Lattices and Boolean Algebra

Section 4.1.1

Partial Ordering:

A binary relation R in a relation or set P is called a partial order relation or a partial ordering in P if and only if R is

(i) reflexive (ie) $aRa \quad \forall a \in P$

(ii) R is antisymmetric
(ie). aRb and $bRA \Rightarrow a = b \quad \forall a, b \in P$

(iii) R is transitive
(ie). aRb and $bRc \Rightarrow aRc \quad \forall a, b, c \in P$

Note: A partial ordering is denoted by the symbol $\leq$

Definition:

If $\leq$ is a partial ordering on P, the ordered pair $(P, \leq)$ is called a partially ordered set or a poset

Definition:

Let $(P, \leq)$ be a poset. If for every $x, y \in P$ we have either $x \leq y$ or $y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P and $(P, \leq)$ is called a totally ordered or simply ordered set or a chain.

Definition: Dual of a poset:

If $(P, \leq)$ is a partially ordered set, then $(P, \geq)$ is also a partially ordered set and $(P, \geq)$ is called the dual of $(P, \leq)$

Definition: Lattice

A Lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound

For any $a, b \in L$.

$GLB\{a, b\} = a * b$

$LUB\{a, b\} = a \oplus b$

$a * b$ is called the meet or product of a, b.

and

$a \oplus b$ is called the join or sum of a, b

Sometimes we write $a \wedge b$ (or) $a \cdot b$ for $a * b$

and $a \vee b$ (or) $(a+b)$ for $a \oplus b$

Note:

A totally ordered set is a lattice because given any $a, b \in L$, is a totally ordered set, $a \leq b$ (or) $b \leq a$

$\therefore a \leq b \Rightarrow a * b = a$ and $a \oplus b = b$

$b \leq a \Rightarrow a * b = b$ and $a \oplus b = a$

Example: 1

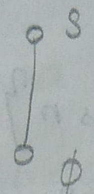
Let S be any set consider its power set $P(S)$, with the partial ordering $\subseteq$.

Then $(P(S), \subseteq)$ is a lattice in which

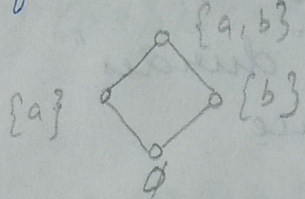
$A \oplus B = LUB\{A, B\} = A \cup B$

$A * B = GLB\{A, B\} = A \cap B$

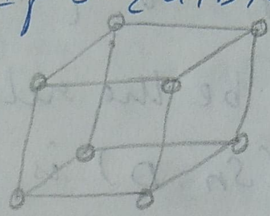
1) If $S = \{a\}$, then the Hasse diagram is a chain given below



2) If $S = \{a, b\}$



If $S = \{a, b, c\}$



Example : 2

Let I_+ is the set of all the integers. Let D be the relation "divides" in I_+.

ie) If $a, b \in I_+$ then

$aDb \Leftrightarrow a$ divides b.

(for eg: $2 D 6$ as 2 divides 6)

But 2 is not related to 5 as 2 does not divide 5)

(I_+, D) is a lattice since for any two integers m, n.

$m * n = GLB\{m, n\} = GCD\{a, b\}$

$m \oplus n = LUB\{m, n\} = LCM\{a, b\}$

For eg: $6 * 20 = 2$

$6 \oplus 20 = 60$

Example 3

Let n be a positive integer.

$a * b = glb$

$a \oplus b = lub$

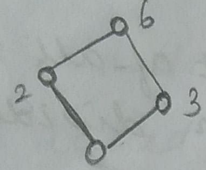
Let $S_n = \{ m \mid m \text{ divides } n \}$

For (e.g) $S_6 = \{1, 2, 3, 6\}$ if $n = 6$

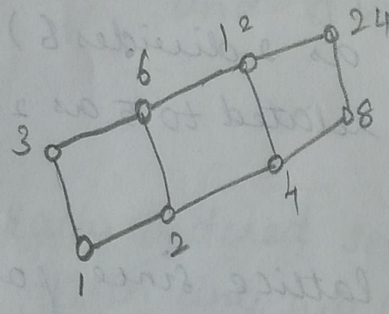
Let D be the relation "divides"

Then (S_n, D) is a lattice

The Hasse diagram for (S_6, D) is



For $n = 24$, the lattice (S_{24}, D) is given by $\{1, 2, 3, 4, 6, 8, 12, 24\}$



Principle of Duality of lattices:

Let $(L, \leq)$ be a lattice Any statement about the lattice involving the operations $*$ and $\oplus$ and the relation $\leq$ and $\geq$ remains true if $*$ is replaced by $\oplus$, $\oplus$ by $*$, $\leq$ by $\geq$ and $\geq$ by $\leq$

The Operations $*$ and $\oplus$ are called duals of each other; the relations $\leq$ and $\geq$ are duals of each other and the lattices $(L, \leq)$ and $(L, \geq)$ are called duals of each other.

Section 4.12 Some properties of lattices

Properties of $*$ and $\oplus$

(L-1) $a * a = a$ (L-1)' $(a \oplus a) = a$ [Idempotent]

(L-2) $a * b = b * a$ (L-2)' $(a \oplus b) = b \oplus a$ (Commutative)

(L-3) $(a*b)*c = a*(b*c)$ (L-3)' $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Associative)

(L-4) $a * (a \oplus b) = a$ (L-4)' $a \oplus (a * b) = a$ (Absorption)

Theorem:

Let $(L, \leq)$ be a lattice

$*, \oplus$ are the operations of meet and join

Then for any $a, b \in L$

$[a \leq b \Leftrightarrow a * b] = a \Leftrightarrow a \oplus b = b$

10m/5m

Proof:

1) $a \leq b \Leftrightarrow a * b = a$
2) $a \leq b \Leftrightarrow a \oplus b = b$
3) $a * b = a \Leftrightarrow a \oplus b = b$

i) To prove that $a \leq b \Leftrightarrow a * b = a$

Assume that $a \leq b$ and also $a \leq a$ To prove

$\Rightarrow a$ is a lower bound for $\{a, b\}$

$\Rightarrow a \leq glb\{a, b\} \Rightarrow a \leq a * b \rightarrow ①$

By the defn of $a * b$, it is a lower bound for $\{a, b\}$ $\therefore a * b \leq a \rightarrow ②$

From ① & ② $a * b = a$

Conversely, let $a * b = a$

$\Rightarrow a * b = a$ is a lower bound for $\{a, b\}$

$\Rightarrow a \leq a$ and $a \leq b$

$\Rightarrow \therefore a \leq b$

$\therefore a * b = a \Leftrightarrow a \leq b$

ii) To prove that $a \leq b \Leftrightarrow a \oplus b = b$

$\Leftrightarrow a \leq b$ also $b \leq b$

$\Rightarrow b$ is an upper bound for $\{a, b\}$

$\Rightarrow LUB\{a, b\} \leq b$

$\Rightarrow a \oplus b \leq b \rightarrow (3)$

By defn $a \oplus b$ is an upperbound for $\{a, b\}$

$\therefore b \leq a \oplus b \rightarrow (4)$. $b = a \oplus b$

Conversely

let $a \oplus b = b$.

By defn $b = a \oplus b$ is an upper bound for $\{a, b\}$

$\therefore a \leq b$ and $b \leq b$

$\therefore a \leq b$] $\therefore a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

Theorem 4.12 - Isotonicity

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, the following properties called isotonicity hold.

$$b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$$

Proof:

We have $b \leq c \Leftrightarrow b * c = b \rightarrow (1)$

Hence to prove $a * b \leq a * c$, enough to prove that (having $a*b$ instead of b & $a*b$ instead of c in ①)

$(a * b) * (a * c) = a * b \rightarrow$ ②

$(a*b) * (a*c) = (b*a) * (a*c)$ (Commutative)

$= b * (a * (a*c))$ (associative)

$= b * ((a*a) * c)$

$= b * (a*c)$ associative

$= (b*a) * c$ Commutative

$= a * (b*c) = (a*b) * c$

$= a * b$ $[\because b \leq c \Rightarrow b*c = b]$

Hence ② is proved.

Now we prove $b \leq c \Rightarrow a \oplus b \leq a \oplus c \rightarrow$ ③

As above consider $(a \oplus b) \oplus (a \oplus c) = a \oplus c$

$(a \oplus b) \oplus (a \oplus c) = (b \oplus a) \oplus (a \oplus c)$

$= b \oplus (a \oplus (a \oplus c))$

$= b \oplus ((a \oplus a) \oplus c) = (b \oplus a) \oplus c$

$= b \oplus (a \oplus c) = a \oplus (b \oplus c)$

$= a \oplus c$ $\because [b \leq c \Rightarrow b \oplus c = c]$

Hence ③ is proved.

Theorem:

$a \leq b \wedge a \leq c \Rightarrow$ i) $a \leq b \oplus c$ ✓
ii) $a \leq b * c$ ✓

III^{ly} $a \geq b \wedge a \geq c \Rightarrow$ (i) $a \geq b \oplus c$ ✓
(ii) $a \geq b * c$ ✓

Proof:

i) $a \leq b$ But $b \leq b \oplus c$

$\Rightarrow a \leq b \oplus c$ (By transitivity)

ii) $a \leq b \wedge a \leq c \Rightarrow a$ is a lower bound for $\{b, c\}$

$\Rightarrow a \leq glb\{b, c\} = b * c$

$\Rightarrow a \leq b * c$

Distributive Inequalities:

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$

(i) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

(ii) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

Proof:

By dfn $a \leq a \oplus b$

$a \leq a \oplus c$

$a \leq (a \oplus b) * (a \oplus c) \rightarrow ①$ $\left[\because a \leq b, \ a \leq c \Rightarrow a \leq b * c\right]$

Also

glb $b * c \leq b \leq a \oplus b$ lub

$b * c \leq c \leq a \oplus c$

$\therefore b * c \leq (a \oplus b) * (a \oplus c) \rightarrow ②$

From ① & ②,

$(a \oplus b) * (a \oplus c) \geq a$ and

$(a \oplus b) * (a \oplus c) \geq b * c$

we have

$(a \oplus b) * (a \oplus c) \geq a \oplus (b * c)$

Hence ① is Proved.

$a \geq a * b$ and $a \geq a * c$

$\therefore a \geq (a * b) \oplus (a * c) \rightarrow ③$

Also $b \oplus c \geq b \geq a * b$

$b \oplus c \geq c \geq a * c$

$\therefore b \oplus c \geq (a*b) \oplus (a*c) \rightarrow ④$

From ③ & ④,

$a * (b \oplus c) \geq (a*b) \oplus (a*c)$

[using $a \leq b$ & $a \leq c$ $\Rightarrow a \leq b * c$]

Hence ii) is proved.

Modular Inequality:

Let $(L, \leq)$ be a lattice. Then for any $a, b, c \in L$

$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$.

Proof:

$a, b, c \in L$

Assume that $a \leq c$

But $a \leq c \Leftrightarrow a \oplus c = c$

$a \oplus (b * c) \leq (a \oplus b) * \underbrace{(a \oplus c)}_{c}$ [By distributive inequality]

$\leq (a \oplus b) * c$ $\quad \therefore a \leq c \Leftrightarrow a \oplus c$

$\therefore a \leq c$

$\Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$

Conversely, Assume that

$a \oplus (b * c) \leq (a \oplus b) * c$

By defn of LUB,

$a \leq a \oplus (b * c) \leq (a \oplus b) * c$

$\Rightarrow a \leq GLB \{a \oplus b, c\} \leq c$

$\Rightarrow a \leq c$ [$\because$ '$\leq$' is transitive]

The other ways of expressing the modular inequalities are as follows

i) $(a*b)\oplus(a*c) \le a*[b\oplus(a*c)]$

(ii) $(a\oplus b)*(a\oplus c) \ge a\oplus[b*(a\oplus c)]$

[Dual of (i)]

Proof:

By defn

$a \ge a*b$

$a \ge a*c$

$\therefore a \ge (a*b)\oplus(a*c)$

$\Rightarrow (a*b)\oplus(a*c) \le a \rightarrow ①$

IIIly $a*b \le b$

$\Rightarrow (a*b)\oplus(a*c) \le b\oplus(a*c) \rightarrow ②$

From ① & ②

$(a*b)\oplus(a*c) \le a*[b\oplus(a*c)]$

using result $a\le b \wedge a\le c \Rightarrow a\le b*c$

i) By replacing $\oplus$ by $*$, $*$ by $\oplus$

$\le$ by $\ge$ and $\ge$ by $\le$, we get (ii)

4.1.3 Lattice as Algebraic systems.

Defn: (Of a Lattice)

A lattice is an algebraic system $(L, *, \oplus)$ with two binary operations $*$, $\oplus$ on L both of which satisfy

i) $a*b = b*a$ and $a\oplus b = b\oplus a$

(Commutative)

4.2 - Boolean Algebra

Definition: 4.2.1

2m" [A Boolean Algebra is Complemented, distributive lattice]

A Boolean algebra will generally denoted by $\underline{(B, *, \oplus, ', 0, 1)}$ only in order"

Note:

From 1) In a Boolean algebra every element has a unique Complement

2) In a Boolean algebra, De Morgan's laws hold.

Complement
$a \in L$ least - 0
$a' \in L$ high - 1

$\left\{ \begin{array}{l} a * a' = 0 \\ a \oplus a' = 1 \end{array} \right\}$ satisfies

In general, a Boolean Algebra is denoted by $(B, *, \oplus, ', 0, 1)$ is a Lattice with in which (i) $(B, *, \oplus)$ binary operations $*, \oplus$

ii) The Corresponding partially ordered set will be denoted by $(B, \leq)$

iii) The bounds of the lattice are denoted by 0 and 1 where 0 is the least and 1 is the greatest element of $(B, \leq)$

iv) Since it is a Complemented and distribution lattice, every element has a unique Complement

If $a \in B$, there its Complement is denoted by $a' \in B$

$a * b = a * c$
$a \oplus b = a \oplus c$ Translation property
$\Rightarrow b = c$

A Boolean algebra $(B, *, \oplus, ', 0, 1)$ satisfies the following properties which have been already proved:

I) $(B, *, \oplus)$ as a lattice satisfies:

(L-1) $a * a = a$ (L-1)' $a \oplus a = a$

(L-2) $a * b = b * a$ (L-2)' $a \oplus b = b \oplus a$

(L-3) $(a * b) * c = a * (b * c)$ (L-3)' $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(L-4) $a * (a \oplus b) = a$ (L-4)' $a \oplus (a * b) = a$

II $(B, *, \oplus)$ as a distributive lattice satisfies:

(D-1) $a * (b \oplus c) = (a * b) \oplus (a * c)$

(D-2) $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

(D-3) $(a * b) \oplus (b * c) \oplus (c * a) = (a \oplus b) * (b \oplus c) * (c \oplus a)$

(D-4) $a * b = a * c$ and $a \oplus b = a \oplus c \Rightarrow b = c$

III) $(B, *, \oplus, 0, 1)$ as a bounded lattice, in which for any $a \in B$, the following hold.

(B-1) $0 \leq a \leq 1$

(B-2) $a * 0 = 0$ (B-2)' $a \oplus 1 = 1$

(B-3) $a * 1 = a$ (B-3)' $a \oplus 0 = a$

IV $(B, *, \oplus, ', 0, 1)$ as a uniquely complemented lattice satisfies the following identities.

(C-1) $a * a' = 0$ (C-1)' $a \oplus a' = 1$

(C-2) $0' = 1$ (C-2)' $1' = 0$

(C-3) $(a * b)' = a' \oplus b'$ (C-3)' $(a \oplus b)' = a' * b'$

(De Morgan's laws)

$\underline{V}$ From the binary operations $*$, $\oplus$ $(B, *, \oplus)$ has a partial ordering relation $\leq$ on B

(P-1) $a * b = glb\{a, b\}$ (P-1)' $a \oplus b = lub\{a, b\}$

(P-2) $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

(P-3) $a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow b' \leq a' \Leftrightarrow a' \oplus b = 1$

(P-3) - has to be proved

Prove that in a Boolean Algebra.

$a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$

Proof:

I) To prove that

$a \leq b \Leftrightarrow a * b' = 0$

$a \leq b \Leftrightarrow a * b = a$

$0 = a * 0 = a * (b * b')$

$= \underbrace{(a * b)}_{a} * b' = a * b'$

$\therefore a \leq b \Rightarrow a * b' = 0 \rightarrow$ 1

Assume that

$a * b' = 0$

But $[\because b \oplus b' = 1]$

$a = a * 1 = a * (b \oplus b')$

$= (a * b) \oplus (a * b')$

$= (a * b) \oplus 0$

$= a * b$

$\Rightarrow a \leq b \rightarrow$ 2

From 1 & 2, we get

$a \leq b \Leftrightarrow a * b' = 0$

II) We prove now that $a \le b \Leftrightarrow a' \oplus b = 1$

We have $a \le b \Leftrightarrow a \oplus b = b$

$$1 = 1 \oplus b = (a' \oplus a) \oplus b$$
$$= a' \oplus (a \oplus b)$$
$$= a' \oplus b \rightarrow ③$$

Conversely,

Assume that

$$a' \oplus b = 1$$

$$b = b \oplus 0 = b \oplus (a * a')$$
$$= (b \oplus a) * (b \oplus a')$$
$$= (b \oplus a) * (a' \oplus b)$$
$$= (b \oplus a) * 1 \rightarrow ④$$
$$= b \oplus a$$

$$b = b \oplus a \Leftrightarrow a \le b$$

From ③ & ④ $a \le b \Leftrightarrow a' \oplus b = 1$

III) We now prove that

$$a \le b \Leftrightarrow b' \le a'$$

$a \le b \Leftrightarrow a \oplus b = b \Leftrightarrow a * b = a$

To prove $b' \le a'$ enough to prove

$$b' * a' = b'$$

$b' * a' = (b \oplus a)'$ (De Morgans law)

$= b'$ $[\because a \le b \Leftrightarrow a \oplus b = b]$

$\therefore b' \le a' \rightarrow ⑤$

|||ly if $b' \le a'$ then $b' * a' = b'$

A.T

To prove $a \le b$ $\Rightarrow (b \oplus a) = (b' * a')' = (b')' = b$

$\therefore a \leq b \rightarrow$ ⑥

From ⑤ & ⑥, we get

$a \leq b \Leftrightarrow b' \leq a'$

Hence the proof.

Example: 1

(of a two-element Boolean Algebra)

Let $B = \{0, 1\}$ be a set $\quad 0' = 1, \; 1' = 0$

consider the Boolean algebra $(B, *, \oplus, ', 0, 1)$ given by the following table.

$*$	0	1	$\oplus$	0	1	x	x'
0	0	0	0	0	1	0	1
1	0	1	1	1	1	1	0

$0 * 0 = glb\{0, 0\} = 0$ $\quad 0 \oplus 0 = 0$

$0 * 1 = $ " $\{0, 1\} = 0$ $\quad 0 \oplus 1 = 1$

$1 * 0 = $ " $\{1, 0\} = 0$ $\quad 1 \oplus 0 = 1$

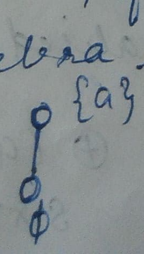
$1 * 1 = $ " $\{1, 1\} = 1$ $\quad 1 \oplus 1 = 1$

Note: A two-element Boolean algebra is the only Boolean algebra which is a chain.

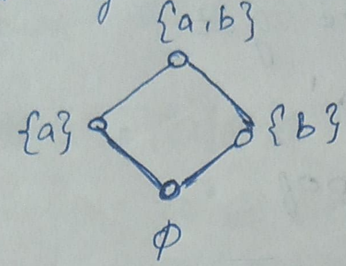
Example: 2

Let S be any non-empty set and $P(S)$, its power set. The set algebra $(P(S), \cap, \cup, \sim, \phi, S)$ is a Boolean Algebra in which for any $A \subseteq S$, $\sim A = S \setminus A$

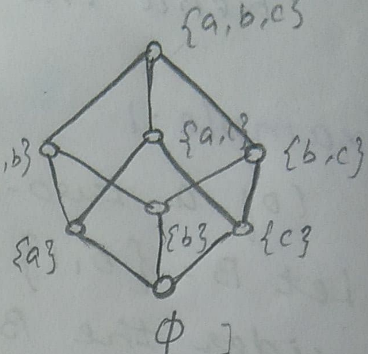
In particular, if $S = \{a\}$, then the Boolean Algebra is given by



If $S = \{a, b\}$, then the Boolean algebra is given by



|||ly for $S = \{a, b, c\}$



Generalization of the laws:

(L-3), (L-3)', (D-1), (D-2), (C-3), (C-3)' in a Boolean Algebra:

Let $S = \{a_1, a_2, \ldots a_n\}$ and $T = \{b_1, b_2, \ldots b_n\}$ where $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ are the elements of a Boolean algebra.

Then,

1) $(\underset{S}{*} a_i) * (\underset{T}{*} b_i) = \underset{S \cup T}{*} c_i$

where

$\underset{S}{*} a_i = a_1 * a_2 * \ldots * a_n$

$\underset{T}{*} b_i = b_1 * b_2 * \ldots * b_n$

$\underset{S \cup T}{*} c_i = a_1 * a_2 * \ldots * a_n * b_1 * b_2 * \ldots * b_n$

2) $(\underset{S}{*} a_i) \oplus (\underset{T}{*} b_j) = \underset{S \times T}{*} (a_i \oplus b_j)$

These are generalised Distributive laws

$(\underset{S}{\oplus} a_i) * (\underset{T}{\oplus} b_j) = \underset{S \times T}{\oplus} (a_i * b_j)$

3) Generalised Demorgan's laws:

$$\left(\underset{S}{*} a_i\right)' = \underset{S}{\oplus} a_i' \text{ and } \left(\underset{S}{\oplus} a_i\right)' = \underset{S}{*} a_i'$$

From the above two results, we get

$$\left[\left(\underset{S}{*} a_i\right) \oplus \left(\underset{T}{*} b_j\right)\right]' = \underset{S \times T}{\oplus} (a_i' * b_j')$$

and

$$\left[\left(\underset{S}{\oplus} a_i\right) * \left(\underset{T}{\oplus} b_j\right)\right]' = \underset{S \times T}{*} (a_i' \oplus b_j')$$

4.2.2 Subalgebra, Direct Product and Homomorphism:

Definitions: Let $(B, *, \oplus, ', 0, 1)$ be a Boolean algebra and let $S \subseteq B$. Then $(S, *, \oplus, ', 0, 1)$ is called a Subalgebra (or) Sub-Boolean algebra. if

(i) $0, 1 \in S$ (ii) S is closed under the operations $*, \oplus, '$.

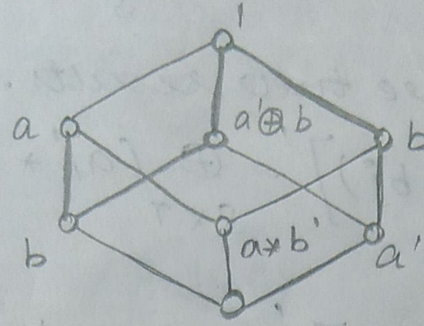
Note:

A subset of a Boolean algebra may be a Boolean algebra but it may not be a sub-algebra because it is not closed w.r. to the operations of B.

Give an example of a subset of a Boolean algebra which has a Boolean algebra but not a sub-algebra.

Example :

Consider the Boolean algebra below.



Consider

$S_1 = \{a, a', 0, 1\}$. This is a sub-Boolean Algebra.

$S_2 = \{a'\oplus b, a*b', 0, 1\}$

This is a sub-Boolean algebra because

(1) $(a'\oplus b) * (a*b') = (b\oplus a') * (a*b')$

$= ((b\oplus a') * a) * b'$

$= (a*b) \oplus (a*a') * b'$

$= ((a*b) \oplus 0) * b'$

$= (a*b) * b' \rightarrow ①$

$= a * 0 = 0$

$\therefore S_2$ is closed under $*$.

$|||^{ly}$ we have $(a'\oplus b) \oplus (a*b') = 1 \rightarrow ②$

Hence S_2 is closed under $*$ & $\oplus$

Also $(a'\oplus b)' = a*b'$ and
$(a*b')' = a'\oplus b$ } From ① & ②

Hence S_2 is closed under $'$ also.

and so $(S_2, *, \oplus, ', 0, 1)$ is a sub algebra

Let $S_3 = \{a * b', b', a, 1\}$

It is not a sub-algebra because

i) $0 \notin S_3$ ii) $a, b' \in S_3$

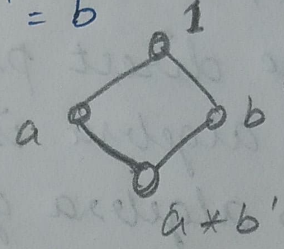
but $(b')' = b \notin S_3$

$\therefore S_3$ is not closed under $'$.

But S_3 is a Boolean algebra because in S_3.

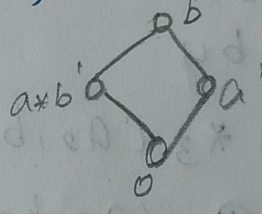
$(b')' = a$ & $a' = b'$

S_3 is given by



$S_4 = \{b', a * b', a', 0\}$ is not a sub-Boolean algebra because

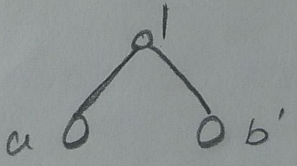
$(a')' = a \notin S_4$



But S_4 is a Boolean algebra.

as $(a')' = a * b'$ and $(a * b)' = a'$

$S_5 = \{a, b', 0, 1\}$ is not even a Boolean algebra since



S_5 is not a lattice at all.

Example 2:

For any Boolean algebra $(B, *, \oplus, ', 0, 1)$ the subsets $\{0, 1\}$ and B are sub-Boolean algebra.

Example: 3

If $a \in B$ Show that $a \neq 0, a \neq 1$, then $S = \{a, a', 0, 1\}$ is a sub Boolean algebra of $(B, *, \oplus, ', 0, 1)$.

Direct product of Boolean Algebras:

Let $(B_1, *_1, \oplus_1, ', 0_1, 1_1)$ and $(B_2, *_2, \oplus_2, '', 0_2, 1_2)$ be two Boolean algebras. Then the direct product of the two Boolean algebra is defined to be a Boolean algebra given by $(B_1 \times B_2, *_3, \oplus_3, ''', 0_3, 1_3)$ in which the operations are defined by

$$(a_1, b_1) *_3 (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$$

$$(a_1, b_1) \oplus_3 (a_2, b_2) = (a_1 \oplus_1 a_2, b_1 \oplus_2 b_2)$$

$$(a_1, b_1)''' = (a_1', b_1''), \quad 0_3 = (0_1, 0_2) \;\&$$

$$1_3 = (1_1, 1_2)$$

$\forall \; a_1, a_2 \in B_1 \;\&\; b_1, b_2 \in B_2$