Unit - V.

Continuous functions on Metric spaces.

5.1. Functions continuous at a point on the real line:-

Continuous in real line.

5.1.A ⇒ Defn: Suppose f is a real valued function whose domain contains all points of some open interval including a itself and let $a \in R^1$, we say that the function f is continuous at a if $\lim_{n \to a} f(x) = f(a)$.

Ex: If $f(x) = \frac{\sin x}{x}$. $(x \in R^1, x \neq 0)$ is not continuous at $x = 0)$.

soln:

$$f(x) = \frac{\sin x}{x} \quad (x \in R^1, x \neq 0).$$

$$\text{here } f(0) = \frac{\sin 0}{0} = \frac{0}{0}.$$

$\therefore f(0)$ is not exists.

$$\text{but } \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

$f(x) = \frac{\sin x}{x}$ is not continuous at $h = 0$.

2) $g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0. \\ 1, & \text{if } x = 0. \end{cases}$

is continuous at $x = 0$.

soln:

PAGE:
DATE:

$$g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

and $g(0) = 1$.

$$\lim_{x \to 0} \frac{\sin x}{x} = g(0) = 1.$$

$\therefore g(x)$ is continuous at $x = 0$.

3) $\lim_{x \to 3} (x^2 + 2x) = 15$ is continuous at $x = 3$.

Because $\lim_{x \to 3} x^2 + 2x = 15$.

and $f(3) = 3^2 + 2 \times 3 = 15$.

5.1.B Theorem:-

If the real valued functions f and g are continuous at $a \in \mathbb{R}^1$, then so are $f+g$, $f-g$ and fg. If $g(a) \neq 0$ then f/g is also continuous at a.

proof:

Given f and g are continuous at $a \in \mathbb{R}^1$

$$\lim_{x \to a} f(x) = f(a) \quad \text{—— (1)}$$

and $\lim_{x \to a} g(x) = g(a)$ —— (2).

prove that, $f+g$, $f-g$, fg are continuous at a.

and if $g(a) \neq 0$ then f/g are continuous at a

use the theorem "If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then,

$$\lim_{x \to a} [f(x) + g(x)] = L + M.$$

$$\lim_{x \to a} [f(x) - g(x)] = L - M.$$

$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M.$$

and if $M \neq 0$, $\lim_{x \to a} \frac{f(x)}{g(x)} = L/M$."

we get,

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} [f(x) + g(x)].$$

$= f(a) + g(a)$ from ① and ②.

$= (f+g)(a)$.

$\therefore \lim_{x \to a} (f+g)(x) = (f+g)(a)$

similarly we get,

$$\lim_{x \to a} (f-g)(x) = (f-g)(a).$$

$$\lim_{x \to a} (fg)(x) = (fg)(a).$$

and $\lim_{x \to a} (f/g)(x) = (f/g)(a)$.

Hence the theorem is proved.

5.1.c ⇒ Theorem: If f and g are real valued function If f is Continuous at a and If g is Continuous at $f(a)$, then $g \circ f$ is continuous at a.

proof.

PAGE:
DATE:

Gives f is continuous at a and g is continuous at $f(a)$.

Prove that $g \circ f$ is continuous at a

(i.e) $\lim_{x \to a} (g \circ f)(x) = (g \circ f)(a)$.

(i.e) $\lim_{x \to a} g[f(x)] = g[f(a)]$.

(i.e) given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$|g(f(x)) - g[f(a)]| < \varepsilon \quad (0 < |x-a| < \delta) \text{ — (1)}$$

Let $b = f(a)$. Now by hypothesis,

$$\lim_{y \to b} g(y) = g(b).$$

Hence there exists $\eta > 0$ such that,

$$|g(y) - g(b)| < \varepsilon \quad (0 < |y-b| < \eta) \text{ — (2)}.$$

But, also by hypothesis

$$\lim_{x \to a} f(x) = f(a).$$

there exists δ such that,

$$|f(x) - f(a)| < \eta \quad (|x-a| < \delta)$$

$$\Rightarrow |f(x) - b| < \eta \quad |x-a| < \delta \text{ — (3)}.$$

Thus if $|x-a| < \delta$ then sub $f(x)$ for y in (2),

we get $|g[f(x)] - g(b)| < \epsilon$ $|x-a| < \delta$

$\Rightarrow$ $|g(f(x)] - g[f(a)]| < \epsilon$ $(0 < |x-a| < \delta)$

Hence the theorem is proved.

Note: The above theorem is stated as " A continuous function of a continuous function is continuous".

5.2 Reformulation.

5.2.A. Theorem:-

The real valued function f is continuous at $a \in \mathbb{R}^1$. If and only if given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ $(|x-a| < \delta)$.

5.2.B $\Rightarrow$ Defn:- open Ball in $\mathbb{R}^1$.

if $a \in \mathbb{R}^1$ and $r > 0$. we define $B[a;r]$ to be the set of all $x \in \mathbb{R}^1$ whose distance to a is less than r.

(i.e) $B[a;r] = \{x \in \mathbb{R}^1 / |x-a| < r]$.

we call $B[a;r]$ the open Ball of radius r about a.

Note: Theorem 5.2.A reads " f is continuous at a if and only if given $\varepsilon>0$ there exists $\delta>0$ such that

$$[f(x) \in B[f(a); \varepsilon] \text{ if } x \in B[a; \delta]$$ "

(i.e.) "The entire open ball $B[a;\delta]$ is mapped by f into the open ball $B[f(a);\varepsilon]$.

5.2.C; Theorem:

The real valued function f is continuous at $a \in \mathbb{R}^1$. If and only if the inverse image under f of any open ball $B[f(a);\varepsilon]$ about $f(a)$ contains an open ball $B[a;\delta]$ about a.

(i.e) given $\varepsilon>0$ there exists $\delta>0$ such that $f^{-1}(B[f(a);\varepsilon]) \supset B[a;\delta]$.

Note:

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to a if and only if given $\varepsilon>0$ there exists $N \in I$ such that $x_n \in B[a;\varepsilon)$ $n \leq N$.

5.2.D. Theorem:-

The real valued function f is continuous at $a \in \mathbb{R}^1$ if and only if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

(i.e.) f is continuous at a if and only if $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f(x) = f(a)$ — (*).

Proof:

Let us first assume that f is continuous at a.

prove that (*) holds.

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of real numbers converging to a.

we must show that:

$$\lim_{n\to\infty} f(x_n) = f(a).$$

(i.e.) given $\varepsilon > 0$, we must find $N \in I$, such that:

$$f(x_n) \in B[f(a); \varepsilon] \; (n \geq N) \text{ — } ①.$$

But since f is continuous at a; there exists $\delta > 0$ such that

$$f(x) \in B[f(a); \varepsilon] \; x \in B[a; \delta] \text{ — } ②$$

and since $\lim_{n\to\infty} x_n = a$, there exists $N \in I$ such that,

$$x_n \in B[a; \delta] \; (n \geq N) \text{ — } ③.$$

for this N, from ② and ③ we get

$$f(x_n) \in B[f(a); \varepsilon] \; (n \leq N)$$

$$\therefore \lim_{n\to\infty} f(x_n) = f(a).$$

Conversely, Assume that suppose (*) holds prove that, f is continuous at a.

Assume the contrary, f is not continuous at a. Then By 5-8.C " The real valued function f is continuous at a if and only if, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f^{-1}(B[f(a); \varepsilon]) \supset B[a; \delta]\text{"}.$$

for some $\varepsilon > 0$, The inverse image under f of $B = E[f(a); \varepsilon]$ contains no open ball about a.

In particular, $f^{-1}(B)$ does not contain $B[a; 1/n]$ for any $n \in I$.

Thus for each $n \in I$, there is a point $x_n \in B[a; 1/n]$ such that $f(x_n) \notin B$.

(i,e) $|x_n - a| < 1/n$ but $|f(x_n) - f(a)| \geq \varepsilon$.

$\Rightarrow \lim_{n\to\infty} x_n = a$ but $\lim_{n\to\infty} f(x_n) \neq f(a)$.

which is a contradiction to (*).

$\therefore$ f must be continuous at a

Hence the theroem proved.

5.3. Functions continuous on a Metric Space.

5.3.A: Defn: Open Ball in Metric Space.

Let $\langle M, \rho \rangle$ be a metric space. If $a \in M$ and $r > 0$ then $B[a; r]$ is defined to be the set of all points in M whose distance to a is less than r.

(i.e) $B[a; r] = \{x \in M / \rho(x, a) < r\}$,

we call $B[a; r]$ the open ball of radius r about a.

Example:

(1) The open ball of radius 1 about the origin in Euclidean 3-space is the set of all points (x, y, z) such that $x^2 + y^2 + z^2 < 1$,

(2) If $M = [0, 1]$ with absolute value metric, then

$$B[1/4; 1/2] = \{x \in [0, 1] / |x - 1/4| < 1/2\}$$
$$= [0, 3/4).$$

(3) If $M = \mathbb{R}_d$ the real line with discrete metric, if a is any point in $\mathbb{R}_d$, then

$$B[a;1] = \{x \in \mathbb{R}_d \mid d(x,a) < 1\}.$$
$$= \{a\}.$$

and $$B[a;2] = \{x \in \mathbb{R}_d \mid d(x,a) < 2\}.$$
$$= \mathbb{R}_d.$$

5.3.B: Defn: Continuous on a metric space.

Let $\langle M_1, \rho_1\rangle$ and $\langle M_2, \rho_2\rangle$ be metric spaces, let $a \in M_1$ and let f be any function from M_1 to M_2, the function f is continuous at $a \in M$, if $\lim_{x\to a} f(x) = f(a)$.

5.3.C: The function f is continuous at $a \in M_1$ if and only if any one of the following conditions hold.

(a) Given $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_2\,[f(x), f(a)] < \epsilon$ $(\rho_1 < x, a < \delta)$

(b) The inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a.

(c) whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

Ruby PAGE: DATE:

Theorem : 5.3.D : Theorem:

Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$, $\langle M_3, \rho_3 \rangle$ be metric spaces and let $f: M_1 \to M_2$, $g: M_2 \to M_3$ if f is continuous at $a \in M_1$, and g is continuous at $f(a) \in M_2$ then $g \circ f$ is continuous at a.

proof:

Given, $f: \langle M_1, \rho_1 \rangle \to \langle M_2, \rho_2 \rangle$ is continous at a

and

$g: \langle M_2, \rho_2 \rangle \to \langle M_3, \rho_3 \rangle$ is continous at $g(a)$.

Prove that, $g \circ f$ is continous at a

using the theorem 5.3.C (c)"

we need to show that,

$$\lim_{n \to \infty} g[f(x_n)] = g[f(a)]$$

whenever $\{x_n\}_{n=1}^{\infty}$ is sequence in M_1

Such that $\lim_{n \to \infty} x_n = a$

since f is continous at a than

$$\lim_{n = \infty} f(x_n) = f(a)$$

and since g is continous at $f(a)$

than

$$\lim g[(f(x_n))] = g[f(a)]$$

$\therefore$ $g \circ f$ is continous at a.

theorem :- Let M be a metric space, and let f and g b real valued fuctions which are continous at $a \in M$

then $f+g$, $f-g$ and fg are continous at a furthermore if $g(a) \neq 0$. then f/g is continous at a.

53.F Definition:

Let M_1 and M_2 be metric Spaces and let $f: M_1 \to M_2$. we say that f is a continous furtions from M_1 into M_2 (or f is continuous on M_1) if f is continous at each point in M_1.

5.3 G. theorem: if f and g are continuous fuctions from a metric Space M_1 into a metric speace M_2 then so $f+g$, $f-g$ and fg Furthermore If, $g(x) \neq 0$ $(x \in M_1)$ then f/g is also continuous on M_1.

proof:- use 5.3 F and 5.3 E we prove the thorom.

5.4 open sets

5.4 A Definition open set.

Let M be a metric Space. we say that the subset G of M is an open subset of M (or G is open) if for every $x \in G$ there exists a number $r > 0$ such that the entre open ball $B[x; r]$ is contained in G.

Ruby PAGE: DATE:

Ex:

(1) The set A of all points in the plane $\mathbb{R}^2$ ~~in~~ inside an ellipse.

Take any point $P \in A$.

Draw a circle with center P which entirely lies in A.

$\therefore$ A is open in $\mathbb{R}^2$.

(2) $M = \mathbb{R}_d$ (discrete metric space)

if $a \in \mathbb{R}_d$ then $\{a\} = \{a; 1]$.

$\therefore \{a\}$ is open in $\mathbb{R}_d$.

(3) $M = \mathbb{R}^1$,

If $a \in \mathbb{R}^1$ then $\{a\}$ is not open in $\mathbb{R}^1$.

Remark: Let (M, ρ) be a Metric space. Every open ball in M is an open set.

Proof: Given $\langle M, \rho \rangle$ be a Metric space

Let $B = B[a; s]$ is an open ball in M.

Prove that B is an open set.

(i.e) to prove, If $x \in B$ we must find $r > 0$ such that $B[x; r] \in B$

(i.e) to prove If $y \in B[x; r] \Rightarrow y \in B_M$.

Let $t = \rho(x, a)$ and
let r be any positive number less that s.t.

If $y \in B[x; r]$

Then $\rho(a, y) \leq \rho(a, x) + \rho(x, y)$

But $\rho(a, x) = t$ and $\rho(x, y) < r$
since $y \in B[x; r]$.
Thus $\rho(a, y) \leq t + \rho(x, y)$.
$< t + r$.
$< t + s - t = s$.

(i.e) $\rho(a, y) < s$.
$\Rightarrow y \in B[a; s] = B$
$B[x; r] \subset B$.
$\therefore$ Every open ball is an open set.

Note: * The half-open interval $[0, 1/2)$.
is not an open subset of R^1.

But $[0, 1/2)$ an open subset of the metric space $[0, 1]$

5.4.B: Theorem :- In any metric space $\langle M, \rho \rangle$ both M and the empty set ϕ are open sets.

proof: If $x \in M$ they every open ball $B[x; r]$ is contained in M. Hence M is open.

The empty set ϕ is open because there are no x in ϕ and hence every $x \in \phi$ satisfies the condition of open set.

5.4.c Theorem:

Let $\mathcal{F}$ be any non-empty family of open subsets of a metric space M. Then $\bigcup_{G \in \mathcal{F}} G$ is also an open subset of M.

Proof:

Given, $\mathcal{F} = \{ G \subset M / G \text{ is open in } M \}$

Let $H = \bigcup_{G \in \mathcal{F}} G$, we may assume that

atleast one $G \in \mathcal{F}$ non-empty

Prove that H is open.

(i,e) to prove if $x \in H$ we must find $r > 0$ such that $B[x; r] \in H$.

But if $x \in H$ then $x \in G$ for some $G \in \mathcal{F}$.

since G is open, then there is some $B[x; r]$ with $B[x; r] \subset G$.

But $G \subset H$ and so $B[x; r] \subset H$.

$\therefore \bigcup_{G \in \mathcal{F}} G$ is open.

5.4.D: Theorem :- Every subset of $\mathbb{R}_d$ is open.

Proof:

Let G be any subset of $\mathbb{R}_d$.

Prove that G is open in $\mathbb{R}_d$.

If $a \in \mathbb{R}_d$ then $\{a\}$ is open in $\mathbb{R}_d$.
but $G = \bigcup_{a \in G} \{a\}$ and Use the theorem

5.4.C " we get G is open in $\mathbb{R}_d$.

5.4.F Theorem; If G_1 and G_2 are open subsets of the metric space M, then $G_1 \cap G_2$ is also open.

Proof: Given G_1 and G_2 are open in M.

Prove that $G_1 \cap G_2$ is open in M. assume that $G_1 \cap G_2 \neq \emptyset$.

If $x \in G_1 \cap G_2$, we must find an $r > 0$ such that $B[x; r] \subset G_1 \cap G_2$.

Since $x \in G_1$ and G_1 is open there exists $r_1 > 0$, such that $B[x_1, r_1] \subset G_1$.

and similarly $x \in G_2$, G_2 is open there is an open ball $B[x; r_2)$ such that $B[x; r_2] \subset G_2$.

Thus if $r = \min(r_1, r_2)$ then $B[x; r]$ is contained in G_1 and G_2.

Thus $B[x; r] \subset G_1 \cap G_2$.

$G_1 \cap G_2$ is open.

Note:- * The intersection of any finite number of open sets is open.

* Example:

In R^1, if I_n denotes the open interval $(-1/n, 1/n)$.

then $\bigcap_{n=1}^{\infty} I_n$ contains only 0.

(i.e) $\bigcap_{n=1}^{\infty} I_n = \{0\}$ is not open in $\mathbb{R}^1$.

The intersection of infinite number of open sets is not open.

5.4.F Theorem: Every open subset G of $\mathbb{R}^1$ can be written $G = \cup I_n$ where $I_1, I_2 \dots$ are a finite number or a countable number of open intervals which are mu disjoint (i.e) $I_m \cap I_n = \phi$ if $m \neq n$).

Proof:

If $x \in G$, Then there is an open interval (open ball) B containing x such that $B \subset G$.

Let I_x be the largest interval containing x such that $I_x \subset G$.

Then $G = \bigcup_{x \in G} I_x$.

Now, If $x \in G$, $y \in G$ then either $I_x = I_y$ or $I_x \cap I_y = \phi$.

For If $I_x \neq I_y$ and $I_x \cap I_y \neq \phi$ the

$I_x \cup I_y$ be the largest open interval contained in G which is larger then I_x.

This contradicts the defn of I_x. Finally, each I_x contains a rational number. since disjoint intervals cannot contain the same rational numbers. and since there are only countably many rationals, there cannot be uncountably ~~many~~ mutually disjoint intervals I_x

Every open subset G of $\mathbb{R}^1$ can be written $G = \cup I_n$ where $I_1, I_2, \dots$ are a finite or a countable number of open intervals which are mutually disjoint.

5.4.G Theorem: Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f: M_1 \to M_2$ Then f is continuous on M_1 if and only if $f^{-1}(G)$ is open in M_1 whenever G is open in M_1 wherever G is open in M_2. (f is continuous if and only if the inverse image of every open set is open).

proof:

Given, $f: M_1 \to M_2$

Assume that f is continuous on M_1

Prove that, if G is open in M_2 then $f^{-1}(G)$ is open in M_1, (i, e) to prove, if $x \in f^{-1}(G)$ we must find an open ball $B[x; r]$ such that, $B[x; r] \subset f^{-1}(G)$.

Now since $x \in f^{-1}(G)$ they $y = f(x) \in G$

Hence there is an open ball $B[y: \delta]$ Contained in G because G is open in M_2. By the theorem, 5.3. C, (b) The function f is continuous at $a \in M_1$ if and only if the inverse image under f, of any open ball $B[f(a) : \varepsilon]$ about $f(a)$ Contains an open ball $B[a; \delta]$ about "a".

L (1).

$f^{-1}(B[y; \delta])$ Contains some $B[x; r]$.

Hence

$f^{-1}(G) \supset f^{-1}(B[y; \delta)] \supset B[x; r]$.

(i.e) $f^{-1}(G) \supset B[x; r]$

$f^{-1}(G)$ is open in M_1.

Conversely, Assume that, $f^{-1}(G)$ is open in M_1, whenever G is open in M_2.

To show that f is Continuous on M_1.

(i.e) it is sufficient to show that f is continuous at an arbitary point $a \in M_1$.

Let $B = B[f(a); \varepsilon]$ be any open ball about $f(a)$.

Then B is open in M_2.

and so by assumption $f^{-1}(B)$ is open in M_1

Since $a \in f^{-1}(B)$ and $f^{-1}(B)$ is open, there is an open ball $B[a; \delta]$ Contained in $f^{-1}(B)$.

(i.e) $f^{-1}(B) \supset B[a; \delta]$.

(i.e) $f^{-1}(B[f(a); \varepsilon]) \supset B[a; \delta]$.

(i.e) inverse image of an open ball $B[f(a); \varepsilon]$ contains an open ball $B[a; \delta]$ $\therefore$ By (1) we get

f is continuous at an arbitary point $a \in M_1$.

$\therefore$ f is continuous on M_1.

Hence the theorem is proved.

5.5. Closed sets.

5.5.A. Defn:

Limit Point.

Let E be a subset of the metric space. M. A point $x \in M$ is called a limit point of E if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E which converges to x.

* The set $\bar{E}$ of all limit points of E and $\bar{E}$ is called the closure of E.

Corollary: If E is any subset of the Metric space M then $E \subset \bar{E}$ ~~Any Point~~ ~~$x \in E$~~ is a ~~limit point~~.

Proof: Any point $x \in E$ is a limit point of E, because the sequence $x_1, x, x \cdots$ of points of E which converges to x.

$\therefore$ If $x \in E$ then $x \in \bar{E} \Rightarrow E \subset \bar{E}$

Ex:

$E = (0,1)$.

$\{1/n\}_{n=1}^{\infty}$ is a sequence of points in $(0,1)$ and converges to 0.

$\therefore 0$ is a limit point of E.

and also 1 is a limit point of E.

But, the closed interval $[0,1]$ contains all, its limit points.

5.5.c Defn.

Closed set.

Let E be a subset of the metric space M. we say that E is closed subset of M if $E = \bar{E}$.

* $E \subset M$ is said to be closed if it contains all its limit points.

Ex: $[0,1]$ is a closed set.

5.1.D Theorem:

Let E be a subset of the metric space M. Then the point $x \in M$ is a limit point of E if and only if every open ball $B[x; r]$ about x contains atleast one point of E.

Proof:

Given, $E \subset M$ and M is metric space assume that $x \in M$, is a limit point of E. Prove that $B[x; r] \cap E \neq \phi$.

Suppose x is a limit point of E.

Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E that converges to x.

If $B[x; r]$ is any open ball about x, then $B[x; r]$ contains x_n for any n such that $\rho(x_n, x) < r$.

Hence $B[x; r] \cap E \neq \phi$.

$\therefore$ $B[x; r]$ contains a point of E.

Conversely, Assume that $B[x; r] \cap E \neq \phi$.

Prove that x is limit point of E.
let $x \in M$ and suppose every $B[x; r]$. contains a point of E.

Then for $n \in I$, the open ball $B[x; 1/n]$ contains a point $x_n \in E$.

The sequence $\{x_n\}_{n=1}^{\infty}$ obviously converges to x because $\rho(x, x_n) < 1/n$.

$\therefore$ x is limit point of E.
Hence the theorem is proved.

Note: * For any metric space M if $x \in M$ then $\{x\}$ is a closed subset of M.

* In particular $\mathbb{R}_d$ metric space if $a \in \mathbb{R}_d$ then $\{a\}$ is closed.

* The set $\{a\}$ is both open and closed in $\mathbb{R}_d$

5.5.E. Theorem :- If E is any subset of a metric space M, then $\bar{E}$ is closed. That is $\bar{E} = \bar{\bar{E}}$

Proof: Given, $E \subset M$, M is metric space.

prove that $\bar{E}$ is closed.

(i.e) to prove $\bar{E} = \bar{\bar{E}}$

$\therefore$ we have ~~it~~ to prove $\bar{E} \subset \bar{\bar{E}}$ and $\bar{\bar{E}} \subset \bar{E}$

since $\bar{E} \subset \bar{\bar{E}}$ we need to prove that $\bar{E} \supset \bar{\bar{E}}$

take any $x \in \bar{\bar{E}}$ to show that $x \in \bar{E}$
it is enough to prove by 5.5.2"
any open ball $B[x;r]$ contains a point of E

(i.e) $B[x;r] \cap E \neq \phi$.

Since $x \in \bar{\bar{E}}$ the ball $B[x;r] \cap \bar{E} \neq \phi$
by ① $\therefore y \in B[x;r]$ and $y \in \bar{E}$

Let $s = \rho(x,y)$ and choose any positive number t with $t < r-s$.

Since $y \in \bar{E}$ the ball $B[y;t]$ contains a point $z \in E$ by ①.
But $\rho(x,y) = s$.
$\rho(y,z) < t < r-s$ and so.

$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$
$< s + r - s = r$.
$\rho(x,z) < r$.
$\Rightarrow z \in B[x;r]$ and $z \in E$
$\therefore B[x;r] \cap E \neq \phi$.

Thus $B[x;r]$ contains a point of E

$\therefore \overline{\overline{E}} \supset \overline{E}$ and $\overline{E} \subset \overline{\overline{E}}$

$\therefore \overline{E} = \overline{\overline{E}}$

$\therefore \overline{E}$ is closed.

5.5.F Theorem :- In any metric space $\langle M, \rho \rangle$ the sets M and ϕ are both closed.

Proof:

M contains all its limit points and ϕ has no limit points.

Both M and ϕ sets are closed

5.5.G: Theorem: If F_1 and F_2 are closed subsets of the metric space M, then $F_1 \cup F_2$ is also closed.

Proof:

Given F_1 & F_2 are closed subsets of M.

Prove that $F_1 \cup F_2$ is closed.

(i.e) to Prove that $F_1 \cup F_2 = \overline{F_1 \cup F_2}$

But $F_1 \cup F_2 \subset \overline{F_1 \cup F_2}$ —— ①

so we have to prove $\overline{F_1 \cup F_2} \subset F_1 \cup F_2$.

It is enough to prove that,

let $x \in \overline{F_1 \cup F_2}$ then $x \in F_1 \cup F_2$

let $x \in \overline{F_1 \cup F_2}$, then there is a sequence $\{x_n\}_{n \geq 1}^{\infty}$ of points in $F_1 \cup F_2$ converging to x.

But $\{x_n\}_{n=1}^{\infty}$ must have a subsequence consisting wholly of points in F_2.

since any subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to x,

This shows that either $x \in \overline{F_1}$ or $x \in \overline{F_2}$

But by hypothesis F_1 & F_2 are closed.

$\Rightarrow$ either $x \in F_1$ or $x \in F_2$.

$\Rightarrow x \in F_1 \cup F_2$.

$\therefore \overline{F_1 \cup F_2} \subset F_1 \cup F_2$ —— (2).

from (1) and (2) we get

$\therefore F_1 \cup F_2$ is closed.

5.5.H : Theorem:

If $\mathcal{F}$ is any family of closed subsets of a metrix space M, then $\bigcap_{F \in \mathcal{F}} F$ is also closed.

Proof:

Given $\mathcal{F} = \{F \subset M \,/\, F$ is closed$\}$.

Prove that $\bigcap_{F \in \mathcal{F}} F = \overline{\bigcap_{F \in \mathcal{F}} F}$.

But, we know that, $\bigcap_{F \in \mathcal{F}} \subset \overline{\bigcap_{F \in \mathcal{F}} F}$ —— (1)

So, it is enough to prove, $\overline{\bigcap_{F \in \mathcal{F}} F} \subset \bigcap_{F \in \mathcal{F}} F$

Let $x \in \overline{\bigcap_{F \in \mathcal{F}} F}$. Then x is a limit point of $\bigcap_{F \in \mathcal{F}} F$

Then any ball $B[x;r]$ contains a point $y \in \bigcap_{F \in \mathcal{F}} F$

Thus for any $F \in \mathcal{F}$, the ball $B[x;r]$ Contains a point of F.

Hence $x \in \overline{F} = F$. Thus x lies in every $F \in \mathcal{F}$ and so $x \in \bigcap_{F \in \mathcal{F}} F$

$\Rightarrow \overline{\bigcap_{F \in \mathcal{F}} F} \subset \bigcap_{F \in \mathcal{F}} F \rightarrow (2) \Rightarrow$ from (1) and (2) we get

$\bigcap_{F \in \mathcal{F}} F$ is closed.

5.5.I: Theorem: Let G be an open subset of the metric space M. Then $G' = M - G$ is closed. Conversely if F is closed subset of M then $F' = M - F$ is open.

Proof: Assume that G is open.

prove that $G' = M - G$ is closed.

Since G is open, if $x \in G$, Then there is a ball $B = B[x;r]$ which lies entirely in G.

Hence B contains no points of G'.

By 5.5.D"

the point x cannot be a limit point of G' Thus no point in G is a limit point of G'. $\therefore G'$ contains all its limit points.

$\therefore G'$ is closed.

Conversely,

Assume that F is closed.

prove that $M - F$ is open

Now, suppose F is closed.

If $y \in F'$ there must be a ball $B[y;r]$. Which contains no points of F.

otherwise, $B[y;r] \cap F \neq \phi$ Then y is limit point of F, since F is closed we get $y \in F$.

which contradicts $y \in F'$.
Thus, for every $y \in F'$ there is a ball $B[y;r]$ lying entirely in F'.

$\therefore F'$ is open.
Hence the theorem is proved.

Note:
Suppose F_1 and F_2 are closed, then by 5.5.I, F_1' and F_2' are open.

By 5.4.E $F_1' \cap F_2'$ are open.
and $F_1' \cap F_2' = (F_1 \cup F_2)'$.
so that $(F_1 \cup F_2)'$ open.
$\therefore F_1 \cup F_2$ is closed by 5.5.I

5.5.I Theorem: Let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f: M_1 \to M_2$. Then f is continuous on M_1 if and only if $f^{-1}(F)$ is closed subset of M_1 whenever F is closed subset of M_2.

Soln: Assume that $f: M_1 \to M_2$ is continuous on M_1. To prove that $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Suppose f is continuous on M_1, $H \in M_2$ is a closed set.

By 5.5.I, F is open.
By 5.4.G $f^{-1}(F')$ is open in M_1
But $F \cup F' = M_2$.

By 1.3.E $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$ we have,
$f^{-1}(F \cup F') = f^{-1}(F) \cup f^{-1}(F') = f^{-1}(M_2)$
(i.e) $f^{-1}(F) \cup f^{-1}(F') = f^{-1}(M_2) = M_1$

Hence $f^{-1}(F)$ is the complement of $f^{-1}(F')$.
Since $f^{-1}(F')$ is open. Then $f^{-1}(F)$ is closed. Conversely

Assume that $f^{-1}(F)$ is closed in M_1, whenever F is closed in M_2.

Prove that f is continuous on M_1.

By 5.4.G " It is enough to prove that $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Let $G \subset M_2$ is open.

By 5.5.I, G' is closed by hypothesis

But $G \cup G' = M_2$.

$\Rightarrow f^{-1}(G \cup G') = f^{-1}(G) \cup f^{-1}(G') = f^{-1}(M_2)$

$\Rightarrow f^{-1}(G) \cup f^{-1}(G') = M_1$

Hence $f^{-1}(G)$ is the complement of $f^{-1}(G')$

since $f^{-1}(G')$ is closed.

$f^{-1}(G)$ is open.

$\therefore f$ is continuous on M_1.

Hence the theorem is proved

5.5.K Theorem: Let f be a 1-1 function from a metric M_1 onto a metric space M_2. Then if f has any one of the following properties it has them all,

(a) Both f and f^{-1} and continuous

(b) The set $G \subset M_1$ is open if and only if its image $f(G) \subset M_2$ is open

(c) The set $F \subset M_1$ is closed if and only if its image $f(F) \subset M_2$ is closed.

PAGE :
DATE :

5.5.2 Defn: Homomorphism.

Let f be a 1-1 function from a metric space M_1 onto a metric space M_2 is said to be homomorphism if f and f^{-1} are ~~conditions~~ continuous.

If a homomorphism, from M_1 onto M_2 exist, we say that M_1 and M_2 are homomorphic.

Ex: $M_1 = [0,1]$, $M_2 = [0,2]$ with absolute value metric.

If $f(x) = 2x$ then f is homomorphism of $[0,1]$ to $[0,2]$.

5.5.M. Defn: Dense.

Let M be a metric space. The subset A of M is said to be dense in M if $\bar{A} = M$

(i.e A is dense in M if every point in M is a limit point of A)

Ex: * The set A of of rationals is dense in R^1.

Ex: $A = (0,1)$ and $M = [0,1]$.

$\bar{A} = \overline{(0,1)} = [0,1] = M$.

$\therefore (0,1)$ is dense in M.

5.6 Discontinuous functions on $\mathbb{R}^1$

Defn: The subset D of $\mathbb{R}^1$ is said to be of type F_σ if $D = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^1$.

Defn: Let $f : \mathbb{R}^1 \to \mathbb{R}^1$. If J is any bounded open interval in $\mathbb{R}^1$. we define $\omega[f; J]$ called oscillation of f over J as,

$$\omega[f; J] = \underset{x \in J}{\text{lub}}\, f(x) - \underset{x \in J}{\text{glb}}\, f(x)$$

If $a \in \mathbb{R}^1$, we define $\omega[f; a]$ called the osillation of f at a to be

$$\omega[f; a] = \text{glb}\ \omega[f : J]$$ where the glb is taken over all bounded open intervals J containing a.

Note:

For any open interval J,
$\omega[f; J] \geq 0$
since $\underset{x \in J}{\text{lub}}\, f(x) \geq \underset{x \in J}{\text{glb}}\, f(x)$.

$\therefore \omega[f, J] \geq 0$ for any J.
and also we have $\omega[f; a] \geq 0$ for any point a.

5.6.c: Theorem:

If $f: R' \rightarrow R'$ and $a \in R'$. then the following statements hold.

(1) if f is Continuous at a then $\omega[f; a] = 0$

(2) if f is not Continuous at a then $\omega[f; a] > 0$

Proof: Since f is continuous at $a \in R'$, Then for given $\varepsilon > 0$, There exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon/2$ whenever $|x - a| < \delta$

(i.e) $|f(x) - f(a)| < \varepsilon/2$ whenever $x \in (a-\delta, a+\delta)$ $\rightarrow$ (1).

Let $I = [a - \delta/2, a + \delta/2]$ and let any $x_1, x_2 \in I$.

Consider,

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= |f(x_1) - f(a) + f(a) - f(x_2)| \\
&= |(f(x_1) - f(a)) - (f(x_2) - f(a))| \\
&\leq |f(x_1) - f(a)| + |f(x_2) - f(a)| \\
&< \varepsilon/2 + \varepsilon/2 \quad (\because \text{by } (1)). \\
&= \varepsilon.
\end{aligned}
$$

(i.e) $|f(x_1) - f(x_2)| < \varepsilon$.

(i.e) $\forall\, x_1, x_2 \in I$ we have $|f(x_1) - f(x_2)| < \varepsilon$

$\Rightarrow \omega[f; I] < \varepsilon$

For every $\varepsilon > 0$ there exists an open interval I such that $\omega[f; I] < \varepsilon$

$\omega[f; a] = \text{g.l.b } \omega[f; I] < \varepsilon$.

$\Rightarrow \omega[f; a] < \varepsilon$

ε is arbitary. $\therefore w[f; a] = 0$.
Hence the theorem is proved.

(2) Assume that f is not continuous $a \in \mathbb{R}'$.
Prove that $w[f; a] > 0$.

Assume the contray, if $w[f; a] = 0$.
then $glb\ w[f; J] = 0$.

$\therefore$ for given $\varepsilon > 0$ there exists a bounded open interval J containing a such that
$w[f; J] < \varepsilon$.
$\therefore$ ~~|f(x)-f(a)|~~
$\therefore |f(x_1) - f(x_2)| < \varepsilon \ \forall\ x_1, x_2 \in J$.
$\Rightarrow |f(x) - f(a)| < \varepsilon \ \forall\ x \in J, a \in J$.
$\Rightarrow f$ is continuous at a.
which is a contradiction to our assumption
$\therefore w[f; a] > 0$, if f is not continuous at $a \in \mathbb{R}'$.

5.6.D Theorem:
Let $f: \mathbb{R}' \to \mathbb{R}'$. For any $r > 0$, let E_r be the set of all $a \in \mathbb{R}'$ such that $w[f; a] \geq 1/r$ then E_r is closed.

Proof:
Given
$r > 0$, $E_r = \{a \in \mathbb{R}' / w[f; a] \geq 1/r\}$

Prove that E_r is closed.
(i.e) E_r contains all its limit points.
Let x be a limit point of E_r.
we must show that $x \in E_r$.
(i.e) $w[f; x]\ 1/r$.

It is sufficient to show that if J is a bounded open interval containing x then

$\omega[f; J] = 1/r$.

Since x is a limit point of E_r, then by theorem 5.5.D.

Then open interval J must contain a point y of E_r.

But $\omega[f; J] \geq \omega[f; y]$ $y \in E_r$

$\geq 1/r$.

$\Rightarrow \omega[f; J] \geq 1/r$.

$\therefore x \in E_r \Rightarrow E_r$ is closed.

5.6.E Theorem:

Let $f: \mathbb{R}^1 \to \mathbb{R}^1$ and D be the set of points in $\mathbb{R}^1$ at which f is not continuous. Then D is of type F_σ.

proof:

Given, D is the set of points in $\mathbb{R}^1$. at which f is not continuous.

To prove that D is of type F_σ.

[(i.e) D is equal to the countable union of closed subsets of $\mathbb{R}^1$].

If $x \in D$, then $\omega[f; x] > 0$, [by theorem 5.6.C(a)]

For some $n \in I$, we have

$\omega[f; x] \geq 1/n$.

$\Rightarrow x \in E_{1/n}$ [where $E_{1/n} = \{x / \omega[f; x] \geq 1/n\}$].

$\Rightarrow x \in \bigcup_{n=1}^{\infty} E_{1/n}$.

$\Rightarrow D \subset \bigcup_{n=1}^{\infty} E_{1/n}$ —— ①

Conversely,

if $x \in \bigcup_{n=1}^{\infty} E_{1/n}$.

Then $x \in E_{1/n}$ for some n.

$\Rightarrow \omega[f; x] \geq 1/n$ for some $n \in I$.

$\Rightarrow \omega[f; x] > 0$

$\Rightarrow x \in D$

$\therefore \bigcup_{n=1}^{\infty} E_{1/n} \subset D$ —— ②

from ① and ② we have.

$$D = \bigcup_{n=1}^{\infty} E_{1/n}.$$

From Theorem 5.6.D.

For any $n \in I$, each $E_{1/n}$ is called.

$\therefore D$ is a countable union of closed sets.

$\therefore D$ is of type F_σ.

Hence the theorem is proved.

5.6.F Defn:

The subset A of $\mathbb{R}^1$ is said to be nowhere dense in $\mathbb{R}^1$ if $\bar{A}$ contains no empty open interval.

Ex:

1) A closed set F in $\mathbb{R}^1$ is nowhere dense, if F itself contains no open interval.

2) The set of positive integers is nowhere dence.

3) Anyone point set in $\mathbb{R}^1$ in the usual metric is nowhere dense.

4) In $\mathbb{R}^1$ with usual metric,

$A = \{1, 1/2, 1/3, 1/4, \ldots\}$ is nowhere dense in $\mathbb{R}^1$. Because $\bar{A} = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$ contains no non-empty open interval. Therefore A is nowhere dense.

5.6.G Defn:

The subset D of R^1 is said to be of the first category if $D = \bigcup_{n\geq 1}^{\infty} E_n$ where each E_n is nowhere dense in $\mathbb{R}^1$.

If D is not of the first category then we say that D is of the second category.

Example:

1) Any countable set D in $\mathbb{R}^1$ is of first category.

Since D is the countable union of one-points and any one-point set is closed and nowhere dense in $\mathbb{R}^1$.

2) The set of rationals is of the first category. Because, the set of rationals are countable

5.6.H. Theorem:

If A and B are sets of the first category, then $A \cup B$ is also of the first category.

Proof: A and B are of the first category.

To prove that $A \cup B$ is of the first category. A & B are of the first category

$A = \bigcup_{n=1}^{\infty} H_n$ and $B = \bigcup_{n=1}^{\infty} E_n$ where

each E_n and each H_n are nowhere dense.

$$A \cup B = \left(\bigcup_{n=1}^{\infty} H_n\right) \cup \left(\bigcup_{n=1}^{\infty} E_n\right).$$

$$\therefore A \cup B \left(\bigcup_{n=1}^{\infty} H_n\right) \cup \left(\bigcup_{n=1}^{\infty} E_n\right)$$

$\therefore$ $A \cup B$ is the union of all the E_n's and H_n's.

Hence $A \cup B$ is of the first category.

Baire category theorem:

Theorem: The set $\mathbb{R}^1$ is of the second category.

proof: prove that, $\mathbb{R}^1$ is of the second category. Assume the contray, the set $\mathbb{R}^1$ is of the first category.

(i.e) $\mathbb{R}^1 = \bigcup_{n=1}^{\infty} F_n$ where each F_n is nowhere dense.

we may assume that, the F_n are closed otherwise we would consider the set $\overline{F_n}$

since $\mathbb{R}^1 = \bigcup_{n=1}^{\infty} \overline{F_n}$ and the $\overline{F_n}$ are closed and nowhere dense.

Hence assume that,

$\mathbb{R}^1 = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed & nowhere close ϵ in $\mathbb{R}^1$

Take any $x_1 \notin F_1$.

Since F_1 is closed, x_1 is not a limit point of F_1. Then there is an open interval I_1 about x_1 which does not intersect F_1.

Let J_1 be a closed interval with $0 < \text{length } J_1 < 1$ such that $J_1 \subset I$.

Then $J_1 \cap F_1 = \phi$.

Now F_2 is nowhere dense and thus does not contain all of the interior of J_1.

Take any x_2 in the interior of J_1 such that $x_2 \notin F_2$. This implies x_2 is not a limit point of F_2.

Then there is an open interval I_2 about x_2 which does not intersect F_2 such that $I_2 \subset J_1$.

Let J_2 be a closed interval with $0 < \text{length } J_2 < 1/2$ such that $J_2 \subset I_2$.
Then $J_2 \cap F_2 = \phi$.

Continuing in this fashion, we may construct a sequence of non-empty closed intervals $J_1 \supset J_2 \supset J_3 \supset \dots$ such that $0 < \text{length } J_n < 1/n$ and $J_n \cap F_n = \phi$.
" Hence by the Nested interval theorem,

For each $n \in I$, let $I_n = \{a_n, b_n\}$ be a non-empty closed bounded interval of real numbers such that

$$I_1 \supset I_2 \supset I_3 \supset \dots \supset I_n \supset I_{n+1} \supset \dots$$ and

$$\lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} (\text{length of } I_n) = 0.$$ Then

$\bigcap_{n=1}^{\infty} I_n$ contains Preciously one point.

There is a point $y \in \mathbb{R}'$ contained in $\bigcap_{n=1}^{\infty} J_n$.

(i.e) $y \in \bigcap_{n=1}^{\infty} J_n$.

$\Rightarrow y \in J_n$ for each $n \in I$.

$\Rightarrow y \notin F_n$ [$\because J_n \cap F_n = \phi$ for each $n \in I$]

$\Rightarrow y \notin \bigcup_{n=1}^{\infty} F_n$.

$\Rightarrow y \notin \mathbb{R}'$

which is a contradiction.

Hence our assumption is wrong

Therefore, the set $\mathbb{R}'$ is of the second category.

5.6.J - Corollary.

The set of all irrationals is of the second category.

proof: Suppose assume that, the set of all irrationals is of the first category.

The set of rationals is of the first category.

" use the theorem, If A and B are sets of the first category then $A \cup B$ is of the first category —— ①.

Since,
$\mathbb{R}^1$ is the union of rationals and irrationals and by the theorem ①, $\mathbb{R}^1$ is of first category.

Which is a contradiction to $\mathbb{R}^1$ is of the second category.
∴ The set of all irrationals is of the second category.

5.6.K- Corollary.
The set of all irrationals is not of type F_σ.
Proof: Let A be the set of all irrationals
Suppose, If A is of type F_σ.

then $A = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed. But each F_n contains only irrationals. Hence F_n contains no non-empty open interval Thus each F_n is closed and nowhere dense.

∴ $A = \bigcup_{n=1}^{\infty} F_n$ is of first category. which is a contradiction - The set of all irrationals is of the 2nd category.

Hence A is not of type F_σ.
5.6.C Theorem:
There is no real valued function f on $\mathbb{R}^1$ which is continuous at each rational but discontinuous at each irrational.
Proof:

Suppose there is a real valued function f on $\mathbb{R}^1$ which is continuous at each rational but discontinuous at each irrational.

Let $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ and D be the set of points of discontinuous of f in $\mathbb{R}$. Then D is thye F_σ"

The set of irrational points is of type G_δ. But it condradicts to, the set of irrational points is not of type F_σ.

Hence, there is no real valued function ~~for~~ f on f" which is continuous at each rational but discontincous at each irrational.