UNIT - 2

2.5 Bounded Sequences:

Definition: (Bounded sequences)

$\{S_n\}_{n=1}^{\infty}$ is bounded iff (if and only if) there exists $M \in R$ such that

$$|S_n| < M \quad (n \in I).$$

Note:

* If a sequence diverges to infinity (or to minus infinity) the sequence is not bounded.
* A sequence that diverges to infinity must be bounded below.

2.5B: Theorem:

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent, then $\{S_n\}_{n=1}^{\infty}$ is bounded.

Proof:

Suppose $L = \lim_{n \to \infty} S_n$.

Then given $\varepsilon = 1$, there exists $N \in I$ such that

$$|S_n - L| < 1 \quad (n \geq N).$$

$$\Rightarrow \quad |S_n| < L + 1 \quad (n \geq N) \quad \text{--- (1)}$$

If we let

$M = \max\{|S_1|, |S_2|, \ldots, |S_{N-1}|$ then

$$|S_n| = |L + (S_n - L)|$$
$$\leq |L| + |S_n - L|$$
$$|S_n| < M + L + 1 \quad (n \in S)$$

$\therefore \{S_n\}_{n=1}^{\infty}$ is bounded.

2.6 Monotone Sequences:

Definition: (Monotone Sequences)

A monotone sequence is a sequence which is either nonincreasing or nondecreasing (or both).

Define nondecreasing:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $S_1 \leq S_2 \leq \cdots \leq S_n \leq S_{n+1} \leq \cdots$, then $\{S_n\}_{n=1}^{\infty}$ is called nondecreasing.

Define nonincreasing:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $S_1 \geq S_2 \geq \cdots \geq S_n \geq S_{n+1} \geq \cdots$, then $\{S_n\}_{n=1}^{\infty}$ is called nonincreasing.

2.6B: Theorem:

A nondecreasing sequence which is bounded above is convergent.

Proof:

Suppose $\{S_n\}_{n=1}^{\infty}$ is nondecreasing and bounded above.

Then the set $A = \{S_1, S_2, \ldots\}$ is nonempty subset of R which is bounded above.

By known theorem

" If A is any nonempty subset of R that is bounded above, then A has a least upper bound in R."

$\therefore$ This set has a l.u.b.

Let $M = \text{lub}\{S_1, S_2, \ldots\}$

$= \text{lub}$ for A.

We shall prove that $S_n \to M$ as $n \to \infty$.

Given $\epsilon > 0$ the number $M - \epsilon$ is not an u.b for A.

Hence for some $N \in I$,

$$S_N > M - \epsilon.$$

But since $\{S_n\}_{n=1}^{\infty}$ is nondecreasing

this implies

$$S_n > M - \epsilon \quad (n \geq N). \text{ --- } ①$$

$M - \epsilon < S_n$

Since M is an u.b for A,

$$M \geq S_n \quad (n \in I) \text{ --- } ②$$

$S_n \leq M < M + \epsilon$

From ① & ② we conclude

$(n \geq N)$

$$M - \varepsilon < S_n < M + \varepsilon \quad (n \geq N)$$

$$\Rightarrow \quad |S_n - M| < \varepsilon$$

$$\Rightarrow \quad \lim_{n \to \infty} S_n = M.$$

This proves $\lim_{n \to \infty} S_n = M$.

2.6c corollary:

The sequence $\{(1 + 1/n)^n\}_{n=1}^{\infty}$ is convergent.

Proof:

By ~~the binomial~~ theorem

Let $S_n = (1 + \frac{1}{n})^n$

By the Binomial theorem

$$S_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 1}{1 \cdot 2 \cdot 3 \cdots n} \cdot \frac{1}{n^n}$$

$$\cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k} +$$

$$= 1 + 1 + \frac{n(n-1)}{1 \cdot 2} \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k} + \cdots + \frac{1}{n^n}$$

For $k = 1, \ldots, n$, the $(k+1)$ st term is

$$\frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k} \cdot \frac{1}{n^k}$$

Which equals to

$$\frac{1}{1\cdot 2\cdots k}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \quad (1)$$

If we expand S_{n+1} we obtain $n+2$ terms and for $k=1,\ldots,n$,

the $(k+1)$st term is

$$\frac{1}{1\cdot 2\cdots k}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)$$

which is greater than or equal to (1).

$\therefore \Rightarrow S_n \leq S_{n+1}$ (that is $\{S_n\}_{n=1}^{\infty}$ is nondecreasing).

But

$$S_n < 1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{1\cdot 2\cdots n}$$

$$< 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right)$$

$$< 1+\frac{1-(1/2)^n}{1-1/2}$$

$$< 1+\frac{1}{1-1/2}$$

$$< 1+\frac{1}{1/2}$$

$$< 1+2$$

$$S_n < 3$$

Thus $\{S_n\}_{n=1}^{\infty}$ is bounded above.

Hence by known theorem

"A nondecreasing sequence which is bounded above is convergent."

$\therefore \{S_n\}_{n=1}^{\infty}$ is convergent.

Theorem 2.6D:

A nondecreasing sequence which is not bounded above diverges to infinity.

Proof:

Suppose $\{S_n\}_{n=1}^{\infty}$ is nondecreasing but not bounded above.

Given $M > 0$, we must find $N \in I$ ∋

$S_n > M. \quad (n \geq N) \longrightarrow (1)$

Since M is not an upper bound for $\{S_1, S_2, \ldots\}$.

there must exist $N \in I$ ∋

$S_N > M.$

Then for this N, the $\{S_n\}_{n=1}^{\infty}$ is nondecreasing.

$\therefore$ Hence the proof.

2.6E: Theorem:

A nonincreasing sequence which is bounded below is convergent.

A nonincreasing sequence which is not

bounded below diverges to minus infinity.

2.7. operations on convergent sequences:

2.7A Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$, then $\lim_{n\to\infty}(s_n + t_n) = L+M$. In words the limit of the sum (of two convergent sequences) is the sum of the limits

Proof:

Given $\varepsilon > 0$ we must find $N \in I$ ∋

$$|(s_n + t_n) - (L+M)| < \varepsilon \quad (n \geq N) \quad \text{— (1)}$$

Now

since $\lim_{n\to\infty} s_n = L$, there exists $N_1 \in I$ such that

$$|s_n - L| < \frac{\varepsilon}{2} \quad (n \geq N_1)$$

Also since $\lim_{n\to\infty} t_n = M$, there exists $N_2 \in I$ ∋

$$|t_n - M| < \varepsilon/2 \quad (n \geq N_2)$$

Hence if we let $N = \max(N_1, N_2)$.

Now
$$|(s_n + t_n) - (L+M)| = |(s_n - L) + (t_n - M)|$$
$$\leq |s_n - L| + |t_n - M|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$< 2\varepsilon/2$$
$$|(s_n + t_n) - (L+M)| < \varepsilon \quad (n \geq N) \quad \text{— (2)}$$

Thus for this N eq ② and hence eq ①

will be true.

$$\therefore \lim_{n\to\infty} (s_n + t_n) = L + M.$$

Hence the proof.

2.7B: Theorem:

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, if $c \in \mathbb{R}$ and if $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} cs_n = cL$.

Proof:

If $c = 0$, the theorem is obvious.

We now assume $c \neq 0$.

Given $\varepsilon > 0$ we must find $N \in I$ ∋

$$|cs_n - cL| < \varepsilon \qquad (n \geq N).$$

Now since $\lim_{n\to\infty} s_n = L$, there exists $N \in I$ ∋

$$|s_n - L| < \frac{\varepsilon}{|c|} \qquad (n \geq N)$$

$$|c|\,|s_n - L| < \varepsilon \qquad (n \geq N)$$

$$\therefore |cs_n - cL| < \varepsilon \qquad (n \geq N)$$

$$\therefore \lim_{n\to\infty} cs_n = cL$$

Hence the proof.

2.7C theorem:

a) If $0 < x < 1$, then $\{x^n\}_{n=1}^{\infty}$ converges to 0.

b) If $1 < x < \infty$, then $\{x^n\}_{n=1}^{\infty}$ diverges to ∞.

Proof:

(a). If $0 < x < 1$

Then $x^{n+1} = x \cdot x^n$

$x^{n+1} \cancel{x^{n+1}} < x^n$

$x = 1/2$,

$x^{n+1} = x \cdot x^n < x^n$

$(1/2)^{n+1} = (1/2) \cdot (1/2)^n$

$n \geq 1$

$(1/2)^{1+1} = \frac{1}{4} < (1/2)$

Hence $\{x^n\}_{n=1}^{\infty}$ is nonincreasing.

since $x^n > 0$ for $n \in I$,

$\{x^n\}_{n=1}^{\infty}$ is bounded below.

By known theorem

"A nonincreasing sequence which is bounded below is convergent".

$\therefore \{x^n\}_{n=1}^{\infty}$ is convergent.

Let $L = \lim_{n \to \infty} x^n$.

By known formula

$\lim_{n \to \infty} c s_n = cL$

Instead of c we substitute x. (ie $c = x$).

$\therefore \lim_{n \to \infty} x s_n = xL$

$\lim_{n \to \infty} x \cdot x^n = xL$

(ii) $\{x^{n+1}\}_{n=1}^{\infty}$ converges to xL.

But $\{x^{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{x^n\}_{n=1}^{\infty}$.

By known theorem

"All subsequences of a convergent sequence of real numbers converge to the same limit."

$\therefore \quad L = Lx$

$\Rightarrow \quad L - Lx = 0$

$L(1-x) = 0.$

$L = 0$ and $1 - x = 0$

$x \neq 1$

$\therefore \quad L = 0$, and Hence

$\{x^n\}_{n=1}^{\infty}$ converges to Zero.

(b)

If $x > 1$.

Then $x^{n+1} = x \cdot x^n > x^n$.

so $\{x^n\}_{n=1}^{\infty}$ is nondecreasing.

We shall show that $\{x^n\}_{n=1}^{\infty}$ is not bounded above.

suppose if $\{x^n\}_{n=1}^{\infty}$ were bounded above,

By known theorem

"A nondecreasing sequence which is bounded above is convergent".

$\therefore \{x^n\}_{n=1}^{\infty}$ would converge to some $L \in \mathbb{R}$.

But the same reasoning in (a),

we show that $L = Lx$.

so $L = 0$

$L = \lim_{n \to \infty} x^n$

But $x^n \geq 1$ and so $\{x^n\}_{n=1}^{\infty}$ cannot converge to 0.

$\therefore$ This is a contradiction.

This contradiction proves that $\{x^n\}_{n=1}^{\infty}$ is not bounded above.

$\therefore \{x^n\}_{n=1}^{\infty}$ diverges to infinity.

2.7D Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$, then $\lim_{n\to\infty} (s_n - t_n) = L - M$.

Proof:

Since $\lim_{n\to\infty} t_n = M$.

Then by known theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, if $c \in R$, and if $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} c s_n = cL$."

Then $c = -1$, we have

$\lim_{n\to\infty} (-t_n) = -M$. But

By the theorem

$$\lim_{n\to\infty} (s_n + t_n) = L + M.$$

$$\therefore \lim_{n\to\infty} (s_n - t_n) = \lim_{n\to\infty} [s_n + (-t_n)]$$

$$= \lim_{n\to\infty} s_n + \lim_{n\to\infty} (-t_n)$$

$$= L + (-M)$$

$$= L - M$$

$$\therefore \lim_{n\to\infty} (s_n - t_n) = L - M.$$

Hence the proof.

2.7 E : Corollary.

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers if $s_n \leq t_n$ $(n \in I)$, and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$ then $L \leq M$.

Proof:

By known theorem

$$\lim_{n\to\infty} (t_n - s_n) = M - L.$$

But $(t_n - s_n) \geq 0$ $(n \in I)$. Hence

By known theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers and if $\lim_{n\to\infty} s_n = L$, then $L \geq 0$."

$\therefore M - L \geq 0$.

$M \geq L$

$\therefore L \leq M$

Hence the proof.

2.7 F : Lemma :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers which converges to L, then $\{s_n^2\}_{n=1}^{\infty}$ converges to L^2.

Proof:

We must prove that $\lim_{n\to\infty} s_n^2 = L^2$.

That is given $\varepsilon > 0$, we must find $N \in I$ ∋

$|s_n^2 - L^2| < \varepsilon$ $(n \geq N)$.

(or)

$|s_n + L| \cdot |s_n - L| < \varepsilon$ $(n \geq N)$.

By known theorem,

"All convergent sequences are bounded."

$\therefore \{s_n\}_{n=1}^{\infty}$ is bounded.

Thus for some $M > 0$,

$|s_n| \leq M$ $(n \in I)$.

So,

$|s_n + L| \leq |s_n| + |L|$

$\leq M + |L|$. ——— (1)

Since $\lim_{n \to \infty} s_n = L$, there exists $N \in I$ ∋

$|s_n - L| < \dfrac{\varepsilon}{M + |L|}$. $(n \geq N)$. ——— (2)

By using (1) & (2) we get

$|s_n - L| \cdot |s_n + L| < \cancel{M + |L|} \cdot \dfrac{\varepsilon}{\cancel{M + |L|}}$

$|s_n - L| \cdot |s_n + L| < \varepsilon$.

$|s_n^2 - L^2| < \varepsilon$ $(n \geq N)$.

$\therefore \lim_{n \to \infty} s_n^2 = L^2$

Hence the proof.

2.7G. Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$, and if $\lim_{n\to\infty} t_n = M$, then $\lim_{n\to\infty} s_n t_n = LM$.

Proof:

We can use

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] \quad (a,b) \in R. \quad \text{——— (1)}$$

Now as $n \to \infty$

we know that, $s_n + t_n \to L+M$

$$(s_n+t_n)^2 \to (L+M)^2 \quad \text{——— (2)}$$

also $(s_n - t_n) \to L-M$

$$(s_n - t_n)^2 \to (L-M)^2 \quad \text{——— (3)}$$

From (2) & (3) and using $\lim_{n\to\infty}(s_n - t_n) = L-M$, we get

$$(s_n+t_n)^2 - (s_n-t_n)^2 \Rightarrow (L+M)^2 - (L-M)^2$$

$$= 4LM. \quad \text{——— (4)}$$

From (1) & (4) we get

$$s_n t_n = \frac{1}{4}\left[(s_n+t_n)^2 - (s_n-t_n)^2\right]$$

$$= \frac{1}{4} \cdot 4LM$$

$$\therefore \lim_{n\to\infty} s_n t_n = LM.$$

2.7 H. Lemma :

If $\{t_n\}_{n=1}^{\infty}$ is a Sequence of real numbers, if $\lim_{n\to\infty} t_n = M$. where $M \neq 0$, then $\lim_{n\to\infty} (1/t_n) = \frac{1}{M}$

Proof:

If $M \neq 0$,

Then Either $M > 0$ or $M < 0$.

We can prove this theorem by using $M > 0$. (Because $M < 0$ can be proved by applying the first case to $\{-t_n\}_{n=1}^{\infty}$).

So we assume $M > 0$.

Given $\varepsilon > 0$, we must find $N \in I$ &

$$\left| \frac{1}{t_n} - \frac{1}{M} \right| < \varepsilon \quad (n \geq N). \quad \text{--- (1)}$$

(or)

$$\left| \frac{t_n - M}{t_n M} \right| < \varepsilon \quad (n \geq N).$$

Now there exists $N_1 \in I$ such that

$$|t_n - M| < \frac{M}{2} \quad (n \geq N_1)$$

$$\Rightarrow -\frac{M}{2} < t_n - M < \frac{M}{2} \quad (n \geq N_1)$$

$$t_n - M > -\frac{M}{2} \quad (n \geq N_1)$$

$$t_n > \frac{M}{2} \quad (n \geq N_1) \quad \text{--- (2)}$$

In addition, for $\varepsilon > 0$ there exists $N_2 \in I$ such that

$$|t_n - M| < \frac{M^2 \varepsilon}{2} \quad (n \geq N_2)$$

Thus if $N = \max(N_1, N_2)$ we have for $n \geq N$.

Then $$\left|\frac{t_n - M}{t_n M}\right| = \frac{1}{|t_n M|} |t_n - M|$$

$$< \frac{1}{M^2/2} \cdot \frac{M^2 \varepsilon}{2}$$

$$\frac{|t_n - M|}{|t_n M|} < \varepsilon \quad (n \geq N).$$

$$\therefore \lim_{n \to \infty} \frac{1}{t_n} = \frac{1}{M}.$$

Hence the proof.

2.7I Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n \to \infty} s_n = L$, and if $\lim_{n \to \infty} t_n = M$. where $M \neq 0$, then $\lim_{n \to \infty} \left(\frac{s_n}{t_n}\right) = \frac{L}{M}$.

Proof:

By using $\lim_{n \to \infty} \frac{1}{t_n} = \frac{1}{M}$ and $\lim_{n \to \infty} s_n t_n = LM$ we have

$$\lim_{n \to \infty} s_n \cdot \frac{1}{t_n} = L \cdot \frac{1}{M}$$

$$\lim_{n \to \infty} \frac{s_n}{t_n} = \frac{L}{M}$$

Problem:

Prove $\lim_{n\to\infty} \frac{3n^2-6n}{5n^2+4} = \frac{3}{5}$

Soln:

Let $\lim_{n\to\infty} \frac{3n^2-6n}{5n^2+4} = \lim_{n\to\infty} \frac{n^2(3-6/n)}{n^2(5+4/n^2)}$

$= \lim_{n\to\infty} \frac{3-6/n}{5+4/n^2}$ —— (A)

Since $\lim_{n\to\infty} \frac{1}{n} = 0$

Then $\lim_{n\to\infty} \frac{6}{n} = 6 \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 6 \cdot 0$

$= 0$

and also $\lim_{n\to\infty} 3 = 3$

Then $\lim_{n\to\infty} \left(3-\frac{6}{n}\right) = 3-0$

$= 3$ —— (1)

Since we know that

$\lim_{n\to\infty} \frac{1}{n} = 0$, we have

$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 0 \cdot 0$

$= 0$

And $\lim_{n\to\infty} 5 = 5$

$\therefore \lim_{n\to\infty} \left[5+\frac{4}{n^2}\right] = \lim_{n\to\infty} 5 + 4 \cdot \lim_{n\to\infty} \frac{1}{n^2}$

$$= 5 + 0$$

$$= 5 \quad \text{--- } ②$$

From ① & ② and substitute Ⓐ we get

$$\lim_{n\to\infty} \frac{3n^2 - 6n}{5n^2 + 4} = \frac{\lim_{n\to\infty} 3 - \frac{6}{n}}{\lim_{n\to\infty} 5 + \frac{4}{n^2}}$$

$$= \frac{3}{5}$$

Hence the proof.

2.3 operations on divergent sequences:

2.8 A Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers that diverge to infinity, then so do their sum and product. That is $\{s_n + t_n\}_{n=1}^{\infty}$ and $\{s_n \cdot t_n\}_{n=1}^{\infty}$ diverge to infinity.

Proof:

Given $M > 0$, choose $N_1 \in I$ such that

$$s_n > M \quad (n \geq N_1)$$

and choose $N_2 \in I$ ∋

$$t_n > 1 \quad (n \geq N_2).$$

Then for $N = \max(N_1, N_2)$, we have

$$s_n + t_n > M + 1 > M \quad (n \geq N)$$

and

$$s_n \cdot t_n > M \cdot 1 > M \quad (n \geq N).$$

$\Rightarrow \{s_n + t_n\}_{n=1}^{\infty} \to \infty$ and $\{s_n \cdot t_n\}_{n=1}^{\infty} \to \infty$.

2.8C Corollary:

If $\{s_n\}_{n=1}^{\infty}$ diverges to infinity and if $\{t_n\}_{n=1}^{\infty}$ converges, then $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to infinity

2.9 Limit Superior and limit inferior:

2.9A Definition:

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers that is bounded above, and let $M_n = \text{l.u.b}\{s_n, s_{n+1}, s_{n+2}, \ldots\}$.

(a) If $\{M_n\}_{n=1}^{\infty}$ converges, we define $\lim_{n\to\infty} \sup s_n$ to be $\lim_{n\to\infty} M_n$.

(b) $\{M_n\}_{n=1}^{\infty}$ diverges to minus infinity we write $\lim_{n\to\infty} \sup s_n = -\infty$.

Ex: let $s_n = (-1)^n \ (n \in I)$.

Then $\{s_n\}_{n=1}^{\infty}$ is bounded above.

For this $M_n = 1$ for every $n \in I$.

$\therefore \lim_{n\to\infty} M_n = 1$.

$\therefore \lim_{n\to\infty} \sup s_n = 1$.

2.9B : Definition :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is not bounded above, we write $\lim_{n\to\infty} \sup s_n = \infty$.

2.9C. Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequences of real numbers, then

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} s_n .$$

Proof :

Let $L = \lim_{n\to\infty} s_n$.

Then given $\varepsilon > 0$ there exists $N \in I$ &

$$|s_n - L| < \varepsilon \quad (n \geq N)$$

(or)

$$-\varepsilon < s_n - L < \varepsilon$$

$$L - \varepsilon < s_n < L + \varepsilon \quad (n \geq N).$$

Thus if $n \geq N$,

then $L+\varepsilon$ is an u.b for $\{s_n, s_{n+1}, \dots\}$

and $L-\varepsilon$ is not an u.b.

Hence $L - \varepsilon < M_n = \text{lub}\{s_n, s_{n+1}, \dots\} \leq L + \varepsilon$.

Then by known theorem,

"If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers, if $s_n \leq t_n$ and if $\lim_{n\to\infty} s_n = L$

$\lim_{n\to\infty} t_n = M$, then $L \leq M$."

$$\therefore L-\varepsilon \leq \lim_{n\to\infty} M_n \leq L+\varepsilon.$$

But $\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup s_n$.

$$\therefore L-\varepsilon \leq \lim_{n\to\infty} \sup s_n \leq L+\varepsilon.$$

$$-\varepsilon \leq \lim_{n\to\infty} \sup s_n - L \leq \varepsilon$$

$$\left|\lim_{n\to\infty} \sup s_n - L\right| < \varepsilon \qquad (n \geq N).$$

$$\left|\lim_{n\to\infty} \sup s_n - \lim_{n\to\infty} s_n\right| < \varepsilon$$

$$\therefore \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} s_n.$$

Hence the proof.

Definition: Limit inferior:

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers that is bounded below, and

let $m_n = g.l.b\{s_n, s_{n+1}, \ldots\}$.

a) If $\{m_n\}_{n=1}^{\infty}$ converges, we define $\lim_{n\to\infty} \inf s_n$ to be $\lim_{n\to\infty} m_n$.

b) If $\{m_n\}_{n=1}^{\infty}$ diverges to infinity, we write $\lim_{n\to\infty} \inf s_n = \infty$.

2.9 E : Definition :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is not bounded below, we write $\lim_{n\to\infty} \inf s_n = -\infty$.

2-9 F : Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers, then $\lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} s_n$.

2-9 G :

If we make the notation convention for the symbols $-\infty$ and ∞ that

$$\left.\begin{array}{ll} -\infty < x & (x \in R) \\ x < \infty & (x \in R) \\ -\infty < \infty & \end{array}\right\} \quad \text{--- (1)}$$

Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then

$$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n .$$

Proof :

If $\{s_n\}_{n=1}^{\infty}$ is bounded.

Then, $m_n = glb\{s_n, s_{n+1}, \dots\} \leq lub\{s_n, s_{n+1}, \dots\} = M_n$.

Thus $m_n \leq M_n$.

Then by known theorem,

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers if $s_n \leq t_n$ $(n \in I)$ and if $\lim_{n\to\infty} s_n = L$,

$\lim_{n\to\infty} t_n = M$, then $L \leq M$".

$\therefore$ if $m_n \leq M_n$

then $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$.

If $\{s_n\}_{n=1}^{\infty}$ is not bounded.

Then either $\lim_{n\to\infty} \sup s_n = \infty$ (or) $\lim_{n\to\infty} \sup s_n = -\infty$

Then follows from

$-\infty < \infty$.

Then $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$

Note:

Then by 2.9C and 2.9F

if $\lim_{n\to\infty} s_n = L$, then

$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = L$.

2.9H : Theorem:

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = L$, where $L \in \mathbb{R}$, then $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = L$.

Proof:

By hypothesis we have

$$L = \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n$$

And $L = \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \text{lub}\{s_n, s_{n+1}, s_{n+2}, \dots\}$.

Then given $\epsilon > 0$ there exists $N_1 \in I$ such that

$$|\text{lub}\{s_n, s_{n+1}, \dots\} - L| < \epsilon \quad (n \geq N_1)$$

$$\Rightarrow s_n < L + \epsilon \quad (n \geq N_1). \text{ ——— } (1)$$

Similarly,

since $\lim_{n\to\infty} \inf s_n = L$,

there exists $N_2 \in I$ & $|\text{glb}\{s_n, s_{n+1}, \dots\} - L| < \epsilon \quad (n \geq N_2)$,

$$\Rightarrow s_n > L - \epsilon \quad (n \geq N_2) \text{ ——— } (2)$$

If $N = \max(N_1, N_2)$, then from (1) & (2)

$$L - \epsilon < s_n < L + \epsilon$$

$$|s_n - L| < \epsilon \quad (n \geq N)$$

$\therefore$ This proves $\lim_{n\to\infty} s_n = L$.

2.9I Theorem:

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\lim_{n\to\infty} \sup = \infty = \lim_{n\to\infty} \inf s_n$ then s_n diverges to infinity.

Proof:

If Given $\lim_{n\to\infty} \inf s_n = \infty$,

Given $M > 0$ there exists an $N \in I$ such that

$glb\{s_n, s_{n+1}, s_{n+2}, \ldots\} > M$ $(n \geq N)$

$\Rightarrow$ M is lower bound (but not glb) for $\{s_n, s_{n+1}, \ldots$

so that

$s_n > M$ $(n \geq M)$.

$\therefore$ s_n diverges to infinity.

2.9 J:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded sequences of real numbers, and if $s_n \leq t_n$ $(n \in I)$, then $\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n$ and

$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n$.

Proof:

By hypothesis $s_n \leq t_n$.

$\therefore$ $lub\{s_n, s_{n+1}, \ldots\} \leq lub\{t_n, t_{n+1}, \ldots\}$

and $glb\{s_n, s_{n+1}, s_{n+2}, \ldots\} \leq glb\{t_n, t_{n+1}, \ldots\}$

Taking the limit as $n \to \infty$ on both sides of these inequalities, and using known theorem

"If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers, if $s_n \leq t_n \quad (n \in I)$ and if $\lim\limits_{n\to\infty} s_n = L$, $\lim\limits_{n\to\infty} t_n = M$ then $L \leq M$."

$\therefore$ Hence

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n \quad \text{and}$$

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$

Note :

1. $\limsup\limits_{n\to\infty} (s_n + t_n) = \limsup\limits_{n\to\infty} s_n + \limsup\limits_{n\to\infty} t_n$,

is not always true even for bounded sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$.

ex:

If $s_n = (-1)^n \quad (n \in I)$

$t_n = (-1)^{n+1} \quad (n \in I)$.

Then $s_n + t_n = 0 \quad (n \in I)$.

and $\limsup\limits_{n\to\infty} s_n = 1 = \limsup\limits_{n\to\infty} t_n$.

$\cancel{\liminf\limits_{n\to\infty} s_n}$

But $\limsup\limits_{n\to\infty} (s_n + t_n) = 0$.

$\therefore \limsup\limits_{n\to\infty} (s_n + t_n) \neq \limsup\limits_{n\to\infty} s_n + \limsup\limits_{n\to\infty} t_n$.

2.9k Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded sequences of real numbers then

$$\lim_{n\to\infty} \sup (s_n + t_n) \leq \lim_{n\to\infty} \sup s_n + \lim_{n\to\infty} \sup t_n .$$

$$\lim_{n\to\infty} \inf (s_n + t_n) \geq \lim_{n\to\infty} \inf s_n + \lim_{n\to\infty} \inf t_n .$$

Proof:

a) let $M_n = \text{lub}\{s_n, s_{n+1}, s_{n+2}, \ldots\}$

$P_n = \text{lub}\{t_n, t_{n+1}, t_{n+2}, \ldots\}$

Then

$$s_k \leq M_n \quad (k \geq n), \quad t_k \leq P_n \quad (k \geq n).$$

and

$$s_k + t_k \leq M_n + P_n \quad (k \geq n).$$

Thus $M_n + P_n$ is an upper bound for ~~$\{s_n, s_{n+1}, \ldots$~~

$\{s_n + t_n, s_{n+1} + t_{n+1}, s_{n+2} + t_{n+2}, \ldots\}$ ~~$\leq M_n + P_n$~~.

So $\text{lub}\{s_n + t_n, s_{n+1} + t_{n+1}, s_{n+2} + t_{n+2}, \ldots\} \leq M_n + P_n$.

By known theorems

"1) if $s_n \leq t_n$ then $L \leq M$. and

2) $\lim_{n\to\infty} (s_n + t_n) = L + M$ "

$$\Rightarrow \lim_{n\to\infty} \text{l.u.b}\{s_n + t_n, s_{n+1} + t_{n+1}, s_{n+2} + t_{n+2}, \ldots\}$$

$$\leq \lim_{n\to\infty} (M_n + P_n)$$

$$\leq \lim_{n\to\infty} M_n + \lim_{n\to\infty} P_n$$

$\therefore \lim_{n\to\infty} \sup(s_n+t_n) \leq \lim_{n\to\infty} \sup s_n + \lim_{n\to\infty} \sup t_n.$

Similarly

$\lim_{n\to\infty} \inf(s_n+t_n) \geq \lim_{n\to\infty} \inf s_n + \lim_{n\to\infty} \inf t_n.$

Hence the proof:

2.9L Theorem:

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

1. If $\lim_{n\to\infty} \sup s_n = M$, then for any $\varepsilon > 0$, (a)
   (a) $s_n < M+\varepsilon$ for all but a finite number of values of n;
   (b) $s_n > M-\varepsilon$ for infinitely many values of n.
2. If $\lim_{n\to\infty} \inf s_n = m$, then for any $\varepsilon > 0$,
   (c) $s_n > m-\varepsilon$ for all but a finite number of values of n;
   (d) $s_n < m+\varepsilon$ for infinetly many values of n.

Proof:

We prove part 2.

If c were false, then for some $\varepsilon > 0$, we would have $s_n \leq m-\varepsilon$ for infinitely many n.

But then, for any $n \in I$, the set $\{s_n, s_{n+1}, \ldots\}$ would contain a number $\leq m-\varepsilon$.

This would imply

$$glb\{s_n, s_{n+1}, \ldots\} \leq m-\varepsilon \quad (n \in I)$$

and taking limits, we would obtain.

By known theorem

" If $\{s_n\}_{n=1}^{\infty}$ & $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers if $s_n \leq t_n$ $(n \in I)$; and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$ then $L \leq M$. "

$\therefore \lim_{n\to\infty} \inf s_n \leq m-\varepsilon$ contradicts our hypothesis.

$\therefore$ Hence (c) is true.

(d): Now, suppose (d) is false.

Then for some $\varepsilon > 0$,

$s_n < m + \varepsilon$ for only a finite number of values of n.

But then there exists $N \in I$ such that

$$s_n \geq m + \varepsilon \quad (n \geq N).$$

By known theorem

" If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded sequences of real numbers, and $s_n \leq t_n$ $(n \in I)$, then $\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n$ and $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n$

$\therefore \liminf_{n\to\infty} s_n \geq m+\epsilon$.

Which contradicts our hypothesis.

$\therefore$ Hence (d) is true.

2.94 : Theorem :

Any bounded sequence of real numbers has a convergent subsequence.

Proof :

Suppose $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers and let $M = \limsup_{n\to\infty} s_n$.

We shall contruct a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ which converges to M.

By known theorem $\Rightarrow$ "If $\limsup_{n\to\infty} s_n = M$, then "$s_n > M-\epsilon$ for infinitely many values of n".

$\therefore$ There are infinitely many values of n such that $s_n > M-1$.

Let n_1 be one such value.

That is $n_1 \in I$ and $s_{n_1} > M-1$.

lly Since there are infinitely many values of n such that $s_n > M - 1/2$,

we can find $n_2 \in I$ $\ni$ $n_2 > n_1$ and $s_{n_2} > M - 1/2$.

Continuing then, for each integer $k>1$, we can find $n_k \in I$ such that $n_k > n_{k-1}$ and

$$S_{n_k} > M - 1/k \quad \text{——— (1)}$$

Given $\varepsilon > 0$, by known theorem (a) of 2.9L,
we can find $N \in I$ such that

$$S_n < M + \varepsilon \quad (n \geq N). \quad \text{——— (2)}$$

Now choose $K \in I$, so that $\frac{1}{K} \leq \varepsilon$ and $n_K \geq N$.
Then if $k \geq K$, we have $\frac{1}{k} < \varepsilon$ and $n_k \geq N$.

Hence using (1) & (2)

$$M - \varepsilon < M - \frac{1}{k} < S_{n_k} \leq M + \varepsilon \quad \text{for every } k \geq K.$$

$$\Rightarrow |S_{n_k} - M| < \varepsilon \quad \forall\, k \geq K.$$

$$S_{n_k} \rightarrow M \text{ as } k \rightarrow \infty.$$

Hence the proof.

2.10. Cauchy sequences:

2.10 A Definition:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{S_n\}_{n=1}^{\infty}$ is called a cauchy sequence if for any $\varepsilon > 0$, there exists $N \in I$ $\ni$

$$|S_m - S_n| < \varepsilon \quad (m, n \geq I).$$

2.10 B: Theorem:

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ converges then $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence.

Proof:

Let $L = \lim_{n \to \infty} S_n$.

Then given $\varepsilon > 0$, there exists an $N \in I$ such that

$$|S_K - L| < \frac{\varepsilon}{2} \quad (K \geq N).$$

Thus if $m, n \geq N$ we have

$$|S_m - S_n| = |(S_m - L) + (L - S_n)|$$

$$\leq |S_m - L| + |L - S_n|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|S_m - S_n| < \varepsilon \quad (m, n \geq N)$$

$\therefore$ Hence this proves that $\{S_n\}_{n=1}^{\infty}$ is cauchy.

2.10 C Lemma:

If $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers, then $\{S_n\}_{n=1}^{\infty}$ is bounded.

Proof:

Given $\varepsilon = 1$, choose $N \in I$ &

$$|S_m - S_n| < 1 \quad (m, n \geq N) \quad \text{—} (1)$$

Then $|S_m - S_N| < 1 \quad (m \geq N)$.

Hence if $m \geq N$,

we have $|S_m| = |(S_m - S_N) + S_N|$

$$\leq |S_m - S_N| + |S_N|$$

$$\leq 1 + |S_N| \quad (\because \text{by using eq } (1)) \quad (m \geq N).$$

If $M = \max(|S_1|, \ldots |S_{N-1}|,)$ then

$$|S_m| < M + 1 + |S_N| \quad (m \in I).$$

so that $\{S_n\}_{n=1}^{\infty}$ is bounded.

2.10 D. Theorem:

If $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers then $\{s_n\}_{n=1}^{\infty}$ is convergent.

Proof:

By known theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers then $\{s_n\}_{n=1}^{\infty}$ is bounded."

We know that $\lim_{n\to\infty} \sup s_n$ and $\lim_{n\to\infty} \inf s_n$ are real numbers. By known theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and if $\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = L$. where $L \in \mathbb{R}$, then $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = L$."

then to prove the theorem it is sufficient to show that

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n.$$

But by the theorem

"If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n$"

$\therefore$ we know that $\lim_{n\to\infty} \sup s_n \geq \lim_{n\to\infty} \inf s_n$ —— (1)

Thus we need to prove

$$\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \inf s_n. \quad \text{—— (2)}$$

Since $\{s_n\}_{n=1}^{\infty}$ is cauchy.

then given $\varepsilon > 0$ there exists $N \in I$ &

$$|s_m - s_n| < \frac{\varepsilon}{2} \quad (m, n \geq N).$$

and $|s_N - s_n| < \frac{\varepsilon}{2} \ (n \geq N)$.

Then $s_N + \frac{\varepsilon}{2}$ and $s_N - \frac{\varepsilon}{2}$ are respectively upper and lower bounds for the set $\{s_N, s_{N+1}, \ldots\}$.

Hence if $n \geq N$, $s_N + \varepsilon/2$ and $s_N - \varepsilon/2$ are upper and lower bounds for $\{s_n, s_{n+1}, \ldots\}$.

This implies for $n \geq N$,

$$s_N - \frac{\varepsilon}{2} \leq glb\{s_n, s_{n+1}, \ldots\} \leq lub\{s_n, s_{n+1}, \ldots\} \leq s_N + \frac{\varepsilon}{2}$$

$$lub\{s_n, s_{n+1}, \ldots\} - glb\{s_n, s_{n+1}, \ldots\} \leq s_N + \frac{\varepsilon}{2} - s_N + \frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$lub\{s_n, s_{n+1}, s_{n+2}, \ldots\} \leq glb\{s_n, s_{n+1}, \ldots\} + \varepsilon$$

Taking limit on both sides and using the theorem

"If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers if $s_n \leq t_n \ (n \in I)$ and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$ then $L \leq M$."

$$\therefore \lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \inf s_n + \varepsilon.$$

Since ε was arbitrary,

$$\therefore \lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \inf s_n \quad \text{——} \ (2)$$

Then by eq ① & ② we have

$$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n$$

Hence the proof.

2.10 E: Theorem:

For each $n \in I$ let $I_n = [a_n, b_n]$ be a (non empty) closed bounded interval of real numbers such that

$$I_1 \supset I_2 \supset \dots \supset I_n \supset I_{n+1} \supset \dots,$$

and $\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} (\text{length of } I_n) = 0$.

Then $\bigcap_{n=1}^{\infty} I_n$ contains precisely one point.

Proof:

By hypothesis (a) we have

$I_n \supset I_{n+1}$ and so

$a_n \le a_{n+1} \le b_{n+1} \le b_n$.

This shows that the sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are respectively nondecreasing and nonincreasing.

By (a) again all terms of both these sequence lie I_1.

and so the sequences are both bounded.

∴ By known theorem both sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent.

Let $x = \lim_{n \to \infty} a_n$ and

let $y = \lim_{n \to \infty} b_n$.

Then for any n we have

$a_n \leq x$ and $y \leq b_n$.

But By known theorem

"If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n \to \infty} s_n = L$ and if $\lim_{n \to \infty} t_n = M$, then $\lim_{n \to \infty} (s_n - t_n) = L - M$.

And by hypothesis (b) we have

$$y - x = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n$$

$$= \lim_{n \to \infty} (b_n - a_n)$$

$$= 0.$$

Thus $x = y$.

But then $a_n \leq x \leq b_n$ for each n, which shows that $x \in \bigcap_{n=1}^{\infty} I_n$.

clearly no $z \neq x$ can lie in $\bigcap_{n=1}^{\infty} I_n$.

Hence $\bigcap_{n=1}^{\infty} I_n$ contains x and no other point.

Hence the theorem.