① AUGUST 02 THURSDAY

Predicates :-

Consider the two statements,

John is a bachelor - ①

Smith is a bachelor - ②.

These two statements "is a bachelor" is called a Predicate.

The statement ① & ② denoted by symbolically as,

B : is a bachelor,

j : John

s : Smith.

then statement ① & ② written as

$B(j)$ and $B(s)$.

* Any statement of the type "p is Q" where Q is a predicate and p is the subject can be denoted by $Q(p)$.

This painting is red — ③

the statement symbolized as

R : " is red "
p : " This painting "

③ can be written as $R(p)$.

* " John is a ~~batch~~ bachelor and this painting is red".

which can be written as $B(j) \wedge R(p)$.

* Statements involving the names of two objects, such as.

Jack is taller than Jill — ④

Here " is taller than" is 2-place predicates.

G : " is taller than "

j_1 : Jack & j_2 : Jill

Important Notes

④ can be written as

$$G(j_1, j_2)$$

③

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and

Canada is to the north of the United States — ⑤

Here, " is to the north of " — is 2- place Predicates.

N : " is to the north of "

C : Canada & S - United States .

⑤ can be written as $N(c, s)$.

\# An n- place predicate requires n names of objects to be inserted in fixed positions in order to obtain a statement. The position of these names ~~of~~ is important.

If S is an n- place predicate letter and $a_1, a_2, \ldots a_n$ are the names of Objects then $S(a_1, a_2, \ldots a_n)$ is a statement.

The statement Function, Variables and Quantifiers.

Ex:

Let H is a predicate "is a mortal"

and b : Mr. Brown

c : Canada and s : A shirt.

Symbolized as $H(b)$, $H(c)$ and $H(s)$.

These statments have a common form

if we write $H(x)$ then $H(b)$, $H(c)$, $H(s)$.

* Use small letters as individual or object Variables.

A simple statement function:-

A Simple Statement function of one Variable is defined to be an expression consisting of a predicate Symbol and an individual Variable.

5

If let $M(x)$ is "x is a man" and $H(x)$ is "x is a mortal" then we can form compound statement function such as,

$M(x) \wedge H(x)$: x is a man and x is a mortal

(or) x is a man and a mortal

$M(x) \rightarrow H(x)$: If x is a man then x is a mortal.

$\neg H(x)$: x is not a mortal

$M(x) \vee \neg H(x)$: x is a man and x is not a mortal.

etc...

1. $G(x, y)$: x is taller than y

If both x and y are replaced by the names of objects, we get a statement.

m represents Mr. Miller and f Mr. Fox

Then we have

$G(m,f)$: Mr. Miller is taller than Mr. Fox.

$G(f,m)$: Mr. Fox is taller than Mr. Miller.

* $M(x)$: x is a man

$H(y)$: y is a mortal

Then we may write,

$M(x) \wedge H(y)$: x is a man and y is a mortal.

Universal Quantifier

Ex:-

* 1. All men are mortal

2. Every apple is red

3) Any integer is either positive or negative.

Important Notes

~~the~~ above these statement write in the following manner as,

1a. For all x, If x is man, then x is a mortal.

2a. For all x, if x is an apple then x is red.

3a. For all x, if x is an integer then x is either positive or negative.

we symbolize "For all x" by the symbol "$(\forall x)$" or "(x)".

let $M(x)$: x is man, $H(x)$: x is a mortal
$A(x)$: x is an apple $R(x)$: x is red
$N(x)$: x is an integer
$P(x)$: x is either positive or negative.

we write 1a, 2a & 3a as,

1b $(x)\,(M(x) \rightarrow H(x))$

2b $(x)\,(A(x) \rightarrow R(x))$

3b $(x)\,(N(x) \rightarrow P(x))$

(or)

1b $(\forall x)\,(M(x) \rightarrow H(x))$
2b $(\forall x)\,(A(x) \rightarrow R(x))$
3b $(\forall x)\,(N(x) \rightarrow P(x))$.

The symbol "(x)" or "$(\forall x)$" are called "Universal quantifiers".

Existential Quantifier

Ex:—

Consider the following statements

4. There exist a man
5. Some men are clever
6. Some real numbers are rational.

these statement expressed as

4a. There exist an x such that x is a man.

5a. There exist an x such that x is a man and x is clever

~~6b. There exist at least one x such that~~

6a. There exist an x suchat x is a real number and rational.

We symbolize "there is at least one x such that" or "there exists an x such that" or "for some x" by the symbol "$(\exists x)$".

The symbol "$\exists x$" is called the existential quantifier

let $M(x)$: x is a man
$C(x)$: x is clever
$R_1(x)$: x is a real number
$R_2(x)$: x is rational.

We write 4a, 5a and 6a as,

4b. $(\exists x)(M(x))$
5b. $(\exists x)(M(x) \wedge C(x))$
6b. $(\exists x)\ (R_1(x) \wedge R_2(x))$

Predicate Formulas:-

* $P(x_1, x_2, \ldots x_n)$ denotes an n-place predicate formula in which the capital letter P is an n-place predicate and $x_1, x_2, x_3 \ldots x_n$ are individual Variables.

* $P(x_1, x_2, \ldots x_n)$ will be called an atomic formula of predicate Calculus.

Example of atomic formulas :-

R, $Q(x)$, $P(x,y)$, $A(x,y,z)$, $P(a,y)$ and $A(x,a,x)$.

A Well-formed formula of Predicate Calculus

1. An atomic formula is a well-formed formula.
2. If A is a well-formed formula then $\neg A$ is a well-formed formula.
3. If A and B are well-formed formula then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \rightleftarrows B)$ are also well-formed formulas.
4. A is a well formed formula and x is any variable then $(x) A$ and $(\exists x) A$ are well-formed formulas
5. Only those formulas obtained by using rules ① to ④ are well-formed formulas.

Free and Bound Variables

Given a formula containing a part of the form $(x)\,P(x)$ or $(\exists x)\,P(x)$, such a part is called an <u>x-bound part of the formula</u>.

* Any occurance of x in an x-bound part of a formula is called a <u>bound occurance</u> of x,

* Any occurance of x or of any variable that is not a bound occurance is called <u>free occurance</u>.

* The formula $P(x)$ either in $(x)\,P(x)$ or in $(\exists x)\,P(x)$ is described as the <u>scope of the quantifier</u>.

②

Note :- The scope of a quantifier is the formula immediately following the ~~qua~~ quantifier.

Illustrations :-

1) $(x)\ P(x,y)$

$P(x,y)$ is the scope of the quantifier, x is Bound occurance and y is free occurance

2. $(x)\left(P(x) \to Q(x)\right)$

The scope of the universal quantifier is $P(x) \to Q(x)$. and

All occurances of x are bound.

3. $(x)\left(P(x) \to (\exists y)\ R(x,y)\right)$

The scope of (x) is $P(x) \to (\exists y) R(x,y)$

but, the scope of $(\exists y)$ is $R(x,y)$

All occurances of both x and y are bound occurances.

4. $(x)(P(x) \rightarrow R(x)) \vee (x)(P(x) \rightarrow Q(x))$

The scope of the first quantifier is $P(x) \rightarrow R(x)$, and

the scope of the second quantifier is $P(x) \rightarrow Q(x)$.

All occurances of x are bound occurances.

5. $(\exists x)\, P(x) \wedge Q(x)$.

The Scope of $(\exists x)$ is $P(x)$.

The last occurance of x in $Q(x)$ is free.

Example:-

1. Let $P(x)$: x is a person

$F(x,y)$: x is the father of y
$M(x,y)$: x is the mother of y.

Write the predicate " x is the father of the mother of y".

Soln:- To Symbolize the predicate, we name a person called z as the mother of y.

Now, we want to say that, x is the father of Z and Z the mother of y.

It is assumed that such a person Z exist.

$\therefore (\exists Z)(P(Z) \wedge F(x,Z) \wedge M(Z,y))$

Example 2:-

Symbolize the expression "

"All the world loves a lover"

Soln:-

let $P(x)$: x is a person

$L(x)$: x is a lover

$R(x,y)$: x ~~lovey~~ loves y.

$\therefore$ The required expression is

$(x)(P(x) \rightarrow (y)(P(y) \wedge L(y)) \rightarrow R(x,y)))$

Important Notes

The Universe of Discourse

* Some simplification can be introduced by limiting the class of individuals or object under consideration. This limitation means that the variables which are quantified stand for only those objects which are members of a particular class or set. Such a restricted class is called the Universe of discourse or the domain of individuals or the Universe.

For example:-

(i) If the discussion refers to human beings only then the universe of ~~discourse~~ discourse is the class of human beings.

(ii) In a elementary algebra or number theory, the Universe of discourse could be numbers (real, rational complex, etc).

Ex 1:– Symbolize the statement
" All men are giants ".

Soln:–

Let $G(x)$: x is a giant
$M(x)$: x is a man.

Symbolized as $(x)\,(M(x) \rightarrow G(x))$.

But, if we restrict the variable x to the universe which is the class of men, then the statement is

$$(x)\; G(x).$$

Ex 2:– Consider the statement, " Given any positive integer, there is a greater positive integer". — ①

Symbolize this with and without using the set of positive integers as the Universe of discourse.

~~soln:-~~
~~Case (i) Using Universe of discourse~~

Soln:-
Case (i) :- Let the variables x and y be restricted to the set of positive integers.

① can be written as,

For all x, there exists a y such that y is greater than x.

If $G(x,y)$: "x is greater than y" then the given statement is,

$(x)\ (\exists y)\ G(y,x)$.

Case (ii) :- We do not impose the restriction on the Universe of discourse,

If $P(x)$: "x is a positive integer"

① can be written as,

$(x)\left(P(x) \rightarrow (\exists y)\left(P(y) \wedge G(y,x)\right)\right)$

Example 3:-

Consider the Predicate

$Q(x)$: x is less than 5.

And the statements $(x)\,Q(x)$ and $(\exists x)\,Q(x)$.

If the Universe of discourse is given by the sets

1. $\{-1, 0, 1, 2, 4\}$
2. $\{3, -2, 7, 8, -2\}$
3. $\{15, 20, 24\}$.

Then $(x)\,Q(x)$ is true for the Universe of discourse ① and false for ② & ③.

The statement $(\exists x)\,Q(x)$ is true for both ① & ② but $(\exists x)\,Q(x)$ is false for ③.

Important Notes

Inference Theory of the Predicate Calculus

Valid Formulas and Equivalances

Formulas :-

$$\neg\neg A(x) \rightleftarrows A(x) \qquad E_1$$

$$C(x,y) \wedge B(x) \rightleftarrows B(x) \wedge C(x,y) \qquad E_2$$

$$A(x) \to B(x) \rightleftarrows \neg A(x) \vee B(x) \qquad E_{16}$$

Let $A(x)$, $B(x)$ and $C(x,y)$ denote any Prime formulas of the Predicate calculss.

* A substitution instance is one in which any variable in a formula is consistently replaced by any other formula throughout the statement.

* A Predicate formula is a "Prime formula" if no sentential connectives appear it.

②

Some Valid Formulas over Finite Universes

Let the Universe of discourse be denoted by a finite set S given by

$$S = \{a_1, a_2, \dots a_n\}$$

$$(x) A(x) \Leftrightarrow A(a_1) \wedge A(a_2) \wedge \dots \wedge A(a_n) \quad \text{— ①}$$

$$(\exists x) A(x) \Leftrightarrow A(a_1) \vee A(a_2) \vee \dots \vee A(a_2) \quad \text{— ②}$$

~~① & ② can be proved easily with~~

De' Morgan's Laws are,

$$\neg((x) A(x)) \Leftrightarrow (\exists x) \neg A(x) \quad \text{— ③}$$

$$\neg((\exists x) A(x)) \Leftrightarrow (x) \neg A(x) \quad \text{— ④}$$

Proof of ③ :-

$$\neg((x) A(x)) \Leftrightarrow \neg[A(a_1) \wedge A(a_2) \wedge \dots \wedge A(a_n)]$$

$$\Leftrightarrow \neg A(a_1) \vee \neg A(a_2) \vee \dots \vee \neg A(a_n)$$

$$\Leftrightarrow (\exists x) \neg A(x)$$

Proof of (4) :-

$$\neg((\exists x)\, A(x)) \Leftrightarrow \neg\, [A(a_1) \vee A(a_2) \vee \cdots \vee A(a_n)]$$
$$\Leftrightarrow \neg A(a_1) \wedge \neg A(a_2) \wedge \cdots \wedge \neg A(a_n)$$
$$\Leftrightarrow (x)\, \neg A(x).$$

Example 1:- Negate the following statements

(a) Ottawa is a small town

(b) Every city in Canada is clean.

Soln :-

(a) It is not the case that Ottawa is a small town.

(or)

Ottawa is not a small town.

(b) It is not the case that every city in canada is clean.

(or)

Not every city in Canada is clean.

(or)

Some city in canada is not clean.

1-6.4 Theory of Inference for The Predicate Calculus

The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulas involving quantifiers. The rules **P** and **T**, regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulas already introduced, remain the same. If the conclusion is given in the form of a conditional, we shall also use the rule of conditional proof called **CP**. Occasionally, we may use the indirect method of proof in introducing the negation of the conclusion as an additional premise in order to arrive at a contradiction.

The equivalences and implications of the statement calculus can be used in the process of derivation as before, except that the formulas involved are generalized to predicates. But these formulas do not have any quantifiers in them, while some of the premises or the conclusion may be quantified. In order to use the equivalences and implications, we need some rules on how to eliminate quantifiers during the course of derivation. This elimination is done by *rules of specification* called rules **US** and **ES**. Once the quantifiers are eliminated, the derivation proceeds as in the case of the statement calculus, and the conclusion is reached. It may happen that the desired conclusion is quantified. In this case, we need *rules of generalization* called rules **UG** and **EG**, which can be used to attach a quantifier.

The rules of generalization and specification follow. Here $A(x)$ is used to denote a formula with a free occurrence of x. $A(y)$ denotes a formula obtained by the substitution of y for x in $A(x)$. Recall that for such a substitution $A(x)$ must be free for y.

Rule **US** (Universal Specification) From $(x)A(x)$ one can conclude $A(y)$.

Rule **ES** (Existential Specification) From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time **ES** is used. (The conditions of **ES** are more restrictive than ordinarily required, but they do not affect the possibility of deriving any conclusion.)

Rule **EG** (Existential Generalization) From $A(x)$ one can conclude $(\exists y)A(y)$.

Rule **UG** (Universal Generalization) From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of **ES**, then no variables introduced by that use of **ES** appear free in $A(x)$.

We shall now show, by means of an example, how an invalid conclusion could be arrived at if the second restriction on rule **UG** were not imposed. The other restrictions on **ES** and **UG** are easy to understand.

Let $D(u, v)$: u is divisible by v. Assume that the universe of discourse is $\{5, 7, 10, 11\}$, so that the statement $(\exists u)D(u, 5)$ is true because both $D(5, 5)$

and $D(10, 5)$ are true. On the other hand, $(y)D(y, 5)$ is false because $D(7, 5)$ and $D(11, 5)$ are false. Consider now the following derivation.

{1}	(1)	$(\exists u)D(u, 5)$	**P**
{1}	(2)	$D(x, 5)$	**ES**, (1)
{1}	(3)	$(y)D(y, 5)$	**UG**, (2) (neglecting second restriction)

In step 3 we have obtained from $D(x, 5)$ the conclusion $(y)D(y, 5)$. Obviously x is not free in the premise, and so the first restriction is satisfied. But x is free in step 2 which resulted by use of **ES**, and that x has been introduced by use of **ES** and appears free in $D(x, 5)$; hence it cannot be generalized. This is the reason why we obtained a false conclusion from a true premise.

We now give several examples with comments to explain the method of derivation. In the first two examples we use the principles **UG** and **US**, but not **EG** and **ES**.

EXAMPLE 1 Show that $(x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as the "Socrates argument" which is given by:

All men are mortal.

Socrates is a man.

Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal, and s: Socrates, we can put the argument in the above form.

SOLUTION

{1}	(1)	$(x)(H(x) \rightarrow M(x))$	**P**
{1}	(2)	$H(s) \rightarrow M(s)$	**US**, (1)
{3}	(3)	$H(s)$	**P**
{1, 3}	(4)	$M(s)$	**T**, (2), (3), I_{11}

Note that in step 2 first we remove the universal quantifier. ////

EXAMPLE 2 Show that

$$(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$$

SOLUTION

{1}	(1)	$(x)(P(x) \rightarrow Q(x))$	**P**
{1}	(2)	$P(y) \rightarrow Q(y)$	**US**, (1)
{3}	(3)	$(x)(Q(x) \rightarrow R(x))$	**P**
{3}	(4)	$Q(y) \rightarrow R(y)$	**US**, (3)
{1, 3}	(5)	$P(y) \rightarrow R(y)$	**T**, (2), (4), I_{13}
{1, 3}	(6)	$(x)(P(x) \rightarrow R(x))$	**UG**, (5)

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EXAMPLE 3 Show that $(\exists x)M(x)$ follows logically from the premises

$$(x)(H(x) \to M(x)) \qquad \text{and} \qquad (\exists x)H(x)$$

SOLUTION

{1}	(1)	$(\exists x)H(x)$	**P**
{1}	(2)	$H(y)$	**ES**, (1)
{3}	(3)	$(x)(H(x) \to M(x))$	**P**
{3}	(4)	$H(y) \to M(y)$	**US**, (3)
{1, 3}	(5)	$M(y)$	**T**, (2), (4), I_{11}
{1, 3}	(6)	$(\exists x)M(x)$	**EG**, (5)

Note that in step 2 the variable y is introduced by **ES**. Therefore a conclusion such as $(x)M(x)$ could not follow from step 5 because it would violate the rules given for **UG**. ///

EXAMPLE 4 Prove that

$$(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$$

SOLUTION

{1}	(1)	$(\exists x)(P(x) \wedge Q(x))$	**P**
{1}	(2)	$P(y) \wedge Q(y)$	**ES**, (1), y fixed
{1}	(3)	$P(y)$	**T**, (2), I_1
{1}	(4)	$Q(y)$	**T**, (2), I_2
{1}	(5)	$(\exists x)P(x)$	**EG**, (3)
{1}	(6)	$(\exists x)Q(x)$	**EG**, (4)
{1}	(7)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	**T**, (4), (5), I_9

///

It is instructive to try to prove the converse which does not hold. The derivation is

(1)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	**P**
(2)	$(\exists x)P(x)$	**T**, (1), I_1
(3)	$(\exists x)Q(x)$	**T**, (1), I_2
(4)	$P(y)$	**ES**, (2)
(5)	$Q(z)$	**ES**, (3)

Note that in step 4, y is fixed, and it is no longer possible to use that variable again in step 5.

EXAMPLE 5 Show that from

(*a*) $(\exists x)(F(x) \wedge S(x)) \to (y)(M(y) \to W(y))$
(*b*) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \to \neg S(x))$ follows.

SOLUTION

{1}	(1)	$(\exists y)(M(y) \wedge \neg W(y))$	P
{1}	(2)	$M(z) \wedge \neg W(z)$	ES, (1)
{1}	(3)	$\neg(M(z) \to W(z))$	T, (2), E_{17}
{1}	(4)	$(\exists y) \neg(M(y) \to W(y))$	EG, (3)
{1}	(5)	$\neg(y)(M(y) \to W(y))$	E_{26}, (4)
{6}	(6)	$(\exists x)(F(x) \wedge S(x)) \to (y)(M(y) \to W(y))$	P
{1, 6}	(7)	$\neg(\exists x)(F(x) \wedge S(x))$	T, (5), (6), I_{12}
{1, 6}	(8)	$(x) \neg(F(x) \wedge S(x))$	T, (7), E_{25}
{1, 6}	(9)	$\neg(F(x) \wedge S(x))$	US, (8)
{1, 6}	(10)	$F(x) \to \neg S(x)$	T, (9), E_9, E_{16}, E_{17}
{1, 6}	(11)	$(x)(F(x) \to \neg S(x))$	UG, (10)

EXAMPLE 6 Show that

$$(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$$

SOLUTION We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	P (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	T, (1), E_9
{1}	(3)	$\neg(x)P(x)$	T, (2), I_1
{1}	(4)	$(\exists x)\neg P(x)$	T, (3), E_{26}
{1}	(5)	$\neg(\exists x)Q(x)$	T, (2), I_2
{1}	(6)	$(x)\neg Q(x)$	T, (5), E_{25}
{1}	(7)	$\neg P(y)$	ES, (4)
{1}	(8)	$\neg Q(y)$	US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	T, (7), (8), I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	T, (9), E_9
{11}	(11)	$(x)(P(x) \vee Q(x))$	P
{11}	(12)	$P(y) \vee Q(y)$	US, (11)
{1, 11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	T, (10), (12), I_9 contradiction

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1-6.5 Formulas Involving More Than One Quantifier

So far we have considered only those formulas in which the universal and existential quantifiers appear singly. We shall now consider cases in which the quanti-