1.3 Functions

Definition: Cartesian product:

If A, B are any two sets, then the cartesian product of A and B is the set of all ordered pairs $\langle a, b \rangle$ where $a \in A$ and $b \in B$.

(ie) $A \times B = \{ \langle a, b \rangle / a \in A, b \in B \}$

1.3C Definition: Function

Let A and B be any two sets. A function f from A into B is a subset of $A \times B$ with the property that each $a \in A$ belongs to precisely one pair $\langle a, b \rangle$.

1.5A Definition: one-to-one function:

If $f : A \to B$, then f is called one-to-one function if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ $(a_1, a_2 \in A$

Onto Function: Definition:

If $f : A \to B$, then f is called on-to function if for each $b \in B$ there exists atleast one $a \in A$ such that

$$f(a) = b$$

Definition: one-one correspondence:

Let A and B are any two sets if $f : A \to B$ and if f is both one-one and onto function then, it is said to be a one-one correspondence between A and B.

Definition: Inverse of a function:

Let f from A to B, if f is both one-one and onto, then the inverse of $f^{-1} : B \to A$ is defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.

~~1.3D~~ 1.3E. Theorem.

If $f : A \to B$ and if $X \subset B$, $Y \subset B$, then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$. In words the inverse image of the union of two sets is the union of the inverse images.

Proof:

Suppose $a \in f^{-1}(X \cup Y)$.

Then $f(a) \in (X \cup Y)$.

Hence either $f(a) \in X$ (or) $f(a) \in Y$.

So either $a \in f^{-1}(X)$ (or) $a \in f^{-1}(Y)$. But

$a \in f^{-1}(X) \cup f^{-1}(Y)$.

Thus $f^{-1}(X \cup Y) \subset f^{-1}(X) \cup f^{-1}(Y)$. —— (1)

Conversely,

if $b \in f^{-1}(X) \cup f^{-1}(Y)$.

Then either $b \in f^{-1}(X)$ (or) $b \in f^{-1}(Y)$.

Hence either $f(b) \in X$ (or) $f(b) \in Y$.

So that $f(b) \in X \cup Y$

But $b \in f^{-1}(X \cup Y)$.

So $f^{-1}(X) \cup f^{-1}(Y) \subset f^{-1}(X \cup Y)$. —— (2)

from equation ① & ② we get

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

Hence the proof.

Theorem : 1.3G

If $f: A \to B$ and $X \subset A$, $Y \subset A$ then

$$f(X \cup Y) = f(X) \cup f(Y).$$

In words, the image of the union of two sets is the union of the images.

Proof:

If $b \in f(X \cup Y)$, then $b = f(a)$ for some $a \in X \cup Y$

Either $a \in X$ or $a \in Y$. Thus

either $b \in f(X)$ or $b \in f(Y)$. Hence $b \in f(X) \cup f(Y)$.

$\Rightarrow f(X \cup Y) \subset f(X) \cup f(Y)$. —— ①

conversely,

if $c \in f(X) \cup f(Y)$ then either

$c \in f(X)$ or $c \in f(Y)$.

Then c is the image of some point in X (or)

c is the image of some point in Y.

Hence c is the image of some point in $X \cup Y$,

(ie) $c \in f(X \cup Y)$.

so $f(X) \cup f(Y) \subset f(X \cup Y)$. —— ②

from equation ① & ② we get

$$f(X \cup Y) = f(X) \cup f(Y).$$

Hence the proof

Definition : The composition of Functions:

If $f: A \to B$ and $g: B \to C$, then we define the function $g \circ f$ by

$$g \circ f(x) = g[f(x)] \quad (x \in A)$$

Thus, the image of x under $g \circ f$ is defined to be the image of $f(x)$ under g. The function $g \circ f$ is called the composition of f with g.

For example

if $f(x) = 1 + \sin x \quad (-\infty < x < \infty)$,

$g(x) = x^2 \quad (0 \leq x < \infty)$,

then $g \circ f(x) = g[f(x)]$

$= g[1 + \sin x]$

$= (1 + \sin x)^2$

$= 1 + 2\sin x + \sin^2 x. \quad (-\infty < x < \infty)$

1.4 Real-valued functions:

Definition: Real valued function:

If $f: A \to R$ we call f as a real valued function. If $x \in A$, then $f(x)$ is also called the value of f at x.

1.4B : Definition:

If $f: A \to R$ and $g: A \to R$, we define $f+g$ as the function whose value at $x \in A$ is equal to $f(x) + g(x)$

(ii) $(f+g)(x) = f(x) + g(x) \quad \forall x \in A$. 5

In set notation,

$$f+g = \{ \langle x, f(x)+g(x) \rangle / x \in A \}.$$

Similarly, we define

$$(f-g)x = f(x) - g(x), \quad x \in A$$

$$(fg)x = f(x)g(x), \quad x \in A$$

Finally if $g(x) \neq 0$ for all $x \in A$, we can define f/g by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad x \in A$.

1.4C: Definition:

If $f: A \to R$ and c is a real numbers $(c \in R)$, the function cf is defined by

$$(cf)(x) = c[f(x)], \quad x \in A.$$

Thus the value of $3f$ at x is 3 times the value of f at x.

Definition:

If $f: A \to R$, $g: A \to R$ then $\max(f,g)$ is the function defined by

$$\max(f,g)(x) = \max[f(x), g(x)], \quad x \in A.$$

and $\min(f,g)$ is the function defined by

$$\min(f,g)(x) = \min[f(x), g(x)], \quad x \in A.$$

(6)

Definition:

If $f: A \to \mathbb{R}$ then $|f|$ is the function defined by $|f|(x) = |f(x)|$, $x \in A$.

If a, b are real numbers, the formulae

$$\max(a,b) = \frac{|a-b| + a + b}{2}$$

$$\min(a,b) = \frac{-|a-b| + a + b}{2}$$

Definition: characteristic function of A.

If $A \subset S$, then χ_A (called the characteristic fn of A) is defined as,

$\chi_A(x) = 1$, $x \in A$

$\chi_A(x) = 0$, $x \in A'$

Note:

If $A = B$ iff $\chi_A = \chi_B$.

If A, B are subsets of S, then

$\chi_{A\cup B} = \max(\chi_A, \chi_B)$.

$\chi_{A\cap B} = \min(\chi_A, \chi_B) = \chi_A \chi_B$.

$\chi_{A-B} = \chi_A - \chi_B$

$\chi_{A'} = 1 - \chi_A$

$\chi_S = 1$, $\chi_\emptyset = 0$.

Finite Set :

A Set A is said to be a finite set if A has finite number of elements.

Infinite Set :

A set A is said to be an infinite set if for every positive integer n, A has a subset containing precisely n elements.

1.5E. Definition : countable set :

The Set A is said to be countable (or denumerable) if A is equivalent to the set I of positive integers.

Definition : uncountable Set

An uncountable set is an infinite set which is not countable.

1.5. Equivalence. countability :

Theorem : 1.

The set of all integers is countable. (Example for countable)

Proof :

Let Z denotes the set of all integers.

Define $f : I \to Z$ as $f(n) = \frac{n-1}{2}$, if n is odd

$= \frac{-n}{2}$, if n is even.

$f(1) = \frac{1-1}{2} = 0$

$f(2) = -2/2 = -1$

$f(3) = \frac{3-1}{2} = 1$

$f(4) = \frac{-4}{2} = -2$

$f(5) = \frac{5-1}{2} = 2$

$\vdots$

Hence f is both one-one and onto function.

Hence $\mathbb{Z}$ has a 1–1 correspondence with I.

Hence $\mathbb{Z}$ is countable.

1.5F Theorem :

If $A_1, A_2, \ldots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

In words the countable union of countable set is countable.

Proof :

If $A_1 = \{a_1^1, a_2^1, a_3^1, \ldots\}$, $A_2 = \{a_1^2, a_2^2, a_3^2 \ldots\}$ ... $A_n = \{a_1^n, a_2^n, \ldots\}$.

So that a_k^j is the k^{th} element of the set A_j.
Define the height of a_k^j to be $j+k$.

Then a_1^1 is the only element of height 2; likewise a_2^1 and a_1^2 are the only elements of height 3; and so on.

Since for any positive integer $m \geq 2$ there are only $m-1$ elements of height m.

We may arrange the elements of $\cup_{n=1}^{\infty} A_n$ according to their height as

$$a_1^1,\ a_1^2,\ a_2^1,\ a_3^1,\ a_2^2,\ a_1^3,\ a_1^4, \ldots,$$

Then pictorially we are listing the elements of $\cup_{n=1}^{\infty} A_n$ in the following array

$$\begin{matrix} a_1^1 & a_2^1 \rightarrow & a_3^1 & a_4^1 & \cdots \\ \downarrow & & & & \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots \\ & & & & \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & \cdots \\ & & & & \\ a_1^4 & a_2^4 & \cdots & & \end{matrix}$$

This counting scheme eventually counts every a_k^j.

Hence $\cup_{n=1}^{\infty} A_n$ is countable.

Hence the proof.

1.5G. Corollary.

The set of all rational numbers is countable.

Proof:

The set of all rational numbers is the union $\cup_{n=1}^{\infty} E_n$.

where E_n is the set of rationals which can be written with denominator n.

(ie) $E_n = \{ 0/n, -1/n, 1/n, -2/n, 2/n \}$.

Each E_n is clearly equivalent to the set of all integers.

Hence each E_n is countable.

$\therefore$ By the theorem

If $A_1, A_2, \ldots$ are countable.

Then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Hence $\bigcup_{n=1}^{\infty} E_n$ is countable.

Hence the set of all rationals is countable.

1.5.H Theorem:

If B is an infinite subset of the countable set A, then B is countable.

Proof:

Since A is countable, It can be written as

$$A = \{a_1, a_2, \ldots\}.$$

Now B is an infinite subset of A, then every element of B is also an element of A.

Then it can be written as a_i for some $i \in I$.

Let n_1 be the smallest subscript for which $a_{n_1} \in B$,

Let n_2 be the next smallest subscript for which $a_{n_2} \in B$ and so on.

Then $B = \{a_{n_1}, a_{n_2}, \ldots\}$.

The elements of B are thus labeled with 1, 2, ... and so.

Hence B is countable.

1.5 I. corollary:

The set of all rational numbers in [0, 1] is countable.

Proof:

The set of all rationals in [0, 1] is an infinite subset of the set of all rational numbers.

Since the set of all rational numbers are countable, and by the theorem,

"An infinite subset of a countable set is countable."

Hence the set of all rationals in [0, 1] is countable.

~~1.6 Real~~

1.6 Real Numbers :

1.6 A. Theorem :

10 mark

The Set $[0,1] = \{x / 0 \leq x \leq 1\}$ is uncountable.

Proof :

Suppose $[0,1]$ were countable.

Then $[0,1] = \{x_1, x_2, \ldots\}$ where every number in $[0,1]$ occurs among the x_i.

Expanding each x_i in decimals we have

$$x_1 = 0 \cdot a_1^1 a_2^1 a_3^1 \ldots$$
$$x_2 = 0 \cdot a_1^2 a_2^2 a_3^2 \ldots$$
$$x_3 = 0 \cdot a_1^3 a_2^3 a_3^3 \ldots$$
$$\vdots$$
$$x_n = 0 \cdot a_1^n a_2^n a_3^n \ldots a_n^n \ldots$$
$$\vdots$$

Let b_1 be any integer from 0 to 8 such that $b_1 \neq a_1^1$.

Then let b_2 be any integer from 0 to 8 such that $b_2 \neq a_2^2$.

In general for each $n = 1, 2, \ldots$
Let b_n be any integer from 0 to 8 such that $b_n \neq a_n^n$.

Let $y = 0 \cdot b_1 b_2 \ldots b_n \ldots$. Then for any n, the decimal expansion for y differs from the decimal expansion for x_n, since $b_n \neq a_n^n$.

Moreover, the decimal expansion for y is unique since no b_n is equal to 9.

Hence $y \neq x_n$ for every n and $0 \leq y \leq 1$, which contradicts the assumption that every number in $[0,1]$ occurs among the x_i.

Hence our assumption is wrong.

$\therefore$ The set $[0,1]$ is uncountable.

1.6B. Corollary:

The set R of all real numbers is uncountable.

Proof:

Suppose R were countable.

Then the set $[0,1]$ would be countable.

since by known theorem

"An infinite subset of a countable set is countable."

But we know the set $[0,1]$ is uncountable.

$\therefore$ our assumption is wrong.

$\therefore$ The set R of all real numbers is uncountable.

Hence the proof.

1.7. Least upper bounds :

Definition: Upper bound.

Let A be the subset of all R, if there exists a $M \in R$ such that

$$x \leq M \quad \forall x \in A.$$

Then A is said to be bounded above and M is said to be an upper bound of A.

Definition: Lower bound :

Let A be the subset of R, if there exists $N \in R$ such that $x \geq N \quad \forall x \in A$.

Then A is said to be bounded below and N is said to be an Lower bound for A.

1.7. c Definition: Least upper bound :

The subset A of R be bounded above. The number L is called the least upper bound for A if

(i) L is an upper bound for A, and

(ii) no number smaller than L is an upper bound for A.

Definition : greatest Lower bound :

The subset A of R be bounded below. The number l is called the greatest lower bound for A if

(i) l is a lower bound for A and

(ii) no number greater than l is a lower bound for A.

Least upper bound axiom :

If A is any nonempty subset of R that is bounded above, then A has a least upper bound in R.

greatest lower bound axiom:

1.7.E : Theorem :

If A is any nonempty subset of R that is bounded below, then A has a greatest lower bound in R.

Proof :

Let $B \subset R$ be the set of all $x \in R$ such that $-x \in A$. (ie elements of B are the negatives of the elements of A).

If M is a lower bound for A, then $-M$ is an upper bound for B.

If $x \in B$ then $-x \in A$ and so

$M \leq -x$, $x \leq -M$.

Hence B is bounded above, so by known theorem

"If A is any nonempty subset of $\mathbb{R}$ that is bounded above, then A has a least upper bound

Then B has a lub.

If Q is the l.u.b for B then $-Q$ is the g.l.b for A.

Hence A has a greatest lower bound in $\mathbb{R}$.

Definition: Bounded

If A is both bounded below and bounded above then the set A is said to be bounded.

Sequence of Real Numbers

2.1. Definition of sequence and subsequence:

Sequence Definition:

A sequence $S = \{S_n\}_{n=1}^{\infty}$ is a function from I the set of all positive integers to R the set of all real numbers. Here S_n denotes the n^{th} term of the sequence.

Definition: subsequence of the sequence of positive integers.

A subsequence N of the sequence of positive integers is a function from I to I such that $N(i) < N(j)$ whenever $i < j$.

Subsequence of a sequence:

Let $S = \{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $N = \{n_i\}_{i=1}^{\infty}$ be a subsequence of the sequence of positive integers, then

$$S \circ N = S \circ N(i)$$

$$S \circ N = S[N(i)]$$

$$= S\{n_i\}_{i=1}^{\infty}$$

$$= \{S_{n_i}\}_{i=1}^{\infty}$$ is the subsequence of S.

2.2 Limit of a sequence:

2.2A Definition:

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that s_n approaches the limit L (as n approaches infinity), if for every $\varepsilon > 0$ there is a positive integer N such that

$$|s_n - L| < \varepsilon \qquad (n \geq N)$$

If s_n approaches the limit L, we write

$$\lim_{n\to\infty} s_n = L$$

(or)

$$s_n \to L \quad (n \to \infty).$$

Let $\{s_n\}_{n=1}^{\infty} = \left\{\frac{2n}{n+4n^{1/2}}\right\}_{n=1}^{\infty}$

To prove that $\lim_{n\to\infty} s_n = 2$.

Proof:

Given $\varepsilon > 0$, we must find $N \in I$ such that

$$\left|\left\{\frac{2n}{n+4n^{1/2}}\right\} - 2\right| < \varepsilon \qquad (n \geq N) \quad —① $$

$$\left|\frac{\cancel{2n} - \cancel{2n} - 8n^{1/2}}{n+4n^{1/2}}\right| < \varepsilon \qquad (n \geq N)$$

$$\frac{8n^{1/2}}{n+4n^{1/2}} < \varepsilon \qquad (n \geq N) \quad —② $$

the left side of eq (2) is less than $8 \frac{n^{1/2}}{n}$

(i.e) $\frac{8}{n^{1/2}}$.

Hence eq (2) will be true if $\frac{8}{n^{1/2}} < \varepsilon \quad (n \geq N)$ — (3)

If we choose N so that

$$\frac{8}{N^{1/2}} < \varepsilon,$$

(ii) $$\frac{64}{N} < \varepsilon^2$$

$$\therefore \; N > \frac{64}{\varepsilon^2}.$$

Then eq (3) will be true.

Hence if N is any positive integer greater that

$\frac{64}{\varepsilon^2}$.

Then eq (3) and eq (2) and eq (1) will be true.

Hence $\lim_{n\to\infty} S_n = 2$.

Theorem 2.2B

(x) 5 mark

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers and if $\lim_{n\to\infty} S_n = L$, then $L \geq 0$.

Proof:

Suppose the contrary.

(ie) $L < 0$.

since $\lim_{n\to\infty} S_n = L$.

For given $\varepsilon>0$, there exists $N \in I$ such that

$|S_n - L| < \varepsilon \quad \forall (n \geq N)$.

Choose $\varepsilon = -\frac{L}{2}$.

Then there exists $N \in I$ ∋

$$|S_n - L| < -\frac{L}{2} \quad \forall (n \geq N).$$

In particular

$$|S_N - L| < -\frac{L}{2} \quad \cancel{\forall} (n \geq N)$$

$$\Rightarrow \frac{L}{2} < S_N - L < -\frac{L}{2} \quad \cancel{(n \geq N)}.$$

$$\Rightarrow S_N - L < -\frac{L}{2} \quad \cancel{(n \geq N)}$$

$$\Rightarrow S_N < -\frac{L}{2} + L \quad \cancel{(n \geq N)}.$$

$$S_N < \frac{L}{2} < 0 \quad \cancel{(n \geq N)}$$

But by hypothesis $S_N \geq 0$.

∴ our assumption is wrong.

∴ This contradiction shows that $L \geq 0$.

Hence the proof.

2.3 Convergent sequences:

2.3A Definition (convergent sequences)

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ has the limit L, we say that $\{s_n\}_{n=1}^{\infty}$ is convergent to L.

Ex:

the sequence $\{1, 1, 1, \ldots\}$ and $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ are convergent ~~to~~ the limit 1 and 0 respectively.

Note:

If $\{s_n\}_{n=1}^{\infty}$ does not have a limit, we say that $\{s_n\}_{n=1}^{\infty}$ is divergent.

Theorem 2.3B.

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ is convergent to L. then $\{s_n\}_{n=1}^{\infty}$ cannot also converge to a limit distinct from L. That is if $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} s_n = M$. then $L = M$.

(or)

Prove that a sequence cannot converge to more than one limit.

Proof:

Assume the contrary.

Then choose $L \neq M$.

So that $|M - L| > 0$.

Then let $\varepsilon = \frac{1}{2}|M - L|$

Since by hypothesis

$\lim_{n \to \infty} S_n = L$ there exists $N_1 \in I$ &

$$|S_n - L| < \epsilon_1 \quad (n \geq N_1)$$

Since by hypothesis

$\lim_{n \to \infty} S_n = M$ there exists $N_2 \in I$ &

$$|S_n - M| < \epsilon_1 \quad (n \geq N_2).$$

Let $N = \max(N_1, N_2)$.

Then $N \geq N_1$ and $N \geq N_2$.

Let

$$|M - L| = |M - S_N + S_N - L|$$

$$= |(S_N - L) - (S_N - M)|$$

$$\leq |S_N - L| + |S_N - M|$$

$$< \epsilon + \epsilon$$

$$< 2\epsilon_1$$

$$< 2 \cdot \frac{1}{2} |M - L|$$

$\Rightarrow \quad |M - L| < |M - L$

This is a contradiction.

This contradiction shows that $M = L$.

Hence $L = M$.

Theorem 2.3C :

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L.

2.3D Corollary:

All subsequences of a convergent sequence of real numbers converge to the same limit.

Proof:

If the sequence S converges to L then By known theorem

"A sequence cannot converge more than one limit":

$\therefore S$ converges to no other limit.

and by known theorem

"If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is Convergent to L, then any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L."

Then all subsequences of S converge to L.

2.4 Divergent sequences:

2.4A Definition: divergent Sequences:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that S_n apporaches infinity as n approaches infinity if for any real number $M>0$ There is a Positive integer N such that

$$S_n \geq M \quad (n \geq N).$$

In this case we write $S_n \to \infty$ as $n \to \infty$.

2.4B. Definition: (Divergent Sequences)

Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence of real numbers. we say that S_n approaches minus infinity as n approaches infinity if for any real number $M > 0$, there is a positive integer N such that

$$S_n < -M \quad (n \geq N).$$

In this case we write $S_n \to -\infty$ as $n \to \infty$. and say $\{S_n\}_{n=1}^{\infty}$ diverges to minus infinity.

2.4C. Definition: Oscillates:

If the sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers diverges but does not diverge to infinity and does not diverge to minus infinity, we say that $\{S_n\}_{n=1}^{\infty}$ oscillates.

Ex:

$$S_n = \{(-1)^n\}_{n=1}^{\infty}$$