UNIT IV

Sound:

Equation of motion for a body executing angular simple harmonic oscillations–Definition of free, damped and forced vibrations – Theory of forced vibrations – Resonance – Sharpness of resonance – Fourier's theorem – application for Saw– tooth wave and square wave. –Sonometer – determination of A.C. frequency using sonometer

11 CHAPTER Waves and Oscillations

11.1. SIMPLE HARMONIC MOTION

Let P be a particle moving on the circumference of a circle of radius a with a uniform angular velocity ω (Fig. 11.1). O is the centre of the circle.

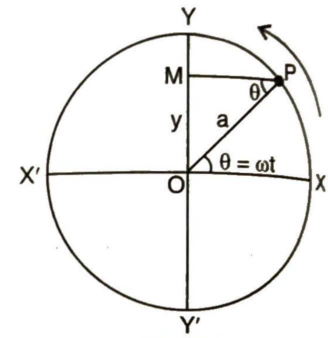

Fig. 11.1

A perpendicular PM is drawn from the particle on the diameter YY' of the circle. As the particle P moves round the circle, the foot of the perpendicular M vibrates along the diameter YY'. Since the motion of P is uniform, the motion of M is periodic. As the particle P completes one revolution, the foot of the perpendicular M completes one *vertical oscillation*. The distance OM is called the displacement and is denoted by y.

The particle moves from X to P in time t.

$$\angle POX = \angle MPO = \theta = \omega t$$

From the ΔMPO,

$$\sin\theta = \sin\omega t = \frac{OM}{a}$$

or

$$OM = y = a\sin\omega t$$

OM is called the displacement of the vibrating particle.

The displacement of a vibrating particle at any instant can be defined as its distance from the mean position of rest.

The maximum displacement of a vibrating particle is called its *amplitude*.

Displacement $= y = a\sin\omega t$ ...(1)

Fig. 11.2 shows the changes in the displacement of a vibrating particle in one complete vibration.

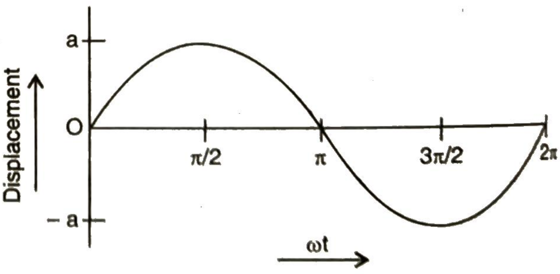

Fig. 11.2

$$\text{Velocity} = v = \frac{dy}{dt} = a\omega\cos\omega t = \omega\sqrt{(a^2 - y^2)} \quad ...(2)$$

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -a\,\omega^2 \sin\omega t = -\omega^2 y \quad ...(3)$$

Thus, acceleration is directly proportional to displacement and directed towards a fixed point. This type of motion is called *simple harmonic motion*.

Following are the characteristics of simple harmonic motion:

(*i*) The motion is periodic.

(*ii*) The motion is along a straight line about the mean or equilibrium position.

(*iii*) The acceleration is proportional to displacement.

(*iv*) Acceleration is directed towards the mean or equilibrium position.

Definition : *If a particle moves in a straight line, so that its acceleration is always directed towards a fixed point on the line, and is proportional to its displacement from the fixed point, the particle is said to move with simple harmonic motion.*

Eq. (3) can be written as

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ...(4)$$

This is the differential equation of S.H.M.

$$\text{The period} = T = \frac{2\pi}{\omega}$$

$$\text{The frequency} = n = \frac{1}{T} = \frac{\omega}{2\pi}.$$

Examples of S.H.M.

1. The vertical oscillations of a spiral spring suspended from a rigid support, and loaded at the lower end. This is linear type of S.H.M.
2. The vibrations of a simple pendulum. This is angular type of S.H.M.

Phase : Consider a particle starting from S and moving on the circumference of a circle (Fig. 11.3). $\angle SOX = \alpha$. The particle moves from S to P in time t.

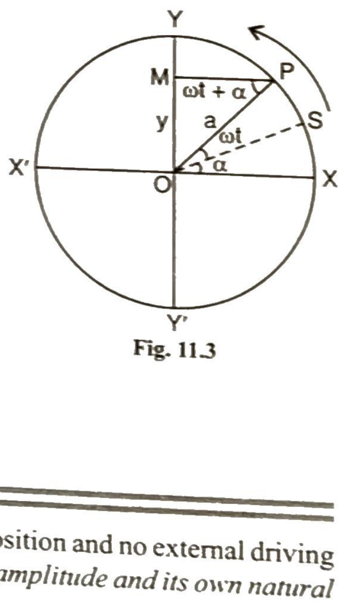

Fig. 11.3

$$\angle OPM = (\omega t + \alpha).$$

$$\text{Displacement } y = a \sin(\omega t + \alpha).$$

α is called *initial phase or epoch of the S.H.M.*

The angle $(\omega t + \alpha)$ is called the *phase* of the S.H.M.

11.2. FREE VIBRATIONS OF A BODY

When a body, free to oscillate, is displaced from its equilibrium position and no external driving or resisting force is acting on it, it continues to oscillate with a *constant amplitude and its own natural frequency*. Such vibrations of a body are called free vibrations.

Example. A simple pendulum oscillating in vacuum. When the bob of simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion.

The simple pendulum vibrates with a time period T given by

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

The time period (T) depends only on the length of the pendulum (l) and the acceleration due to gravity (g) at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. The amplitude of swing remains constant. In such cases, there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations.

Differential Equation of an Undamped Vibration

Let m be the mass of a particle executing S.H.M.

Kinetic energy of the particle for displacement $y = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2$.

At the same instant, the potential energy of the particle $= \frac{1}{2}Ky^2$.

Here, K is the restoring force per unit displacement.

The total energy at any instant $= \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}Ky^2$.

For an undamped harmonic oscillator, this total energy remains constant.

$$\therefore \quad \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}Ky^2 = \text{constant} \qquad \text{...(1)}$$

Differentiating Eq. (1) with respect to time,

$$m\frac{d^2y}{dt^2} + Ky = 0 \qquad \text{...(2)}$$

or

$$\frac{d^2y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \qquad \text{...(3)}$$

or

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \text{...(4)}$$

Here

$$\omega^2 = \frac{K}{m}$$

The solution for Eq. (4) is

$$y = a\sin(\omega t - \alpha)$$

$$y = a\sin\left[\left(\sqrt{\frac{K}{m}}\right)t - \alpha\right]$$

The period of vibration is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{K}{m}\right)}}$

Thus, in the case of undamped free vibratitons, the differential equation is

$$\frac{d^2y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \qquad \text{...(5)}$$

11.3. DAMPED VIBRATIONS

When a body executing simple harmonic vibrations is left to itself, the amplitude of vibrations gradually decreases. After some time, the vibrations completely die out. This is because the motion of the body is resisted by various frictional effects. Retarding forces are called into play due to the viscosity or internal friction of the body and the resistance of the air. These forces reduce the amplitude and thus damp the oscillations. Such oscillations of a body are called damped vibrations.

Example. In actual practice, when a simple pendulum vibrates in air medium, there are frictional forces (resistance of air). So energy is dissipated in each vibration. The amplitude of swing decreases continuously with time. Finally the oscillations die out. Such vibrations are called free **damped** vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium.

Differential Equation of a Damped Vibration

Consider a body of mass m. Let y be the displacement of the body from the equilibrium position at any instant.

The instantaneous velocity is $\frac{dy}{dt}$.

When a body is oscillating in a resisting medium, two forces are acting on the body.

1. A restoring force directly proportional to the displacement y, but acting in the opposite direction. It may be written as

$$-Ky.$$

Here, K is the restoring force per unit displacement.

2. A frictional (or damping) force proportional to the velocity, but opposite to the direction of motion. It may be written as

$$-\mu\frac{dy}{dt}.$$

Here, μ is a positive constant depending upon the force of resistance.

Therefore, the differential equation in the case of free-damped vibrations is,

$$m\frac{d^2y}{dt^2} = -Ky - \mu\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + Ky + \mu\frac{dy}{dt} = 0$$

or $$\frac{d^2y}{dt^2} + \left(\frac{\mu}{m}\right)\frac{dy}{dt} + \left(\frac{K}{m}\right)y = 0 \qquad ...(1)$$

Put $\frac{\mu}{m} = 2b$ and $\frac{K}{m} = k^2$.

Then, $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + k^2y = 0 \qquad ...(2)$$

General Solution. Let us try the solution,

$$y = Ae^{pt}$$

Then, $\frac{dy}{dt} = Ape^{pt}$ and $\frac{d^2y}{dt^2} = Ap^2e^{pt}$

Substituting these values in Eq. (2),

$Ap^2e^{pt} + 2bApe^{pt} + k^2Ae^{pt} = 0$

or $$p^2 + 2bp + k^2 = 0$$

or $$p = -b \pm \sqrt{(b^2 - k^2)}$$

The general solution is

$$y = Ae^{\{-b+\sqrt{(b^2-k^2)}\}t} + Be^{\{-b-\sqrt{(b^2-k^2)}\}t}$$

$$y = e^{-bt}[Ae^{\{\sqrt{(b^2-k^2)}\}t} + Be^{-\{\sqrt{(b^2-k^2)}\}t}]$$

If $b^2 < k^2$, $\sqrt{(b^2-k^2)}$ becomes imaginary.

$\therefore \quad \sqrt{(b^2-k^2)} = \sqrt{-1(k^2-b^2)} = i\sqrt{(k^2-b^2)} = i\beta$

Here $\quad i = \sqrt{-1}$ and $\beta = \sqrt{(k^2-b^2)}$

The solution becomes,

$$y = e^{-bt}[Ae^{i\beta t} + Be^{-i\beta t}]$$

$$y = e^{-bt}[A(\cos\beta t + i\sin\beta t) + B(\cos\beta t - i\sin\beta t)]$$

$$y = e^{-bt}[(A+B)\cos\beta t + i(A-B)\sin\beta t]$$

Put $(A+B) = C\sin\delta$ and $i(A-B) = C\cos\delta$.

$$y = Ce^{-bt}\sin(\beta t + \delta) \qquad ...(3)$$

Here C and δ are constants.

Eq. (3) shows that the motion is oscillatory.

The amplitude of oscillation is Ce^{-bt}.

The amplitude is not constant.

The amplitude decreases exponentially with time.

Finally, the amplitude becomes zero after a long time (Fig. 11.4).

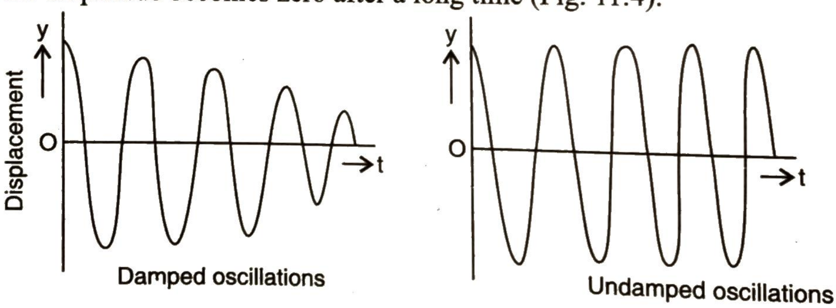

Fig. 11.4

The period of damped vibration is

$$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2-b^2}} = \frac{2\pi}{\sqrt{\frac{K}{m} - \frac{\mu^2}{4m^2}}}$$

The period of damped vibration is greater than the period of undamped vibration.

11.4. FORCED VIBRATIONS

A body set in vibration gradually loses its amplitude due to retarding forces and thus loses energy. If some external periodic force is constantly applied on the body, the body continues to oscillate under the influence of such external forces. Such vibrations of the body are called *forced vibrations*.

When an external periodic force is applied to a body executing damped vibrations, the body shows a tendency to vibrate with its natural frequency. But the external periodic force applied tries to impress on the body its own frequency of vibration. As time advances, the natural vibrations die out due to frictional forces. The body finally vibrates with the frequency of vibration of the applied external periodic force.

The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency of the applied force. The amplitude will be large if difference in frequencies is small.

Resonance : The particular case of forced vibration in which the frequency of the applied periodic force is equal to the natural frequency of the body itself is called *resonance.* When the frequencies are equal, the applied force helps to increase the amplitude of the body at each step. So when resonance occurs, the body vibrates with a large amplitude.

Examples of Resonance : (1) A vibrating tuning fork is held just above the open end of a tube containing water. If the level of water in the tube is gradually lowered, the length of the air column increases. The air column is thrown into *forced vibrations.* When the natural frequency of the air column is equal to the frequency of the tuning fork, a very loud sound is produced. This is due to resonance.

(2) Another phenomenon of resonance is the tuning of a radio receiver to a desired broadcasting station. By turning the tuning knob, the local oscillator in the radio continuously changes its frequency of oscillation. At the point when the frequency of the oscillator is equal to the frequency of the station, the sound is maximum.

(3) The shank of an excited tuning fork is pressed on the sounding board of the sonometer. The length of the sonometer wire is adjusted so that its frequency is equal to the frequency of the tuning fork. Now, the wire begins to vibrate with a large amplitude and a paper rider placed on the wire is violently thrown off.

Differential Equation of Forced Vibrations

Consider a system oscillating about an equilibrium position under an external *periodic force.* Let y be its displacement from the equilibrium position at an instant during the oscillation. Its instantaneous velocity is $\frac{dy}{dt}$.

Three forces are acting upon the system at this instant.

1. A restoring force proportional to the displacement y, but acting in the opposite direction. This may be written as

$$-Ky$$

Here, K is the restoring force per unit displacement.

2. A frictional force proportional to the velocity but acting in the opposite direction. This may be written as

$$-\mu\frac{dy}{dt}.$$

Here, μ is a positive constant depending upon the force of resistance.

3. An external periodic force represented by

$$F \sin pt.$$

Here p is the angular frequency of the applied periodic force.

Let m be the mass of the system and $\frac{d^2y}{dt^2}$ the instantaneous acceleration.

The equation of motion of the system is

$$m\frac{d^2y}{dt^2} = -Ky - \mu\frac{dy}{dt} + F \sin pt$$

$$\therefore \quad m\frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + Ky = F \sin pt \quad ...(1)$$

General Solution. The particular solution of Eq. (1) representing the forced vibrations is

$$y = a \sin (pt - \alpha) \quad ...(2)$$

$$\frac{dy}{dt} = ap \cos (pt - \alpha) \quad ...(3)$$

$$\frac{d^2y}{dt^2} = -ap^2 \sin (pt - \alpha) = -p^2y \quad ...(4)$$

Substituting these values in Eq. (1).

$$-mp^2 a \sin (pt - \alpha) + \mu\, ap \cos (pt - \alpha) + Ka \sin (pt - \alpha) = F \sin pt$$

$$-mp^2 a[\sin pt \cos \alpha - \cos pt \sin \alpha] + \mu ap[\cos pt \cos \alpha + \sin pt \sin \alpha]$$
$$+ Ka [\sin pt \cos \alpha - \cos pt \sin \alpha] - F \sin pt = 0 \quad ...(5)$$

When $\sin pt = 1$; $\cos pt = 0$

$$-mp^2 a \cos \alpha + \mu ap \sin \alpha + Ka \cos \alpha - F = 0 \quad ...(6)$$

When $\cos pt = 1$; $\sin pt = 0$

$$+ mp^2 a \sin \alpha + \mu\, ap \cos \alpha - Ka \sin \alpha = 0. \quad ...(7)$$

Dividing Eq. (7) by $\cos \alpha$ and simplifying

$$\tan \alpha = \frac{\mu p}{(K - mp^2)} = \frac{A}{B} \quad ...(8)$$

From Eq. (8).

$$\sin \alpha = \frac{A}{\sqrt{A^2 + B^2}} \quad ...(9)$$

$$\cos \alpha = \frac{B}{\sqrt{A^2 + B^2}} \quad ...(10)$$

Dividing Eq. (6) by $\cos \alpha$.

$$-mp^2 a + \mu\, ap \tan \alpha + Ka - \frac{F}{\cos \alpha} = 0 \quad ...(11)$$

or $$a [(K - mp^2) + \mu p \tan \alpha] = \frac{F}{\cos \alpha}$$

But $(K - mp^2) = B$, and $\mu p = A$.

Substituting the values of $\tan \alpha$ and $\cos \alpha$ in Eq. (11).

$$a\left[B + \frac{A^2}{B}\right] = \frac{F\sqrt{A^2 + B^2}}{B}$$

$$a = \frac{F}{\sqrt{A^2 + B^2}}$$

Substituting the values of A and B

$$a = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad ...(12)$$

$$y = a \sin (pt - \alpha)$$

or
$$y = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha) \quad ...(13)$$

Another solution is obtained when $F = 0$.

$$y = Ce^{-bt} \sin(\beta t + \delta) \quad ...(14)$$

Here,
$$\beta = \sqrt{\frac{K}{m} - \frac{\mu^2}{4m^2}}$$

The general solution is

$$y = Ce^{-bt} \sin(\beta t + \delta) + \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(Pt - \alpha) \quad ...(15)$$

The first term is the transient term and dies away with time.

The second term is called the steady state term. During the steady state, the oscillator performs forced oscillations with the impressed force frequency.

Resonance : The amplitude of the system executing forced vibration is

$$\frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad ...(16)$$

The amplitude is maximum when the denominator is minimum.

This is possible if $K - mp^2 = 0$ or $K = mp^2$

or
$$p = \sqrt{\frac{K}{m}}. \quad ...(17)$$

or
$$p = \omega$$

i.e., when frequency of the external force = natural frequency of the system.

If friction is present, the amplitude at resonance $= \frac{F}{\mu p} = \frac{F}{\mu\sqrt{K/m}}$

or amplitude at resonance $= \frac{F}{\mu}\sqrt{\frac{m}{K}}$

Further, the amplitude will be infinite if μ is also zero.

In practice some damping is always present and μ is never zero. Hence the amplitude at resonance becomes very large but not infinite.

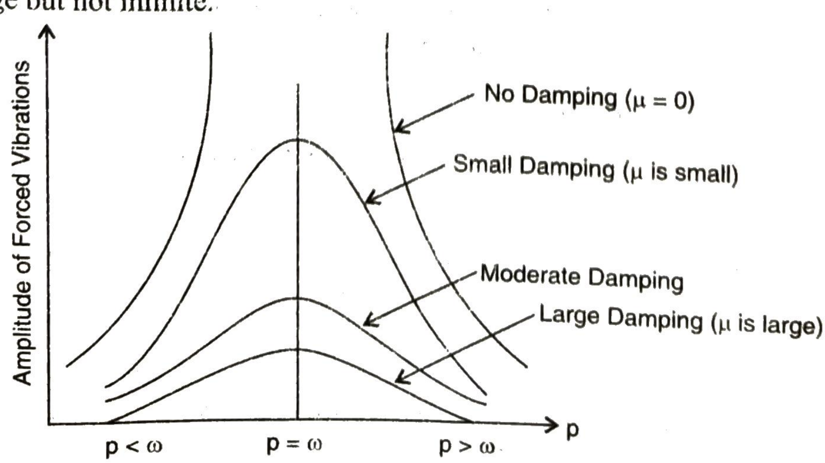

Fig. 11.5

Sharpness of Resonance : *Sharpness of resonance refers to the fall in amplitude with change in frequency on each side of the maximum amplitude.* The resonance is said to be sharp if the amplitude falls off rapidly with the deviation of ω from p. The degree of sharpness depends upon the magnitude of damping. For sharpness of resonance, μ (damping coefficient) must be small. In the sonometer experiment, we get sharp resonance due to the small damping coefficient of the wire.

In the experiment, for determining velocity of sound in air by the resonance column experiment, the resonance is flat due to the large damping coefficient of air.

Typical response curves showing *sharp resonance* and *flat resonance* are shown in Fig. 11.5.

(*i*) When the frictional forces are absent, *i.e.*, $\mu = 0$, the sharpness of resonance is maximum.

(*ii*) The sharpness of resonance decreases with increases in the value of μ.

11.5. FOURIER THEOREM

Statement : *Any single valued periodic function can be expressed as a sum of a number of simple harmonic terms which are multiples of the given function.*

Conditions : The theorem has the following two provisions:

(1) The displacement must be a single valued function and continuous. This condition is satisfied in the case of sound waves. A particle cannot actually have two different displacements simultaneously.

(2) The displacement must always have a finite value. This is true in the case of sound waves. A particle cannot have infinite displacement.

Explanation. The theorem deals with the synthesis of a complex periodic vibration from simple harmonic terms. It also gives a method to analyse a complex vibration into its component vibrations.

In acoustics, Fourier's theorem is applied for the analysis of musical notes. A complex musical *note* may be graphically represented as a periodic function. Fourier's theorem states that it is made up of a number of simple harmonic functions which represent pure tones.

11.6. FOURIER SERIES

Mathematically, Fourier's theorem can be expressed as :

$$y = f(\omega t) = B_0 + \sum_{m=1}^{m=\infty} A_m \sin m\omega t + \sum_{m=1}^{m=\infty} B_m \cos m\omega t \qquad ...(1)$$

Here, $y = f(\omega t)$ is the displacement of a complex periodic motion of angular frequency ω.

Thus the complex motion is a sum of sine and cosine components of amplitudes $A_1, A_2, ...$; $B_1, B_2, ...$and frequencies which are multiples of ω. B_0 is a constant.

Evaluation of Fourier Coefficients

In order to use Fourier theorem for analysing a complex periodic motion or wave-form, we must evaluate the Fourier coefficients B_0, A_m and B_m.

Evaluation of B_0.

Multiply Eq.(1) by dt and integrate from 0 to T.

Here, T is the period of the function.

$$\int_0^T y\,dt = \int_0^T B_0 dt + \int_0^T \left[\sum_{m=1}^{m=\infty} A_m \sin m\omega t\right] dt + \int_0^T \left[\sum_{m=1}^{m=\infty} B_m \cos m\omega t\right] dt \qquad ...(2)$$

Here $$\int_0^T \left[\sum_{m=1}^{m=\infty} A_m \sin m\omega t \right] dt = 0$$

and $$\int_0^T \left[\sum_{m=1}^{m=\infty} B_m \cos m\omega t \right] dt = 0$$

$\therefore$ $$\int_0^T y\, dt = B_0 T$$

or $$B_0 = \frac{1}{T} \int_0^T y\, dt \qquad ...(3)$$

Evaluation of A_m

Multiply Eq.(1) by sin $m\,\omega t$ and integrate the terms for a complete cycle 0 to T. Then,

$$A_m = \frac{2}{T} \int_0^T y \sin (m\omega t)\, dt \qquad ...(4)$$

Evaluation of B_m

Multiply Eq. (1) by cos $m\omega t$ and integrate from 0 to T. Then we have

$$B_m = \frac{2}{T} \int_0^T y \cos (m\,\omega t)\, dt \qquad ...(5)$$

Application in Sound : The quality of a musical note depends upon the harmonics present in it. Hence Fourier's theorem can determine the quality of a note by analysing its oscillation into its harmonic components.

The determination of the Fourier series corresponding to a given complex vibration is called *harmonic analysis* of the vibration. The terms of this series, whose cyclic frequencies are equal to ω, 2ω, 3ω, ..., etc., are called the first (fundamental), second, third, ... etc., harmonics, respectively, of the complex vibration.

11.7. SAW-TOOTH WAVE

Let us apply Fourier's theorem to analyse a saw-tooth wave into its harmonic components. The displacement curve of a periodic saw-tooth wave is shown in Fig. 11.6.

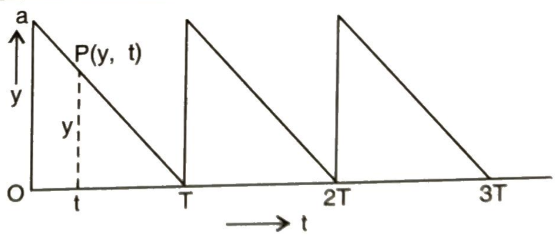

Fig. 11.6

The displacement falls off linearly from $y = a$ to $y = 0$ when t increases from 0 to T. Saw-tooth wave can be represented by the equation

$$y = f(\omega t) = a\left(1 - \frac{t}{T}\right) \text{ for } 0 < t < T \qquad ...(1)$$

According to Fourier Theorem,

$$y = f(\omega t) = B_0 + \sum_{m=1}^{m=\infty} A_m \sin m\,\omega t + \sum_{m=1}^{m=\infty} B_m \cos m\,\omega t \qquad ...(2)$$

Here, $$B_0 = \frac{1}{T} \int_0^T y\, dt \qquad ...(3)$$

$$A_m = \frac{2}{T} \int_0^T y \sin (m\,\omega t)\, dt \qquad ...(4)$$

and $$B_m = \frac{2}{T}\int_0^T y\cos(m\omega t)\,dt \qquad ...(5)$$

Let us obtain the values of the coefficients B_0, A_m and B_m.

(i) $$B_0 = \frac{1}{T}\int_0^T y\,dt$$

$$= \frac{1}{T}\int_0^T a\left(1-\frac{t}{T}\right)dt$$

$$= \frac{a}{T}\left[t-\frac{t^2}{2T}\right]_0^T$$

$$= \frac{a}{T}\left[T-\frac{T^2}{2T}\right] = \frac{a}{2}$$

$\therefore$ $$B_0 = \frac{a}{2} \qquad ...(6)$$

(ii) $$A_m = \frac{2}{T}\int_0^T y\sin(m\omega t)dt$$

$$= \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\sin(m\omega t)dt$$

$$= \frac{2a}{T}\int_0^T \sin(m\omega t)dt - \frac{2a}{T^2}\int_0^T t\sin(m\omega t)dt$$

$$A_m = \frac{2a}{T}\left[-\frac{\cos m\omega t}{m\omega}\right]_0^T - \frac{2a}{T^2}\left[\left(-\frac{t\cos m\omega t}{m\omega}\right)_0^T + \int_0^T \frac{\cos m\omega t}{m\omega}dt\right]$$

$$A_m = \frac{2a}{Tm\omega}\left[-\cos m\omega T + \frac{\cos(m\omega\times 0)}{m\omega}\right]$$

$$+\frac{2a}{T^2}\left[\left(\frac{T\cos m\omega t}{m\omega}\right)\right] - \frac{2a}{T^2}\left[\frac{\sin m\omega t}{m^2\omega^2}\right]_0^T$$

But $-\cos(m\omega T) + \cos 0 = 0$

and $[\sin(\omega t)]_0^T = 0$ $\qquad \left[\because T = \frac{2\pi}{\omega}\right]$

$$A_m = \frac{2a}{T}\left[\frac{\cos 2\pi m}{m\omega}\right] = \frac{2a}{Tm\omega}$$

$\therefore$ $$A_m = \frac{a}{m\pi} \qquad ...(7)$$

(iii) $$B_m = \frac{2}{T}\int_0^T y\cos(m\omega t)dt$$

$$= \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\cos(m\omega t)dt$$

$\therefore$ $$B_m = 0 \qquad ...(8)$$

Hence all the cosine terms of the Fourier series are absent.

Substituting the values of B_0, A_m and B_m in Eq. (2), we have

$$y = f(\omega t) = \frac{a}{2} + \sum_{m=1}^{m=\infty} \frac{a}{m\pi} \sin m\omega t \qquad ...(9)$$

$$\therefore \qquad y = \frac{a}{2} + \frac{a}{\pi}\left(\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 2\omega t + ... + \frac{1}{m}\sin m\omega t ....\right) \qquad ...(10)$$

Fig. 11.7 shows how with the addition of successive terms, the resultant curve gradually approaches the shape of the saw-tooth wave.

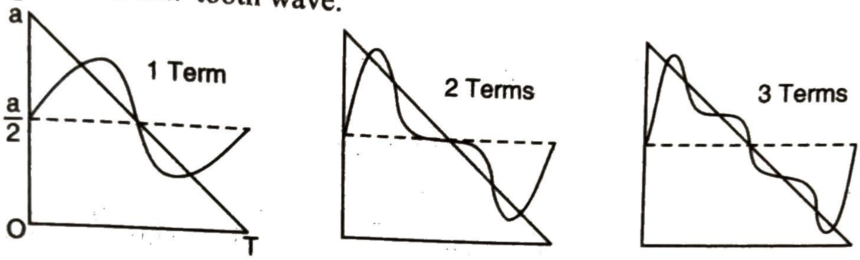

Fig. 11.7

11.8. SQUARE WAVE

Consider a square wave of period T shown in Fig. 11.8.

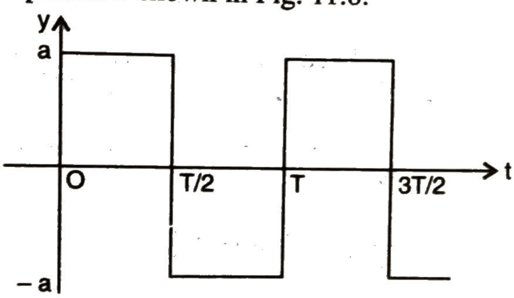

Fig. 11.8

Let $\qquad y = f(\omega t) = 0$ at $t = 0$ and $\frac{T}{2}$.

$$y = f(\omega t) = +a \text{ for } 0 < t < \frac{T}{2} \qquad ...(1)$$

$$y = f(\omega t) = -a \text{ for } \frac{T}{2} < t < T.$$

According to Fourier theorem,

$$y = f(\omega t) = B_0 + \sum_{m=1}^{m=\infty} A_m \sin m\omega t + \sum_{m=1}^{m=\infty} B_m \cos m\omega t \qquad ...(2)$$

Here, $$B_0 = \frac{1}{T}\int_0^T y\,dt \qquad ...(3)$$

$$A_m = \frac{2}{T}\int_0^T y \sin(m\omega t)dt \qquad ...(4)$$

$$B_m = \frac{2}{T}\int_0^T y \cos(m\omega t)dt \qquad ...(5)$$

Let us obtain the values of the coefficients B_0, A_m and B_m.

(i) $$B_0 = \frac{1}{T}\int_0^T y\,dt$$

Substituting the value of y from Eq. (1), we get

$$B_0 = \frac{1}{T}\int_0^{T/2} a\,dt - \frac{1}{T}\int_{T/2}^{T} a\,dt = 0$$

$\therefore$ $$B_0 = 0 \qquad ...(6)$$

(ii) $$A_m = \frac{2}{T}\int_0^T y\sin(m\omega t)dt$$

$$= \frac{2}{T}\int_0^T y\sin\left(\frac{2m\pi t}{T}\right)dt$$

$$= \frac{2}{T}\int_0^{T/2} a\sin\left(\frac{2m\pi t}{T}\right)dt - \frac{2}{T}\int_{T/2}^{T} a\sin\left(\frac{2m\pi t}{T}\right)dt$$

$$= \left(\frac{2a}{T}\right)\frac{T}{2m\pi}\left[-\cos\left(\frac{2m\pi t}{T}\right)\right]_0^{T/2} - \left(\frac{2a}{T}\right)\frac{T}{2m\pi}\left[-\cos\left(\frac{2m\pi t}{T}\right)\right]_{T/2}^{T}$$

$$= \frac{a}{m\pi}[-\cos(m\pi) + \cos 0]\left[-\frac{a}{m\pi}[-\cos(2m\pi) + \cos(m\pi)]\right]$$

$$= \frac{a}{m\pi}[-\cos(m\pi) + 1 + \cos(2m\pi) - \cos(m\pi)]$$

$$= \frac{a}{m\pi}[2 - 2\cos(m\pi)] \qquad [\because \cos(2m\pi) = 1]$$

$$= \frac{2a}{m\pi}[1 - \cos(m\pi)]$$

$\therefore$ $A_m = 0$ for $m = 2, 4, 6, \ldots$ (even integers)

and $A_m = \dfrac{4a}{m\pi}$ for $m = 1, 3, 5$ (odd integers) ...(7)

(iii) $$B_m = \frac{2}{T}\int_0^T y\cos(m\omega t)dt$$

$$= \frac{2}{T}\int_0^T y\cos\left(\frac{2m\pi t}{T}\right)dt \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

Substituting the value of y from Eq. (1), we get

$$B_m = \frac{2}{T}\int_0^{T/2} a\cos\left(\frac{2m\pi t}{T}\right)dt - \frac{2}{T}\int_{T/2}^{T} a\cos\left(\frac{2m\pi t}{T}\right)dt$$

$$= \left(\frac{2a}{T}\right)\left(\frac{T}{2m\pi}\right)\left[\sin\left(\frac{2m\pi t}{T}\right)\right]_0^{T/2} - \left(\frac{2a}{T}\right)\left(\frac{T}{2m\pi}\right)\left[\sin\left(\frac{2m\pi t}{T}\right)\right]_{T/2}^{T}$$

$$= \left(\frac{a}{m\pi}\right)[\sin(m\pi) - \sin 0] - \frac{a}{m\pi}[\sin(2m\pi) - \sin(m\pi)]$$

$$= \left(\frac{a}{m\pi}\right)[2\sin(m\pi) - \sin(2m\pi)] = 0$$

$\therefore$ $$B_m = 0$$

Hence all the cosine terms in Eq. (2) are zero.

Substituting the values of B_0, A_m and B_m in Eq.(2), we have

$$y = f(\omega t) = \frac{4a}{\pi}\left[\sin \omega t + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + ...\right] \quad ...(9)$$

We take the first three terms of the above series representing the square wave and add them together. The result is shown in Fig. 11.9.

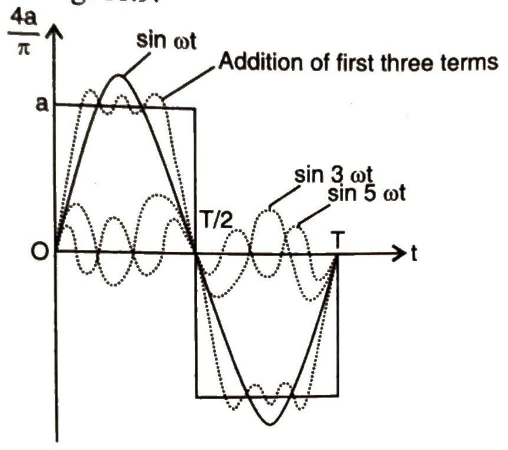

Fig. 11.9

The first harmonic has the frequency of the square wave. The higher frequencies build up the squareness of the wave.

The highest frequencies are responsible for the sharpness of the vertical sides of the wave.

The resultant of the first fifteen terms of Eq. (9) is shown in Fig. 11.10.

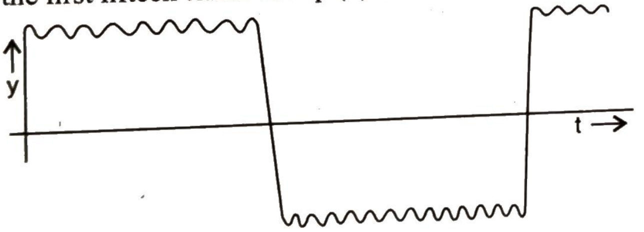

Fig. 11.10

This figure is a near approach of a square wave form but not an exact square form. However, addition of more and more terms will give a resultant curve nearly approaching a square wave form.

ULTRASONICS

11.9. INTRODUCTION

The human ear is sensitive to sound waves in the frequency range from 20 to 20,000 Hz. This range is called audible range. Sound waves of frequency more than 20,000 Hz are called *ultrasonics*. These frequencies are beyond the audible limit.

These waves also travel with the speed of sound (330 ms^{-1}).

These waves exhibit the properties of audible sound waves and also show some new phenomena.

Their wavelengths are small.

Example 1. *What is the wavelength of ultrasonic wave of frequency 330 kHz at 0°C ?* [Given : Velocity of sound at 0°C = 330 ms^{-1}.]

Solution. The relation between velocity (v), frequency (n) and wavelength (λ) is

$$v = n\lambda$$

$$\therefore \quad \lambda = \frac{v}{n} = \frac{330}{(330 \times 10^3)} = 10^{-3}\,\text{m} = 1\,\textbf{mm}$$

11.10. PIEZOELECTRIC EFFECT

If one pair of opposite faces of a quartz crystal is subjected to pressure, the other pair of opposite faces develops equal and opposite electric charges on them (Fig. 11.11). The sign of the charges is reversed when the faces are subjected to tension instead of pressure. The electric charge developed is proportional to the amount of pressure or tension. This phenomenon is called *Piezoelectric effect.*

The effect is *reversible, i.e.*, if an electric field is applied across one pair of faces of the crystal, contraction or expansion occurs across the other pair.

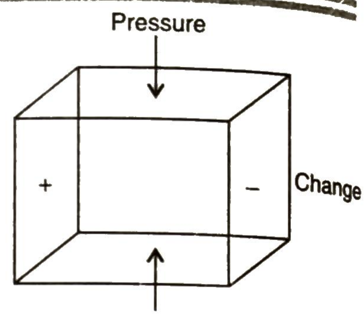

Fig. 11.11

When the two opposite faces of a quartz crystal, their faces being cut perpendicular to the optic axis, are subjected to alternating voltage, the other pair of opposite faces experiences stresses and strains. The quartz crystal will continuously contract and expand. Elastic vibrations are set up in the crystal.

When the frequency of the alternating voltage is equal to the natural frequency of vibration of the crystal or its simple higher multiples, the crystal is thrown into resonant vibrations. The amplitude is large. These vibrations are longitudinal in nature.

Consider a X-cut crystal plate of thickness t. The fundamental frequency of vibration is given by

$$n = \frac{1}{2t}\sqrt{\frac{E}{\rho}}.$$

E is the Young's modulus and ρ is the density of the material of the crystal plate.

Example 1. *A quartz crystal of thickness 0.001 m is vibrating at resonance. Calculate the fundamental frequency. Given E for quartz* = 7.9×10^{10} *Nm*$^{-2}$ *and* ρ *for quartz* = 2650 *kg m*$^{-3}$.

Solution.
$$n = \frac{1}{2t}\sqrt{\frac{E}{\rho}} = \frac{1}{2 \times 0.001}\sqrt{\frac{(7.9 \times 10^{10})}{2650}} = \mathbf{2.73 \times 10^6\ Hz.}$$

Example 2. *A piezoelectric X-cut quartz plate has a thickness of 1.5 mm. If the velocity of propagation of longitudinal sound waves along the X direction is 5760 m/s, calculate the fundamental frequency of the crystal.*

Solution. For the fundamental mode of vibration,

$$\text{thickness} = \frac{\lambda}{2}$$

$$\therefore \quad \lambda = 2 \times \text{thickness} = 2 \times (1.5 \times 10^{-3})\,\text{m} = 3 \times 10^{-3}\text{m}$$

Frequency,
$$n = \frac{v}{\lambda} = \frac{5760}{(3 \times 10^{-3})} = \mathbf{1.92 \times 10^6\ Hz}$$

11.11. PRODUCTION OF ULTRASONIC WAVES — PIEZOELECTRIC CRYSTAL METHOD

Principle. This is based on the inverse piezoelectric effect. When a quartz crystal is subjected to an alternating potential difference along the electric axis, the crystal is set into elastic vibrations along its mechanical axis. If the frequency of the electric oscillations coincides with the natural frequency of the crystal, the vibrations will be of large amplitude. If the frequency of the electric field is in the ultrasonic frequency range, the crystal produces ultrasonic waves.

Construction. The circuit diagram is shown in Fig. 11.12.

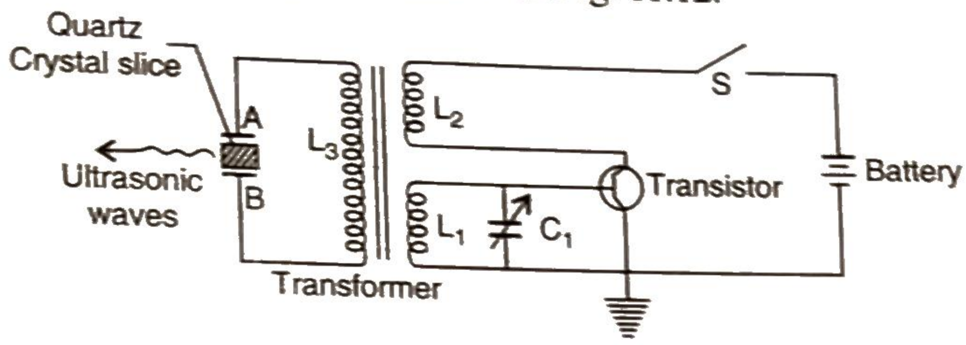

Fig. 11.12

It is a base tuned oscillator circuit. A slice of quartz crystal is placed between the metal plates A and B so as to form a parallel plate capacitor with the crystal as the dielectric. This is coupled to the electronic oscillator through primary coil L_3 of the transformer.

Coils L_2 and L_1 of oscillator circuit are taken from the secondary of the transformer. The collector coil L_2 is inductively coupled to base coil L_1. The coil L_1 and variable capacitor C_1 form the tank circuit of the oscillator.

Working. When the battery is switched on, the oscillator produces high frequency oscillations. An oscillatory e.m.f. is induced in the coil L_3 due to transformer action. So the crystal is now under high frequency alternating voltage.

The capacitance of C_1 is varied so that the frequency of oscillations produced is in resonance with the natural frequency of the crystal. Now the crystal vibrates with large amplitude due to resonance. Thus high power ultrasonic waves are produced.

Advantages

1. Ultrasonic frequencies as high as 500 MHz can be generated.
2. The output power is very high. It is not affected by temperature and humidity.
3. It is more efficient than magnetostriction oscillator.
4. The breadth of the resonance curve is very small. So we can get a stable and constant frequency of ultrasonic waves.

Disadvantages

1. The cost of the quartz crystal is very high.
2. Cutting and shaping the crystal is very complex.