad \dots (1)$$

$$\therefore \quad \frac{dI}{I} = -\frac{R}{L}dt$$

Integrating, $\log_e I = -\frac{R}{L}t + C$ where C is a constant.

When $t = 0$, $I = I_0$, $\therefore \log_e I_0 = C$

$$\therefore \quad \log_e I = -\frac{R}{L}t + \log_e I_0, \text{ or } \log_e \frac{I}{I_0} = -\frac{R}{L}t$$

$$\frac{I}{I_0} = e^{-(R/L)t}$$

$$\therefore \quad I = I_0 e^{-(R/L)t} \qquad \text{... (2)}$$

Eq. (2) represents the current at any instant t during decay. A graph between current and time is shown in Fig. 12.3.

When $t = L/R$, $I = I_0 e^{-1} = \frac{1}{e} I_0 = 0.365\ I_0$

$t = 2L/R$, $I = I_0 e^{-2} = 0.1035\ I_0$

$t = 3L/R$, $I = I_0 e^{-3} = 0.05\ I_0$

Therefore, *the time constant L/R of a R–L circuit may also be defined as the time in which the current in the circuit falls to 1/e of its maximum value* when external source of e.m.f. is removed.

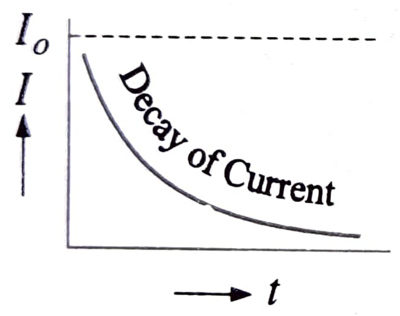

Fig. 12.3

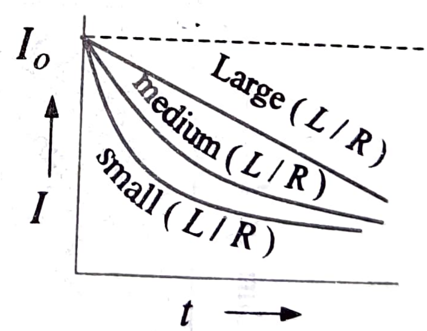

Fig. 12.4

The rate of decay of current is

$$\frac{dI}{dt} = -\frac{R}{L} I_0 e^{-(R/L)t} = -\frac{R}{L} I$$

Thus it is clear that greater the ratio R/L, *or smaller the time constant L/R, the more rapidly does the current die away* (Fig. 12.4).

Fig. 12.5 shows that the growth and decay curves are complementary.

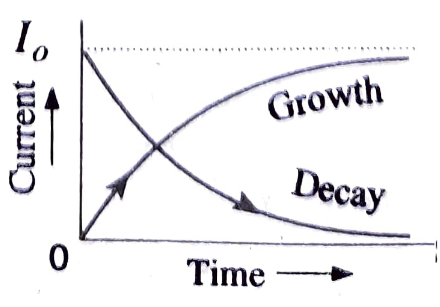

Fig. 12.5

Example 1. *An e.m.f. 10 volts is applied to a circuit having a resistance of 10 ohms and an inductance of 0.5 henry. Find the time required by the current to attain 63.2% of its final value. What is the time constant of the circuit ?*

Sol. $$I = I_0 (1 - e^{-Rt/L})$$

Given $$\frac{I}{I_0} = \frac{63.2}{100}; \frac{R}{L} = \frac{10}{0.5} = 20$$

$$\frac{63.2}{100} = 1 - e^{-20t}$$

$$e^{-20t} = 1 - 0.632 = 0.368$$

$$e^{20t} = \frac{1}{0.368} = 2.717$$

$$20t = \log_e 2.717$$

$$t = \frac{1}{20} \times 2.3026 \times \log_{10} 2.717$$

$$= \frac{2.3026 \times 0.4341}{20} = \mathbf{0.05\ sec.}$$

The time constant of the circuit is

$$\frac{L}{R} = \frac{0.5}{10} = \frac{1}{20} \text{ sec.}$$

Example 2. *An inductance of 500 mH and a resistance of 5 ohms are connected in series with an e.m.f of 10 volts. Find the final current. If now the cell is removed and the two terminals are connected together, find the current after (i) 0.05 sec. and (ii) 0.2 sec.*

Sol. Final current $I_0 = \frac{E}{R} = \frac{10}{5} = 2A$

During discharge, $I = I_0 e^{-Rt/L}$

Now $$\frac{R}{L} = \frac{5}{500 \times 10^{-3}} = 10$$

(*i*) When $t = 0.05$ *sec.*, $I = 2e^{-10 \times 0.05} = 1.213A$

(*ii*) When $t = 0.2$ *sec.*, $I = 2e^{-10 \times 0.2} = 0.271A$

12.3 Charge and Discharge of a Capacitor through a Resistor

(*a*) **Growth of Charge.** A capacitor C and a resistance R are connected to a cell of emf E through a Morse key K (Fig. 12.6). When the key is pressed, a momentary current I flows through R. At any instant t, let Q be the charge on the capacitor of capacitance C.

P.D. across capacitor $= Q/C$

P.D. across resistor $= RI$

The emf equation of the circuit is

$E = (Q/C) + RI$ ... (1)

$E = (Q/C) + R\,(dQ/dt)$

$\because I = dQ/dt.$

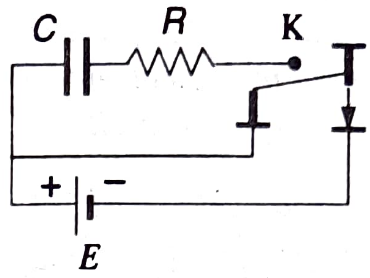

Fig. 12.6

The capacitor continues getting charged till it attains the maximum charge Q_0. At that instant $I = dQ/dt = 0$. The P.D. across the capacitor is $E = Q_0/C$.

i.e., when, $$Q = Q_0,\ \frac{dQ}{dt} = 0 \text{ and } E = \frac{Q_0}{C}$$

$$\therefore \quad \frac{Q_0}{C} = \frac{Q}{C} + R\,\frac{dQ}{dt}$$

$$(Q_0 - Q) = CR\frac{dQ}{dt}$$

$$\left(\frac{dQ}{Q_0 - Q}\right) = \frac{dt}{CR} \qquad \text{... (2)}$$

Integrating, $-\log_e (Q_0 - Q) = \dfrac{t}{CR} + K$

where K is a constant.

When $t = 0$, $Q = 0$ $\therefore -\log_e Q_0 = K$

$$\therefore \quad -\log_e (Q_0 - Q) = \frac{t}{CR} - \log_e Q_0$$

$$\log_e (Q_0 - Q) = -\frac{t}{CR} + \log_e Q_0$$

$$\log_e (Q_0 - Q) - \log_e Q_0 = -\frac{t}{CR}$$

$$\log_e \left(\frac{Q_0 - Q}{Q_0}\right) = -\frac{t}{CR}$$

$$\frac{Q_0 - Q}{Q_0} = e^{-\frac{t}{CR}} \quad \text{or} \quad 1 - \frac{Q}{Q_0} = e^{-\frac{t}{CR}}$$

$$\therefore \quad Q = Q_0\left(1 - e^{-\frac{t}{CR}}\right) \qquad \text{... (3)}$$

The term CR is called time constant of the circuit.

At the end of time $t = CR$, $Q = Q_0\,(1 - e^{-1}) = 0.632Q_0$

Thus, *the time constant may be defined as the time taken by the capacitor to get charged to 0.632 times its maximum value.*

The growth of charge is shown in Fig. 12.7.

The rate of growth of charge is

$$\frac{dQ}{dt} = \frac{Q_0}{CR} e^{-\frac{t}{CR}} = \frac{1}{CR}(Q_0 - Q)$$

Thus it is seen that smaller the product CR, the more rapidly does the charge grow on the capacitor.

The rate of growth of the charge is rapid in the beginning and it becomes less and less as the charge approaches nearer and nearer the steady value.

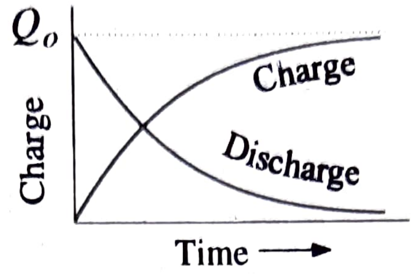

Fig. 12.7

(*b*) **Decay of charge (Discharging of a Capacitor through Resistance)**

Let the capacitor having charge Q_0 be now discharged by releasing the Morse key K (Fig. 12.6). The charge flows out of the capacitor and this constitutes a current. In this case $E = 0$.

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0 \qquad \text{... (1)}$$

or
$$\frac{dQ}{Q} = -\frac{1}{CR}dt$$

Integrating, $\log_e Q = -\frac{t}{CR} + K$, where K is a constant

When $t = 0$, $Q = Q_0$ $\therefore$ $\log_e Q_0 = K$

$$\log_e Q = -\frac{t}{CR} + \log_e Q_0$$

or
$$\log_e \frac{Q}{Q_0} = -\frac{t}{CR} \quad \text{or} \quad \frac{Q}{Q_0} = e^{-t/CR}$$

$\therefore$
$$Q = Q_0 e^{-t/CR} \qquad \text{... (2)}$$

This shows that the charge in the capacitor decays exponentially and becomes zero after infinite interval of time (Fig. 12.7).

The rate of discharge is

$$I = \frac{dQ}{dt} = -\frac{Q_0}{CR} e^{-t/CR} = -\frac{Q}{CR} \qquad \text{... (3)}$$

Thus, smaller the time-constant CR, the quicker is the discharge of the capacitor.

In Eq. (2), if we put $t = CR$, then $Q = Q_0 e^{-1} = 0.368\, Q_0$.

Hence *time constant may also be defined as the time taken by the current to fall from maximum to 0.368 of its maximum value.*

Example 1. *A capacitor is charged by DC supply through a resistance of 2 megaohms. If it takes 0.5 seconds for the charge to reach three quarters of its final value, what is the capacitance of the capacitor ?*

Sol. Here, $R = 2 \times 10^6\ \Omega$, $t = 0.5\text{s}$, $Q/Q_0 = 3/4$.

$$Q = Q_0\,(1 - e^{-t/CR}) \quad \text{or} \quad \frac{Q}{Q_0} = (1 - e^{-t/CR})$$

or $$\frac{3}{4} = \left(1 - e^{-\frac{0.5}{C \times (2 \times 10^6)}}\right)$$

or $$e^{\frac{0.5}{C \times (2 \times 10^6)}} = 4 \quad \text{or} \quad \frac{0.5}{C \times (2 \times 10^6)} = \log_e 4$$

or $$C = \frac{0.5}{(2.3026 \log_{10} 4)\,(2 \times 10^6)} = 0.18 \times 10^{-6}\ \text{F} = 0.18\ \mu\text{F}.$$

Example 2. *A capacitor of capacitance 0.1 μF is first charged and then discharged through a resistance of 10 megaohm. Find the time, the potential will take to fall to half its original value.*

Sol. $$Q = Q_0\, e^{-t/CR} \quad \text{or} \quad \frac{Q}{Q_0} = e^{-t/CR} \quad \text{or} \quad \ln\left(\frac{Q_0}{Q}\right) = \frac{t}{CR}$$

$$\therefore \quad t = CR \ln\left(\frac{Q_0}{Q}\right)$$

$Q = CV$ and $Q_0 = CV_0$. $\qquad \therefore \quad \dfrac{Q_0}{Q} = \dfrac{V_0}{V}$

$$\therefore \quad t = CR \ln\left(\frac{V_0}{V}\right)$$

Here, $C = 10^{-7}$ F; $R = 10^7\ \Omega$, $V_0/V = 2$.

$$\therefore \quad t = 10^{-7} \times 10^7 \times \ln 2 = 0.6931\ \text{s}.$$

Example 3. *A resistance R and a 2μF capacitor in series are connected to a 200 volt direct supply. Across the capacitor is a neon lamp that strikes at 120 volts. Calculate the value of R to make the lamp strike 5 seconds after switch has been closed.*

Sol. The resistance R must be such that the p.d. across the capacitor should rise to 120 volt in 5 seconds after the switch is closed. The lamp would then strike.

The equation of charging is

$$Q = Q_0\,(1 - e^{-t/RC})$$

$$\text{or} \quad CV = CV_0\,(1 - e^{-t/RC})$$

$$\text{or} \quad V = V_0\,(1 - e^{-t/RC})$$

Here, $V = 120$ volt, $V_0 = 200$ volt, $t = 5$ sec and

$$C = 2 \times 10^{-6} \textit{ farad},$$

$$\therefore \quad 120 = 200\,(1 - e^{-5/2\times 10^{-6}R})$$

$$\text{or} \quad e^{5/2\times 10^{-6}R} = \frac{5}{2}$$

$$\frac{5}{2\times 10^{-6}R} = \log_e 5 - \log_e 2 = 1.6094 - 0.6931 = 0.9163$$

$$R = \frac{5}{2\times 10^{-6} \times 0.9163} = 2.73 \times 10^6 \text{ ohm} = \textbf{2.73 megaohm}$$

12.4. Measurement of High Resistance by Leakage

When a capacitor of capacitance C and initial charge Q_0 is allowed to discharge through a resistance R for a time t, the charge remaining on the capacitor is given by

$$Q = Q_0 e^{-t/CR}$$

$$\frac{Q_0}{Q} = e^{t/CR}$$

$$\log_e \frac{Q_0}{Q} = \frac{t}{CR}$$

$$\therefore \quad R = \frac{t}{C \log_e (Q_0/Q)} = \frac{t}{2.3026\, C \log_{10} (Q_0/Q)}$$

If R is high, CR will be high and the rate of discharge of capacitor will be very slow. Thus if we determine Q_0/Q from experiment, then R can be calculated.

Connections are made as shown in Fig. 12.8. C is a capacitor of known capacitance, R is the high resistance to be measured, B.G. is a ballistic galvano-meter, E is a cell, and K_1, K_2, K_3 are tap keys.

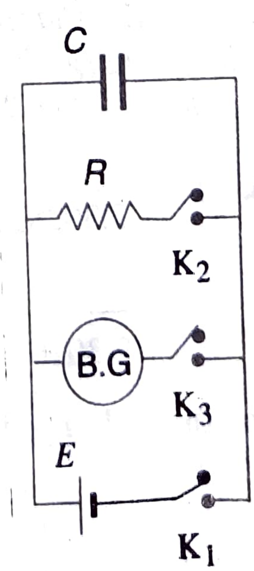

Fig. 12.8

Keeping K_2 and K_3 open, the capacitor is charged by depressing the key K_1. K_1 is then opened and at once K_3 is closed. The capacitor discharges through the galvanometer which records a throw θ_0. The throw θ_0 is proportional to Q_0. The capacitor is again charged to the maximum value keeping K_2 and K_3 open and closing K_1. K_1 is then opened and K_2 is closed for a *known* time t. Some of the charge leaks through R. K_2 is opened and at once K_3 is closed. The charge Q remaining on the capacitor then discharges through the galvanometer. The resulting throw θ is noted. Then $Q \propto \theta$

Now, $$\frac{Q_0}{Q} = \frac{\theta_0}{\theta}$$

$$\therefore \quad R = \frac{t}{2.3026\, C \log_{10} (\theta_0/\theta)}$$

A series of values of t and θ are obtained. A graph is plotted between t and $\log_{10} (\theta_0/\theta)$ which is a straight line. Its slope gives the mean value of $\frac{t}{\log_{10} (\theta_0/\theta)}$. As C is known, the value of R can be calculated.

Example 1. *If the charge on a capacitor of capacitance 2 μF in leaking through a high resistance of 100 megaohms is reduced to half its maximum value, calculate the time of leakage.*

Sol. $$R = \frac{t}{2.3026\, C \log_{10} (Q_0/Q)}$$

Here, $C = 2 \times 10^{-6}$ F, $R = 10^8\ \Omega$, $Q_0/Q = 2$.

$$\therefore \quad t = 2.3026\, CR \log_{10} (Q_0/Q) = 2.3026\, (2 \times 10^{-6})\, 10^8 \log_{10} 2 = 138.7 \text{ s.}$$

Example 2. *In an experiment to determine high resistance by leakage, a capacitor of 0.2 μF is used. It is first fully charged and discharged through a B.G. The observed kick was 12 cm on the scale. The capacitor was fully charged again and allowed to leak through R for 2 sec. The remaining charge in C gave a kick of 6 cm on the same scale when discharged through the B.G. Calculate R.*

Sol. $$R = \frac{t}{2.3026\, C \log_{10} (\theta_0/\theta)}$$

$$= \frac{2}{2.3026 \times (0.2 \times 10^{-6}) \log_{10} (12/6)} = 14.43 \text{ megaohms.}$$

12.5. Growth of charge in a Circuit with Inductance, Capacitance and Resistance

Consider a circuit containing an inductance L, capacitance C and resistance R joined in series to a cell of emf E (Fig. 12.9). When the key K is pressed, the capacitor is charged. Let Q be the charge on the capacitor and I the current in the circuit at an instant t during charging. Then, the p.d.

across the capacitor is Q/C and the self induced emf in the inductance coil is $L\ (dI/dt)$, both being opposite to the direction of E. The p.d. across the resistance R is RI.

The equation of emf's is

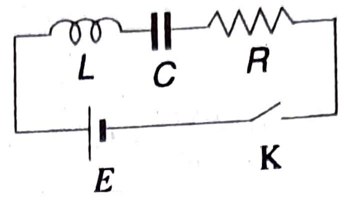

Fig. 12.9

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E \qquad \text{... (1)}$$

But $$I = \frac{dQ}{dt} \text{ and } \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

$\therefore$ $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

or $$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q - CE}{LC} = 0$$

Putting $\frac{R}{L} = 2b$ and $\frac{1}{LC} = k^2$, we have

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + k^2(Q - CE) = 0 \qquad \text{... (2)}$$

Let $x = Q - CE$. Then $\frac{dx}{dt} = \frac{dQ}{dt}$ and $\frac{d^2x}{dt^2} = \frac{d^2Q}{dt^2}$

Eq. (2) becomes, $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0 \qquad \text{... (3)}$$

Hence the most general solution of Eq. (3) is

$$x = Ae^{[-b+\sqrt{(b^2-k^2)}]t} + Be^{[-b-\sqrt{(b^2-k^2)}]t}$$

Now, $CE = Q_0$ = final steady charge on the capacitor.

$\therefore$ $x = Q - CE = Q - Q_0$.

Hence $$Q - Q_0 = Ae^{[-b+\sqrt{(b^2-k^2)}]t} + Be^{[-b-\sqrt{(b^2-k^2)}]t}$$

or $$Q = Q_0 + Ae^{[-b+\sqrt{(b^2-k^2)}]t} + Be^{[-b-\sqrt{(b^2-k^2)}]t} \qquad \text{... (4)}$$

Using initial conditions :

at $t = 0$, $Q = 0$ ... (5)

$\therefore$ $0 = Q_0 + (A + B)$ or $A + B = -Q_0$

$$\frac{dQ}{dt} = A\left(-b + \sqrt{(b^2 - k^2)}\right] e^{[-b + \sqrt{(b^2 - k^2)}]t} +$$

$$B[-b - \sqrt{(b^2 - k^2)}]\, e^{[-b - \sqrt{(b^2 - k^2)}]t}$$

At $t = 0$, $\frac{dQ}{dt} = 0$

$$0 = A\,[-b + \sqrt{(b^2 - k^2)}] + B\,[-b - \sqrt{(b^2 - k^2)}]$$

$$\sqrt{(b^2 - k^2)}\,[A - B] \neq b\,(A + B) = -bQ_0$$

or $$A - B = -\frac{Q_0 b}{\sqrt{(b^2 - k^2)}} \qquad \text{... (6)}$$

Solving Eqs. (5) and (6),

$$A = -\frac{1}{2} Q_0 \left(1 + \frac{b}{\sqrt{(b^2 - k^2)}}\right) \qquad \text{... (7)}$$

$$B = -\frac{1}{2} Q_0 \left(1 - \frac{b}{\sqrt{(b^2 - k^2)}}\right) \qquad \text{... (8)}$$

Substituting the values of A and B in Eq. (4), we have

$$Q = Q_0 - \frac{1}{2} Q_0 e^{-bt}\left[\left(1 + \frac{b}{\sqrt{(b^2 - k^2)}}\right) e^{\sqrt{(b^2 - k^2)}\,.\,t} + \left(1 - \frac{b}{\sqrt{(b^2 - k^2)}}\right) e^{-\sqrt{(b^2 - k^2)}\,.\,t}\right] \qquad \text{... (9)}$$

Case I. If $b^2 > k^2$, $\sqrt{(b^2 - k^2)}$ is real. The charge on the capacitor grows exponentially with time and attains the maximum value Q_0 asymptotically, (curve 1 of Fig. 12.10). The charge is known as *over damped or dead beat.*

Case II. If $b^2 = k^2$, the charge rises to the maximum value Q_0 in a short time (curve 2 of Fig. 12.10). Such a charge is called *critically damped*.

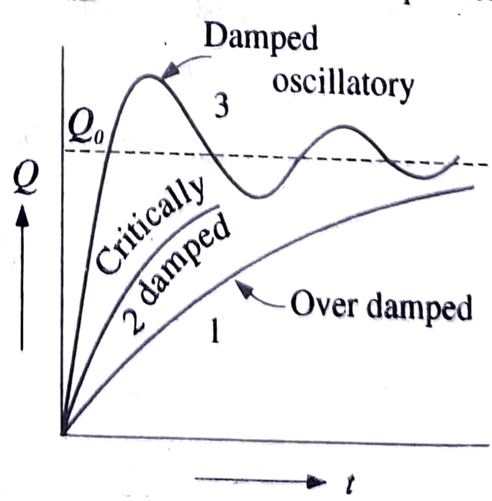

Fig. 12.10

Case III. If $b^2 < k^2$, $\sqrt{(b^2 - k^2)}$ is imaginary.

Let $\sqrt{b^2 - k^2} = i\omega$ where $i = \sqrt{-1}$ and $\omega = \sqrt{k^2 - b^2}$

Eq. (9) may be written as

$$Q = Q_0 - \frac{1}{2} Q_0 e^{-bt} \left[\left(1 + \frac{b}{i\omega}\right) e^{i\omega t} + \left(1 - \frac{b}{i\omega}\right) e^{-i\omega t} \right]$$

$$Q = Q_0 - Q_0 e^{-bt} \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} + \frac{b}{\omega} \frac{(e^{i\omega t} - e^{-i\omega t})}{2i} \right]$$

$$Q = Q_0 - Q_0 e^{-bt} \left(\cos \omega t + \frac{b}{\omega} \sin \omega t \right)$$

$$Q = Q_0 \left[1 - \frac{e^{-bt}}{\omega} (\omega \cos \omega t + b \sin \omega t) \right]$$

Let $\omega = k \sin \alpha$ and $b = k \cos \alpha$ so that $\tan \alpha = \omega / b$.

$$Q = Q_0 \left[1 - \frac{e^{-bt}}{\omega} (k \sin \alpha \cos \omega t + k \cos \alpha \sin \omega t) \right]$$

or $$Q = Q_0 \left[1 - \frac{k e^{-bt}}{\omega} \sin (\omega t + \alpha) \right] \quad \text{... (10)}$$

$$Q = Q_0 \left[1 - \frac{e^{-\frac{R}{2L} t} \sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} . \sin \left[\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \right) t + \alpha \right] \right]$$

This equation represents a damped oscillatory charge as shown by the curve (3). The charge oscillates above and below Q_0 till it finally settles down to Q_0 value. The frequency of oscillation in the circuit is given by

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{k^2 - b^2}}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When $R = 0, \ \nu = \dfrac{1}{2\pi \sqrt{LC}}$

12.6. Discharge of a Capacitor through an inductor and a Resistor in series (Decay of charge in LCR circuit)

Consider a circuit containing a capacitor of capacitance C, an inductance L and a resistance R joined in series (Fig. 12.11). E is a cell. K_2 is kept open. The capacitor is charged to maximum charge Q_0 by closing the key K_1. On opening K_1 and closing key K_2, the capacitor discharges through the inductance L and resistance R. Let I be the current in the circuit and Q be the charge in the capacitor at any instant during discharge. The circuit equation then is

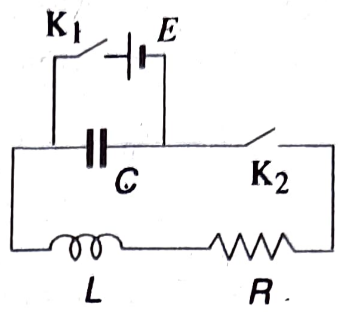

Fig. 12.11

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0$$

But, $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$

$$\therefore \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \qquad \ldots (1)$$

Let $\frac{R}{L} = 2b$ and $\frac{1}{LC} = k^2$, then

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + k^2Q = 0 \qquad \ldots (2)$$

The general solution of this equation is

$$Q = A\,e^{(-b + \sqrt{b^2 - k^2})t} + B\,e^{(-b - \sqrt{b^2 - k^2})t} \qquad \ldots (3)$$

where A and B are arbitrary constants.

When $t = 0$, $Q = Q_0$ and from Eq.(3), $A + B = Q_0$ ... (4)

$$\frac{dQ}{dt} = A(-b + \sqrt{b^2 - k^2})\,e^{(-b + \sqrt{b^2 - k^2})t}$$

$$+ B(-b - \sqrt{b^2 - k^2})\,e^{(-b - \sqrt{b^2 - k^2})t}$$

When, $t = 0$, $\frac{dQ}{dt} = 0$

$$\therefore A(-b + \sqrt{b^2 - k^2}) + B(-b - \sqrt{b^2 - k^2}) = 0$$

$$-b(A + B) + \sqrt{b^2 - k^2}\,(A - B) = 0$$

$$-bQ_0 + \sqrt{b^2 - k^2}\,(A - B) = 0$$

$$\therefore \quad A - B = \frac{bQ_0}{\sqrt{b^2 - k^2}} \qquad \ldots (5)$$

From Eqs. (4) and (5), we get

$$A = \frac{1}{2}Q_0\left(1 + \frac{b}{\sqrt{(b^2 - k^2)}}\right) \text{ and } B = \frac{1}{2}Q_0\left(1 - \frac{b}{\sqrt{(b^2 - k^2)}}\right)$$

Putting these values of A and B in Eq. (3), we get

$$Q = \frac{1}{2} Q_0 e^{-bt} \left[\left(1 + \frac{b}{\sqrt{(b^2 - k^2)}}\right) e^{\sqrt{(b^2 - k^2)}\,t} + \left(1 - \frac{b}{\sqrt{(b^2 - k^2)}}\right) e^{-\sqrt{(b^2 - k^2)}\,t} \right] \quad \dots (6)$$

Case I. If $b^2 > k^2$, $\sqrt{(b^2 - k^2)}$ is real and positive and the charge of the capacitor decays exponentially, becoming zero asymptotically (curve 1 of Fig. 12.12). This discharge is known as *over damped, non-oscillatory or dead beat.*

Case II. When $b^2 = k^2$, $Q = Q_0 (1 + bt)\, e^{-bt}$

This represents a *non-oscillatory* discharge. This discharge is known as *critically damped* (Curve 2 of Fig. 12.12). The charge decreases to zero exponentially in a short time.

Case III. If $b^2 < k^2$, $\sqrt{(b^2 - k^2)}$ is imaginary.

$\sqrt{b^2 - k^2} = i\omega$, where $\omega = \sqrt{(k^2 - b^2)}$

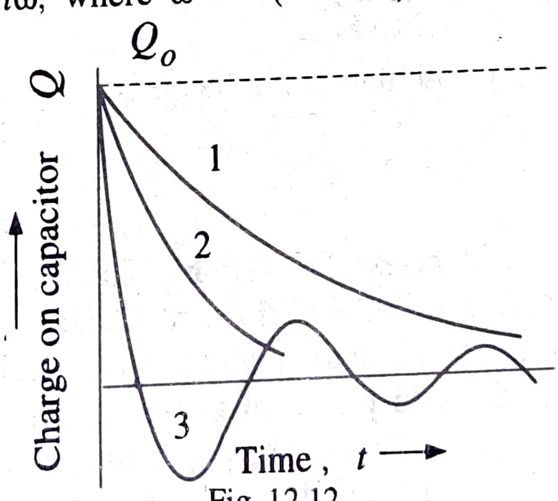

Fig. 12.12

$$\therefore \quad Q = \frac{1}{2} Q_0\, e^{-bt} \left[\left(1 + \frac{b}{i\omega}\right) e^{i\omega t} + \left(1 - \frac{b}{i\omega}\right) e^{-i\omega t} \right]$$

$$= Q_0 e^{-bt} \left[\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) + \frac{b}{\omega} \left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) \right]$$

$$= \frac{Q_0 e^{-bt}}{\omega} (\omega \cos \omega t + b \sin \omega t)$$

Let $\omega = k \sin \alpha$ and $b = k \cos \alpha$ so that $\tan \alpha = \dfrac{\omega}{b}$.

$$Q = \frac{Q_0 e^{-bt} k}{\omega} (\cos \omega t \sin \alpha + \cos \alpha \sin \omega t)$$

$$= \frac{Q_0 e^{-bt} k}{\omega} \sin (\omega t + \alpha)$$

$$Q = \frac{Q_0 e^{-\frac{R}{2L}t}}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\sqrt{LC}} \sin \left(\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\, t + \alpha\right) \quad \dots (7)$$

This equation represents a *damped oscillatory* charge as shown by the curve (3). The charge oscillates above and below zero till it finally settles down to zero value.

The frequency of oscillation in the circuit is given by

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{k^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When $R = 0$, $\nu = \dfrac{1}{2\pi\sqrt{LC}}$

The condition for oscillatory discharge is

$$\frac{R^2}{4L^2} < \frac{1}{LC}, \quad \text{or } R < 2\sqrt{\frac{L}{C}}.$$

Importance in Wireless Telegraphy. The discharge of a capacitor through an inductance is oscillatory if the resistance R of the circuit is less than $2\sqrt{(L/C)}$. During the discharge, the energy of the charged capacitor is stored in the magnetic field produced in the inductance coil, then again back in the electric field between the capacitor plates, and so on. If the fields are caused to alternate rapidly, some energy escapes from the circuit permanently in the form of electro-magnetic waves which travel through space with the speed of light. These waves form the basis of wireless telegraphy.

Messages can be transmitted from one place to another with the help of codes.

Example 1. *If a battery, of emf 100 volts, is connected in series with an inductance of 10 mH, a capacitor of 0.05 μF and a resistance of 100 Ω, find (i) the frequency of the oscillatory current, (ii) the logarithmic decrement and (iii) the final capacitor charge.*

Sol. $\dfrac{R^2}{4L^2} = \dfrac{100^2}{4 \times (10^{-2})^2} = 2.5 \times 10^7$

and $\frac{1}{LC} = \frac{1}{10^{-2} \times 0.05 \times 10^{-6}} = 2 \times 10^9$

$R^2/4L^2 < 1/LC$. *Hence the charging of the capacitor is oscillatory.*

$$\nu = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \frac{1}{2 \times 3.14}\sqrt{(2 \times 10^9 - 2.5 \times 10^7)}$$

$$= 7076.6 \text{ Hz.}$$

The logarithmic decrement $= \frac{RT}{2L} = \frac{100}{2 \times 10^{-2} \times 7076.6} = 0.707.$

The final capacitor charge $Q_0 = EC = 100 \times 0.05 \times 10^{-6} = 5\mu C.$

Example 2. *A charged capacitor of capacitance 0.01 μF is made to discharge through a circuit consisting of a coil of inductance 0.1 henry and an unknown resistance. What should be the maximum value of the unknown resistance, if the discharge of the capacitor is to be oscillatory.*

Sol. If R is the maximum value of the resistance for the discharge to be oscillatory, then

$$\frac{R^2}{4L^2} = \frac{1}{LC} \text{ or } R = \sqrt{\frac{4L}{C}} = \sqrt{\left(\frac{4 \times 0.1}{0.01 \times 10^{-6}}\right)} = 6324\ \Omega$$

Example 3. *(i) Find out whether the discharge of a capacitor through a circuit containing the following elements, is oscillatory.*

C = 0.2 μF, L = 10 mH, R = 250 ohm

(ii) If so, find the frequency.

(iii) Calculate the maximum value of the resistance possible so as to make the discharge oscillatory.

Sol.

(*i*) The condition for oscillations is that

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

or $R^2 < \frac{4L}{C}$, or $R < \sqrt{\frac{4L}{C}}$

We have $\sqrt{\frac{4L}{C}} = \sqrt{\frac{4 \times 10 \times 10^{-3}}{0.2 \times 10^{-6}}} = 447$ ohm

It is given that $R = 250$ ohm

$\therefore R < \sqrt{\frac{4L}{C}}$

Therefore the discharge is oscillatory.