Limits and Metric Spaces

4.1. Limit of a function on the Real line.

4.1 A Definition:

Let $a \in R$ and let f be a real valued function whose domain includes all points in some open interval $(a-h, a+h)$ except possibly the point "a" itself.

$f(x)$ is said to approach L, where $L \in R$ as x approaches a. If given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ ; whenever } 0 < |x-a| < \delta.$$

Theorem 4.1.c:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then $f(x) + g(x)$ has a limit as $x \to a$ and

$$\lim_{x \to a} [f(x) + g(x)] = L + M.$$

Proof:

Given $\lim_{x \to a} f(x) = L$

To Prove: $\lim_{x \to a} [f(x) + g(x)] = L + M$

For given $\epsilon > 0$, we have to find $\delta > 0$ such that

$$|[f(x) + g(x)] - [L + M]| < \epsilon$$

whenever $0 < |x-a| < \delta$ —— ①

since $\lim_{x \to a} f(x) = L$

for given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$|f(x) - L| < \varepsilon/2$ whenever $0 < |x-a| < \delta_1$

since $\lim_{x \to a} g(x) = M$

for the same $\varepsilon > 0$ ~~there~~ there exists $\delta_2 > 0$ s.t.

$|g(x) - M| < \varepsilon/2$, whenever $0 < |x-a| < \delta_2$.

Let $\delta = \min(\delta_1, \delta_2)$

And if $0 < |x-a| < \delta$, then

$|f(x) - L| < \varepsilon/2$, $|g(x) - M| < \varepsilon/2$.

$$\therefore |(f(x)+g(x)) - (L+M)| = |(f(x)-L) + (g(x)-M)|$$
$$\leq |f(x)-L| + |g(x)-M|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$< \varepsilon, \quad 0 < |x-a| < \delta.$$

Hence eq ① holds,

(ii) $\lim_{x \to a} [f(x) + g(x)] = L + M$.

Theorem : 4.1 D:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

a) $\lim_{x \to a} [f(x) - g(x)] = L - M$

b) $\lim_{x \to a} f(x) \cdot g(x) = LM$. and if $M \neq 0$

c) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Definition: 4.1 E.:

$f(x)$ is said to approach L as x approaches ∞. If given $\varepsilon > 0$ there exists $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon \quad (x > M)$$

It is written as $\lim_{x \to \infty} f(x) = L$.

Definition: 4.1 F:

$f(x)$ is said to approach L as x approaches 'a' from the right, if given $\varepsilon > 0$, $\exists$ $\delta > 0$ $\ni$,

$$|f(x) - L| < \varepsilon \text{ whenever } a < x < a + \delta.$$

It is written as $\lim_{x \to a^+} f(x) = L$.

L is called the right hand limit of f at 'a'.

Left hand limit:

$f(x)$ is said to approach M as x approaches 'a' from the left, if given $\varepsilon > 0$, $\exists$ $\delta > 0$ $\ni$

$$|f(x) - M| < \varepsilon, \text{ whenever } a - \delta < x < a.$$

The number M is called the left hand limit of f at 'a'.

It is written as ~~lim~~ $\lim_{x \to a} f(x) = M$.

Definition 4.1 G: Non decreasing & Non-increasing:

If f is a real valued function on an interval I contained in R, then f is said to be non-decreasing on J if

$$f(x) \le f(y), \quad (x<y) \ \forall \ x, y \in J.$$

f is said to be non increasing on J,

$$\text{if } f(x) \ge f(y), \quad (x<y), \ \forall \ x, y \in J.$$

f is said to be monotonic, if f is either nondecreasing (or) non-increasing.

Theorem 4.1 H:

Let f be a nondecreasing function on the bounded open interval (a,b). If f is bounded above on (a,b), then $\lim\limits_{x\to b^-} f(x)$ exists. Also if f is bounded below on (a,b) then $\lim\limits_{x\to a^+} f(x)$ exists.

Proof:

If f is bounded above and non decreasing on (a,b).

Hence by least upper bound axiom,

let $M = \underset{x\in(a,b)}{\text{lub}} f(x)$ — (1)

Then for given $\epsilon>0$, $M-\epsilon$ is not an upper bound for the range of f.

Hence there exists $y \in (a,b)$ such that,

$f(y) > M-\epsilon$

Let $\delta = b - y > 0$, then, $b - \delta = y$.

$\Rightarrow f(b-\delta) = f(y) > M - \epsilon$

$\Rightarrow f(x) > f(b-\delta) > M - \epsilon$ —② $(\because b-\delta < x < b)$.

$f(x) > M - \epsilon$.

$\therefore f(x)$ is nondecreasing.

from eq ①

$f(x) \leq M \ \forall \ x \in (a, b)$.

$M - \epsilon < f(x) \leq M < M + \epsilon$, whenever $b-\delta < x < b$.

$\Rightarrow |f(x) - M| < \epsilon$, whenever $b - \delta < x < b$.

$\Rightarrow \lim_{x \to b^-} f(x) = M$.

Hence $f(x)$ has a left hand limit b.

Similarly

If f is bounded below on (a, b) then the range of f is bounded below

By the theorem,

" If A is any nonempty subset of $\mathbb{R}$ that is bounded below, then A has a greatest lower bound in $\mathbb{R}$.

Let $M = \text{glb } f(x)$, $x \in (a, b)$ —③

Then for given $\epsilon > 0$, $M + \epsilon$ is not an lower bound for the range of f.

Hence their exists $y \in (a,b)$ such that

$f(y) < m+\varepsilon$

let $\delta = y-a > 0$

$\Rightarrow y = a+\delta$.

Since f is nondecreasing,

$a < x < a+\delta$, we have

$f(x) < f(a+\delta) < m+\varepsilon$

$\therefore f(x) < m+\varepsilon \quad (a < x < a+\delta)$

from eq (3)

$f(x) \geq m \quad \forall \; x \in (a,b)$.

$\Rightarrow m-\varepsilon < m \leq f(x)$

Hence $m-\varepsilon < f(x) < m+\varepsilon$, whenever $a < x < a+\delta$

$\Rightarrow |f(x)-m| < \varepsilon$ whenever $a < x < a+\delta$.

$\Rightarrow \lim\limits_{x \to a^+} f(x) = m$

Hence $\lim\limits_{x \to a^+} f(x)$ exists.

Hence the proof.

Theorem: 4.12

Let f be a nonincreasing function on the bounded interval (a,b). If f is bounded below on (a,b) then $\lim\limits_{x \to b^-} f(x)$ exists; if f is bounded above on (a,b) then $\lim_{x \to a^+} f(x)$ exists.

Corollary : 4.1 J

'If f is a monotone fuction on the open interval $c \in (a,b)$. Then $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ ~~exi~~. both exists.

Proof :

suppose f is nondecreasing.

choose $\delta > 0$ such that $(c-\delta, c+\delta)$ is obtained in (a,b).

Then the values of f on the $(c-\delta, c)$ are bounded above by $f(c)$.

And hence by 4.1.H

$\lim_{x \to c^+} f(x)$ exists.

By the theorem,

" If f is nondecreasing function on the bounded open interval (a,b). If f is bounded above on (a,b) then $\lim_{x \to b^-} f(x)$ exists. Also if f is bounded below on (a,b) then $\lim_{x \to a^+} f(x)$ exists." — ①

Similarly, if the values of f on the $(c, c+\delta)$ are bounded below by $f(c)$

Hence $\lim_{x \to c} f(x)$ exists by theorem 4.1 I.

Suppose f is non increasing,

By the theorem,

" Let f be a nonincreasing function on the bounded open interval (a,b). If f is bounded below on (a,b). then $\lim_{x \to b^-} f(x)$ exists, while if f is bounded above on (a,b), then $\lim_{x \to a} f(x)$ exists "

$\Rightarrow \lim_{x \to c^-} f(x)$ and $\lim_{x \to c} f(x)$ exists.

Hence the proof.

Definition: 4.1k

Strictly increasing:

The real valued f on the interval J Contain in $\mathbb{R}$ is said to be strictly increasing. If $f(x) < f(y)$, $(x < y, \forall x, y \in J)$.

Strictly decreasing:

If f is said to be strictly decreasing. If

$f(x) > f(y)$, $(x > y;\ x, y \in J)$

Thus if f is nonincreasing on J. then f is strictly increasing on J if and only if J is 1-1 on J.

4.2 Metric Spaces

Definition: 4.2 B: Metric Space:

Let M be any set. A metric for M is a function ρ with domain $M \times M$ and range contained in $[0, \infty)$, such that

(i) $\rho(x, x) = 0 \quad \forall x \in M$

(ii) $\rho(x, y) > 0 \quad (x, y \in M, x \neq y)$

(iii) $\rho(x, y) = \rho(y, x) \quad (x, y \in M)$

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in M$

(triangle inequality)

If ρ is a metric for M then the ordered pair $\langle M, \rho \rangle$ is called a metric space.

Examples of metric space:

1. Define $\rho : R \times R \to [0, \infty)$ as

$$\rho(x, y) = |x - y|, \quad \forall x, y \in R \quad \text{— (1)}$$

(i) $\rho(x, x) = |x - x|$
$= 0 \quad \forall (x \in R)$

(ii) $\rho(x, y) = |x - y| > 0 \quad (x, y \in R)(\because x \neq y)$

(iii) $\rho(x, y) = |x - y|$
$= |y - x|$

$\rho(x, y) = \rho(y, x) \quad \forall (x, y \in M)$

(iv) $\rho(x,y) = |x-y|$.

$$|x-y| = |x-y+z-z|$$
$$\leq |(x-z| + (z-y))|$$

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y)$$

Hence ρ satisfies all the required condition for a metric.

Example 2:

Define $d : R \times R \to [0,\infty)$ as

$d(x,x) = 0 \quad \forall\, x \in R$

$d(x,y) = 1 \quad \forall (x,y \in R) \; (x \neq y)$.

(i) $d(x,x) = 0 \quad \forall\, x \in R$.

(ii) $d(x,y) = 1 > 0 \quad (x,y \in R,\ x \neq y)$

(iii) $d(x,y) = 1$

$= d(y,x) \qquad (x \in R)$

(iv) $d(x,y) = 1 < 1+1$

$\leq d(x,z) + d(z,y)$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$. $(x,y,z \in R)$

Hence d is the metric for the set of all real numbers.

The metric is called the discrete metric.

The resulting metric space $\langle R, d \rangle$ is denoted by R_d.

Example 3:

Fix $n \in I$, then $R^n = \{ x = \langle x_1, x_2, \dots x_n \rangle / x_1, x_2 \dots x_n \in R \}$

Define ρ from $R^n \times R^n \longrightarrow [0, \infty)$ as

$$\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2 \right]^{1/2} \quad \text{for } x = \langle x_1, x_2, \dots x_n \rangle \in R^n, \; y = \langle y_1, y_2, \dots y_n \rangle \in R^n$$

i) $\rho(x, x) = \left[\sum_{k=1}^{n} (x_k - x_k)^2 \right]^{1/2}$

$= 0 \quad (\because x \in R^n)$

(ii) $\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2 \right]^{1/2} \geq 0$.

(iii) $\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2 \right]^{1/2}$

$= \left[\sum_{k=1}^{n} (y_k - x^2)^2 \right]^{1/2}$

$\rho(x, y) = \rho(y, x) \quad \forall x, y \in R^n$.

(iv) To prove triangle inequality

(ie) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in R^n$.

Let $x = \langle x_1, x_2, \dots x_n \rangle$

$y = \langle y_1, y_2, \dots y_n \rangle$

$z = \langle z_1, z_2, \dots z_n \rangle$

Then $\rho(x,y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2}$

$$= \left(\sum_{k=1}^{n} \left[(x_k - z_k) + (z_k - y_k)\right]^2\right)^{1/2}$$

$$= \left[\sum_{k=1}^{n} (a_k + b_k)^2\right]^{1/2}$$ where

$a_k = x_k - z_k$

$b_k = z_k - y_k$

$$\rho(x,y) \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}$$

By Minkowski's inequality

$$\left[\sum_{n=1}^{\infty} (s_n + t_n)^2\right]^{1/2} \leq \left[\sum_{n=1}^{\infty} s_n^2\right]^{1/2} + \left[\sum_{n=1}^{\infty} t_n^2\right]^{1/2}$$

$$\therefore \rho(x,y) \leq \left[\sum_{k=1}^{n} (x_k - z_k)^2\right]^{1/2} + \left[\sum_{n=1}^{n} (z_k - y_k)^2\right]^{1/2}$$

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad (x, y, z \in R^n)$$

Hence ρ satisfies all the required condition for a metric.

The metric space R^n is called Euclidean n-spaces.

Example 4:

Define $\rho : l^\infty \times l^\infty \longrightarrow [0,\infty)$ as

$$\rho(x,y) = \underset{1 \leq n < \infty}{\text{lub}} |x_n - y_n| \quad (x, y \in l^\infty)$$

(i) $\rho(x, x) = \underset{1 \leq n < \infty}{\text{lub}} |x_n - x_n|$ $(x \in l^\infty)$

$= 0$

(ii) $\rho(x, y) = \underset{1 \leq n < \infty}{\text{lub}} |x_n - y_n|$

≥ 0

$\therefore \rho(x, y) \geq 0, \forall (x, y) \in l^\infty$.

(iii) $\rho(x, y) = \underset{1 \leq n < \infty}{\text{lub}} |x_n - y_n|$

$= \underset{1 \leq n < \infty}{\text{lub}} |y_n - x_n|$

$= \rho(y, x)$ $\forall ((x, y) \in l^\infty)$.

(iv) for $k \in I$

$|x_k - y_k| = |x_k - z_k + z_k - y_k|$

$\leq |x_k - z_k| + |z_k - y_k|$

$\underset{1 \leq n < \infty}{\text{lub}} |x_n - y_n| \leq \underset{1 \leq n < \infty}{\text{lub}} |x_n - z_n| + \underset{1 \leq n < \infty}{\text{lub}} |z_n - y_n|$

$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. $(x, y, z \in l^\infty)$.

Hence ρ satisfies all the conditions.

$\therefore$ The metric space $\langle l^\infty, \rho \rangle$ is simply denoted by l^∞.

Hence l^∞ is a metric space.

Example : 5

Define $\rho : \ell^2 \times \ell^2 \longrightarrow [0, \infty)$ as

$\rho(x, y) = \|x - y\|_2 \quad \forall\, x, y \in \ell^2$

Let $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty}$, $z = \{z_n\}_{n=1}^{\infty}$

(i) ~~$\rho(x, y)$~~

(i) $\rho(x, x) = \|x - x\|_2$

$= 0$

(ii) If $x \neq y$, then $x_n \neq y_n$ for atleast one n, such that

$x_n - y_n \neq 0$

$(x_n - y_n)^2 \neq 0$

$\left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{1/2} > 0$

(ii) $\|x - y\|_2 > 0$

$\rho(x, y) > 0$, $\forall$

iii) $\rho(x, y) = \|x - y\|_2$

$= \left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{1/2}$

$= \left[\sum_{n=1}^{\infty} (y_n - x_n)^2\right]^{1/2}$

$= \|y - x\|_2$

$= \rho(y, x)$

(iv) $\rho(x,y) = \|x-y\|_2$

$$= \left[\sum_{n=1}^{\infty}(x_n-y_n)^2\right]^{1/2}$$

$$= \sum_{n=1}^{\infty}\left[(x_n-z_n)^2+(z_n-y_n)^2\right]^{1/2}$$

$$= \sum_{n=1}^{\infty}(a_n+b_n)^2\Big]^{1/2}$$ where $a_n = x_n - z_n$

$b_n = z_n - y_n$

By Minkowski's inequality,

$$\left[\sum_{n=1}^{\infty}(a_n+b_n)^2\right]^{1/2} \leq \left[\sum_{n=1}^{\infty}a_n^2\right]^{1/2}+\left[\sum_{n=1}^{\infty}b_n^2\right]^{1/2}$$

$\therefore$ we have $\rho(x,y) \leq \left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}+\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2}$

$$\leq \left[\sum_{n=1}^{\infty}(x_n-z_n)^2\right]^{1/2}+\left[\sum_{n=1}^{\infty}(z_n-y_n)^2\right]^{1/2}$$

$$\leq \|x-z\|_2 + \|z-y\|_2$$

$$\rho(x,y) \leq \rho(x,z)+\rho(z,y), \quad x,y,z \in l^2.$$

Hence l^2 satisfies all the required condition for a metric. The metric space $\langle l^2, \rho\rangle$ is simply denoted by l^2.

4.3. Limits in Metric space

Introduction:

Let $\langle M, \rho_1\rangle$ and $\langle M_2, \rho_2\rangle$ are two metric spaces, let $a \in M$, f is a function whose range contained in M_2 and whose domain contains all $x \in M_1$

such that

$$\rho_1(a, x) < h, \text{ for some } h > 0 \text{ except}$$

possibly $x = a$. Also assume that 'a' is a "cluster point" or "Accumulation point" of the domain of f.

Definition : 4.3.A. Limits in metric spaces:

Let $f(x)$ is said to approach L where $L \in \mathbb{R}^2$ as x approaches 'a' if given $\epsilon > 0$, $\exists$ $\delta > 0$ such that,

$$\rho_2(f(x), L) < \epsilon, \text{ whenever } 0 < \rho_1(x, a) < \delta$$

It is written as $\lim_{x \to a} f(x) = L$ (or)

$f(x) \to L$ as $x \to a$.

Theorem: 4.3.B

Let $\langle M, \rho \rangle$ be a metric space and let 'a' be a point in M. Let f and g be real valued functions whose domains are subset of M. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$, then

(i) $\lim_{x \to a} [f(x) + g(x)] = L + N$

(ii) $\lim_{x \to a} [f(x) - g(x)] = L - N$

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = LN$ and

(iv) If $N \neq 0$, $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{N}$

Proof:

Given $\langle M, \rho \rangle$ be a metric space

To prove $\lim_{x \to a} [f(x) + g(x)] = L + N$

(ii) for given $\epsilon > 0$, we have to find $\delta > 0$ such that

$|(f(x) + g(x)) - (L + N)| < \epsilon$, $0 < \rho(x, a) < \delta$ — (1)

Since $\lim_{x \to a} f(x) = L$

for given $\epsilon > 0$, $\exists$ $\delta_1 > 0$ such that

$|f(x) - L| < \epsilon/2$, $0 < \rho(x, a) < \delta_1$

Since $\lim_{x \to a} g(x) = N$

for the same $\epsilon > 0$, $\exists$ $\delta_2 > 0$ such that

$|g(x) - N| < \epsilon/2$, $0 < \rho(x, a) < \delta_2$

Choose $\delta = \min(\delta_1, \delta_2)$

Then for $0 < \rho(x, a) < \delta$

$$|(f(x) + g(x)) - (L + N)| = |(f(x) - L) + (g(x) - N)|$$
$$\leq |f(x) - L| + |g(x) - N|$$

$|(f(x)+g(x)) - (L+N)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\therefore |(f(x)+g(x)) - (L+N)| < \varepsilon, \quad 0 < \rho(x,a) < \delta$.

$\therefore$ eq ① holds

Hence $\lim\limits_{x \to a} [f(x)+g(x)] = L+N$

Hence the proof.

(ii) To prove : $\lim\limits_{x \to a} [f(x) - g(x)] = L - N$

for given $\varepsilon > 0$ we have to find a $\delta > 0$ such that

$|(f(x) - g(x)) - (L-N)| < \varepsilon \quad \forall \; 0 < \rho(x,a) < \delta$ — ②

Since $\lim\limits_{x \to a} f(x) = L$

for given $\varepsilon > 0$, $\exists \; \delta_1 > 0$ such that

$|f(x) - L| < \varepsilon/2 \quad 0 < \rho(x,a) < \delta_1$

Since $\lim\limits_{x \to a} g(x) = N$

for the same $\varepsilon > 0$, their exists $\delta_2 > 0$ such that

$|g(x) - N| < \varepsilon/2 \quad , 0 < \rho(x,a) < \delta_2$

Choose $\delta = \min(\delta_1, \delta_2)$

Then for $0 < \rho(x,a) < \delta$.

$$|(f(x) - g(x)) - (L-N)| = |(f(x) - L) - (g(x) - N)|$$
$$\leq |f(x) - L| + |g(x) - N|$$
$$< \varepsilon/2 + \varepsilon/2 < \varepsilon .$$

$\therefore |((f(x) - g(x)) - (L - N)| < \varepsilon, \quad 0 < \rho(x, a) < \delta.$

$\therefore$ eq (2) holds

Hence $\lim_{x \to a} (f(x) - g(x)) = L - N$

Hence the proof.

(iii) To prove: $\lim_{x \to a} [f(x) \cdot g(x)] = LN.$

Since $\lim_{x \to a} g(x) = N$, we have for some $\delta_1 > 0$,

$$|g(x) - N| < 1 \qquad (0 < \rho(x, a) < \delta_1)$$

Thus $|g(x)| < |N| + 1 := Q \quad (0 < \rho(x, a) < \delta_1)$

$$f(x)g(x) - LN = f(x)g(x) - Lg(x) + Lg(x) - LN$$

$$= g(x)[f(x) - L] + L[g(x) - N]$$

$|f(x)g(x) - LN| \le |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - N|.$

Hence, if $0 < \rho(x, a) < \delta_1$,

$$|f(x)g(x) - LN| \le Q \cdot |f(x) - L| + |L| \cdot |g(x) - N|. \quad (1)$$

given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$Q|f(x) - L| < \frac{\varepsilon}{2} \qquad (0 < \rho(x, a) < \delta_2) \quad (2)$$

and there exists $\delta_3 > 0$ such that

$$|L|\,|g(x) - N| < \frac{\varepsilon}{2} \qquad (0 < \rho(x, a) < \delta_3) \quad (3)$$

If we let $\delta = \min(\delta_1, \delta_2, \delta_3)$, then from (1), (2), (3) it follows that

$$|f(x)g(x) - LN| < \varepsilon \qquad (0 < \rho(x,a) < \delta)$$

This proves $\lim\limits_{x \to a} f(x)\, g(x) = LN$.

(iv) To prove: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{N}$ if $N \neq 0$.

Now, first we have to prove that $\lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{N}$

∴ The given $\varepsilon > 0$, we have to find $\delta > 0$ such that

$$\left| \frac{1}{g(x)} - \frac{1}{N} \right| < \varepsilon, \quad 0 < \rho(x,a) < \delta \quad \text{--- (1)}$$

consider $\left| \dfrac{1}{g(x)} - \dfrac{1}{N} \right| = \left| \dfrac{N - g(x)}{g(x) N} \right|$

$$= \frac{|N - g(x)|}{|g(x)|\,|N|} \quad \text{--- (2)}$$

Since $\lim\limits_{x \to a} g(x) = N$

∴ for given $\varepsilon = \dfrac{|N|}{2} > 0$ ∃ a $\delta_1 > 0$ such that

$$|g(x) - N| < \left|\frac{N}{2}\right|, \qquad 0 < \rho(x,a) < \delta_1$$

consider

$$|g(x)| = |g(x) - N + N|$$

$$= |N - (N - g(x))|$$

$$\geq |N| - |N - g(x)|$$

$$> |N| - \frac{|N|}{2}$$

$$|g(x)| > \frac{|N|}{2}, \quad 0 < \rho(x,a) < \delta_1 \quad \text{--- (3)}$$

Again $\lim_{x \to a} g(x) = N$.

$\therefore$ for given $\epsilon > 0$, $\exists$ $\delta_2 > 0$ such that

$$|g(x) - N| < \frac{|N|^2 \epsilon}{2} \quad \text{— (4)} \qquad 0 < \rho(x, a) < \delta_2.$$

choose $\delta = \min(\delta_1, \delta_2)$

using eq ~~(1), (2), (3), (4)~~ (3) and (4) in (2)

for $0 < \rho(x, a) < \delta$.

$$\left| \frac{1}{g(x)} - \frac{1}{N} \right| = \frac{|N - g(x)|}{|g(x)||N|}$$

$$< \frac{\epsilon |N|^2 \cdot 2}{2 |N| |N|}$$

$$\left| \frac{1}{g(x)} - \frac{1}{N} \right| < \epsilon, \quad 0 < \rho(x, a) < \delta.$$

Hence eq (1) holds,

Hence by the previous result.

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{N}.$$

$$\Rightarrow \lim_{x \to a} f(x) \cdot \frac{1}{g(x)} = L \cdot \frac{1}{N}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{N}$$

Hence the proof.

4.3.C: Definition:

Convergent sequence:

Let $\langle M, \rho \rangle$ be a metric space and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of points in M. $\{S_n\}_{n=1}^{\infty}$ is said to be approach L, where $L \in M$ as n approaches ∞. If given $\varepsilon > 0$, there exists $N \in I$, such that

$$\rho(S_n, L) < \varepsilon, \quad \forall\, n \geq N.$$

It can be written as $\lim\limits_{n \to \infty} S_n = L$ (or) $S_n \to L$ as $n \to \infty$.

Then $\{S_n\}_{n=1}^{\infty}$ is convergent in M to the point L.

Definition: 4.3.D

Cauchy sequence:

Let $\langle M, \rho \rangle$ be a metric space and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of point in M. $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for given $\varepsilon > 0$, $\exists$ $N \in I$ such that

$$\rho(S_m, S_n) < \varepsilon, \quad \forall\, m, n \geq N.$$

Theorem: 4.3.E:

Let (M, ρ) be a metric space. If $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M.

Then $\{S_n\}_{n=1}^{\infty}$ is cauchy.

Proof:

Given $\{S_n\}_{n=1}^{\infty}$ is a convergent sequences of points of M.

Let $\lim_{n \to \infty} S_n = M$.

$\therefore$ for given $\varepsilon > 0$, $\exists$ $N \in I$ such that

$$\rho(S_n, L) < \frac{\varepsilon}{2} \quad \forall n \geq N.$$

Since for $m, n \geq N$.

$$\rho(S_m, S_n) \leq \rho(S_n, L) + \rho(L, S_n) \quad \text{(By triangular inequality)}$$

$$\leq \rho(S_n, L) + \rho(S_n, L)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall m, n \geq N.$$

$$\rho(S_m, S_n) < \varepsilon, \quad \forall m, n \geq N.$$

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ is a cauchy sequence.

Hence the proof.

——— x ——— x ——— x ———