UNIT - 5

The transformation $w = \sin z$.

Since $\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$,

the transformation $w = \sin z$ can be written

$$\left.\begin{aligned} u &= \sin x \cosh y \\ v &= \cos x \sinh y \end{aligned}\right\} \quad \text{——— (1).}$$

Images of the vertical lines $x = c_1$, $0 < c_1 < \pi/2$

when $x = c_1$, $0 < c_1 < \pi/2$.

$u = \sin c_1 \cosh y$, $v = \cos c_1 \sinh y$ $(-\infty < y < \infty)$ ——— (2)

which is the right-hand branch of the hyperbola

$$\Rightarrow \frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1. \quad \text{——— (3).}$$

~~which is a hyperbola~~, with foci at

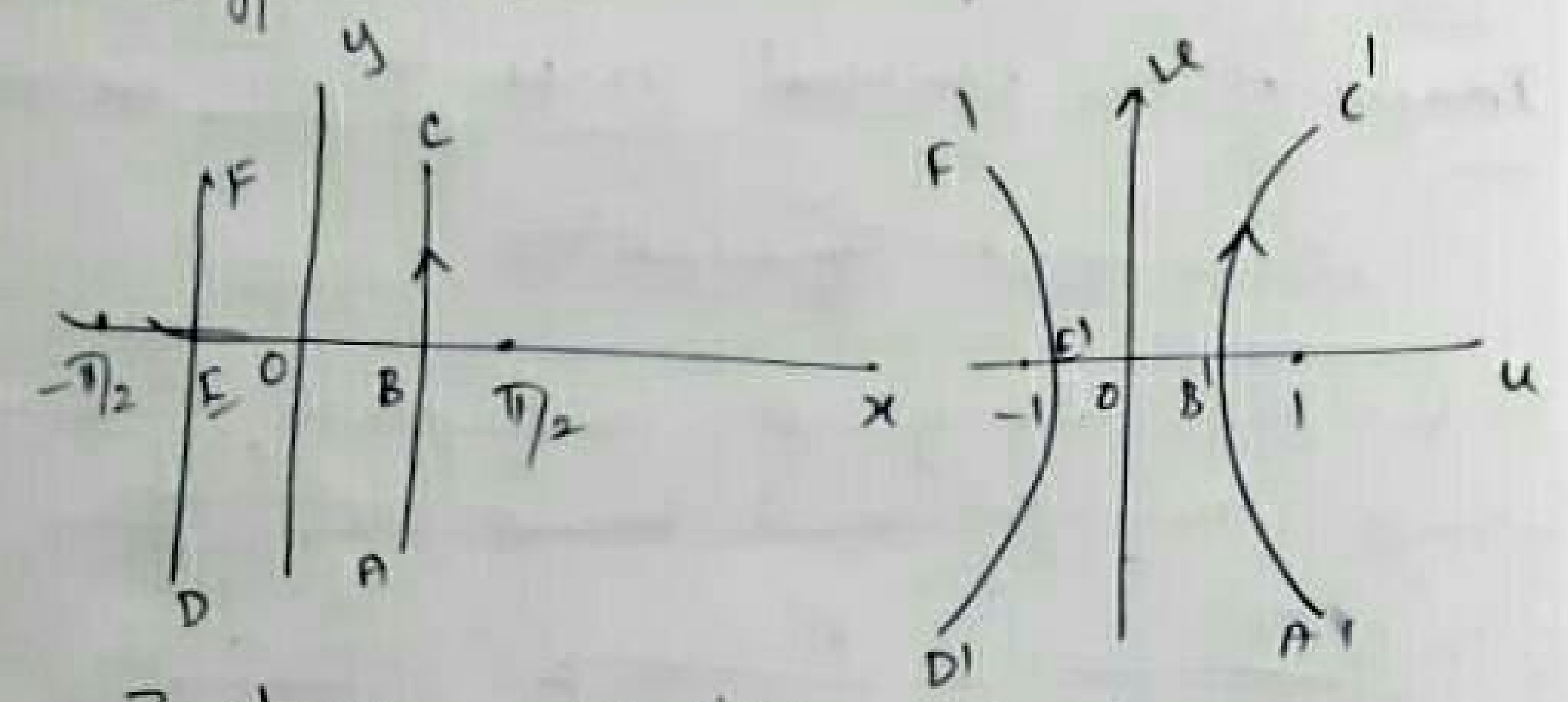
the points,

$$w = \pm\sqrt{\sin^2 c_1 + \cos^2 c_1} = \pm 1.$$

Also ~~When~~ $x = c_1$, $-\pi/2 < c_1 < 0$, is mapped onto the left-hand branch of the same hyperbola (3)

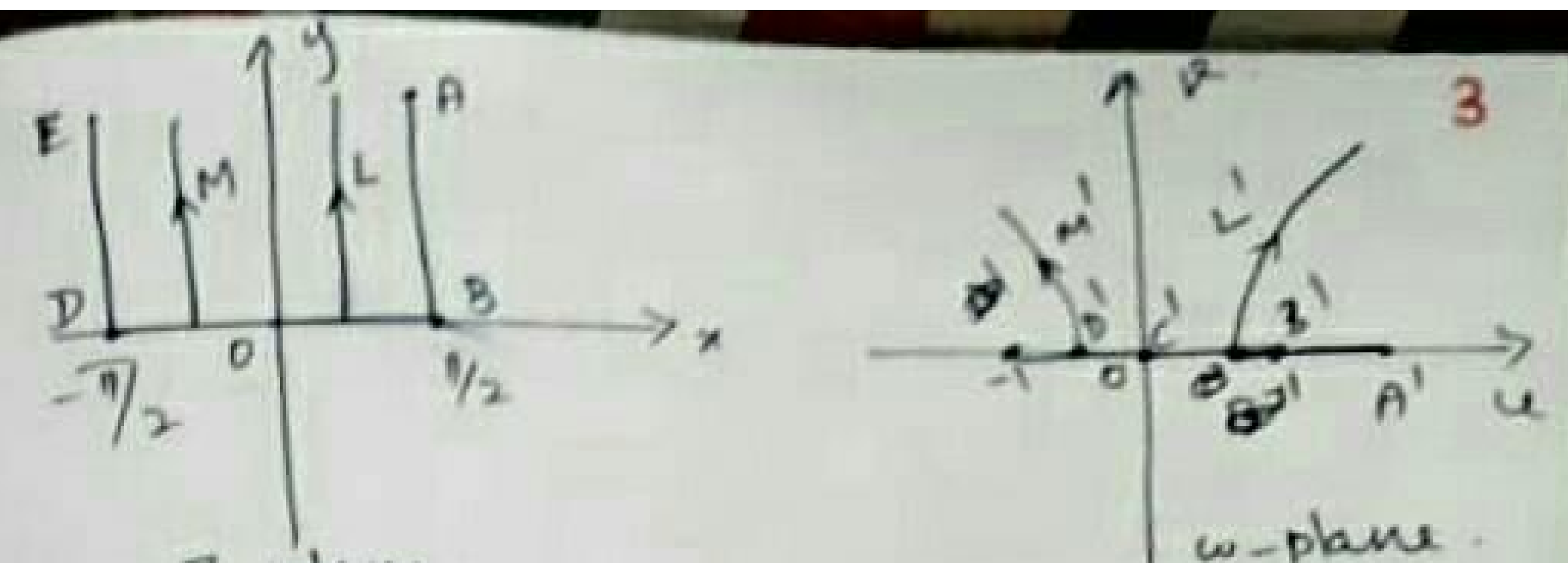
it shows that

From (2), when a point (c_1, y) moves upward along the entire line $x = c_1$, its image moves upward along the entire length of the hyperbola's branch



z-plane $\quad w = \sin z \quad$ w-plane

$w = \sin z$ transforms the semi infinite strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ in the z-plane onto the upper half $v \ge 0$ of the w-plane in the one-one manner.



z-plane $\quad$ w-plane

First we show that the boundary of the strip is mapped onto the real axis in the w-plane in a 1-1 manner as shown in the figure.

Consider the line segment BA in $x = \pi/2$, $y \ge 0$.

$\therefore$ (1) $\Rightarrow$ $u = \cosh y$, $v = 0$.

Hence a typical point $(\pi/2, y)$ on BA is mapped onto the point $(\cosh y, 0)$ in the w-plane

When $(\pi/2, y)$ moves upward along BA, its image $(\cosh y, 0)$ moves to the right from B' along u axis in the w-plane.

Consider, $y = 0$, $-\pi/2 < y < \pi/2$

Then $u = \sin x$

$v = 0$

$\therefore$ A point $(x, 0)$ on the horizontal segment DB has image $(\sin x, 0)$, which moves right from D' to B' as x increases from $x = -\pi/2$ to $x = \pi/2$ (or) $(x, 0)$ moves from D to B.

Finally, consider the line segment DE.

~~(i) $x = c_1$, $\pi/2 \leq c_1$~~

(ii) $x = -\pi/2$, $y \geq 0$.

$\Rightarrow u = -\cosh y$, $v = 0$.

$\therefore (-\pi/2, y)$ on DE is mapped onto $(-\cosh y, 0)$

When $(-\pi/2, y)$ moves upward from D, its image $(-\cosh y, 0)$ moves to the left from D'.

Now each point in the interior $-\pi/2 < x < \pi/2$, $y > 0$ of the strip lies on one of the vertical lines $x = c_1$, $y > 0$, $-\pi/2 < c_1 < \pi/2$. The images of these lines constitute the entire half plane $v > 0$.

Hence the strip $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$ is mapped onto the half plane $v \geq 0$.

The image of the horizontal line segments $y = c_2$ $(-\pi \leq x \leq \pi)$, $c_2 > 0$

When $y = c_2$, $-\pi \leq x \leq \pi$, $c_2 > 0$,

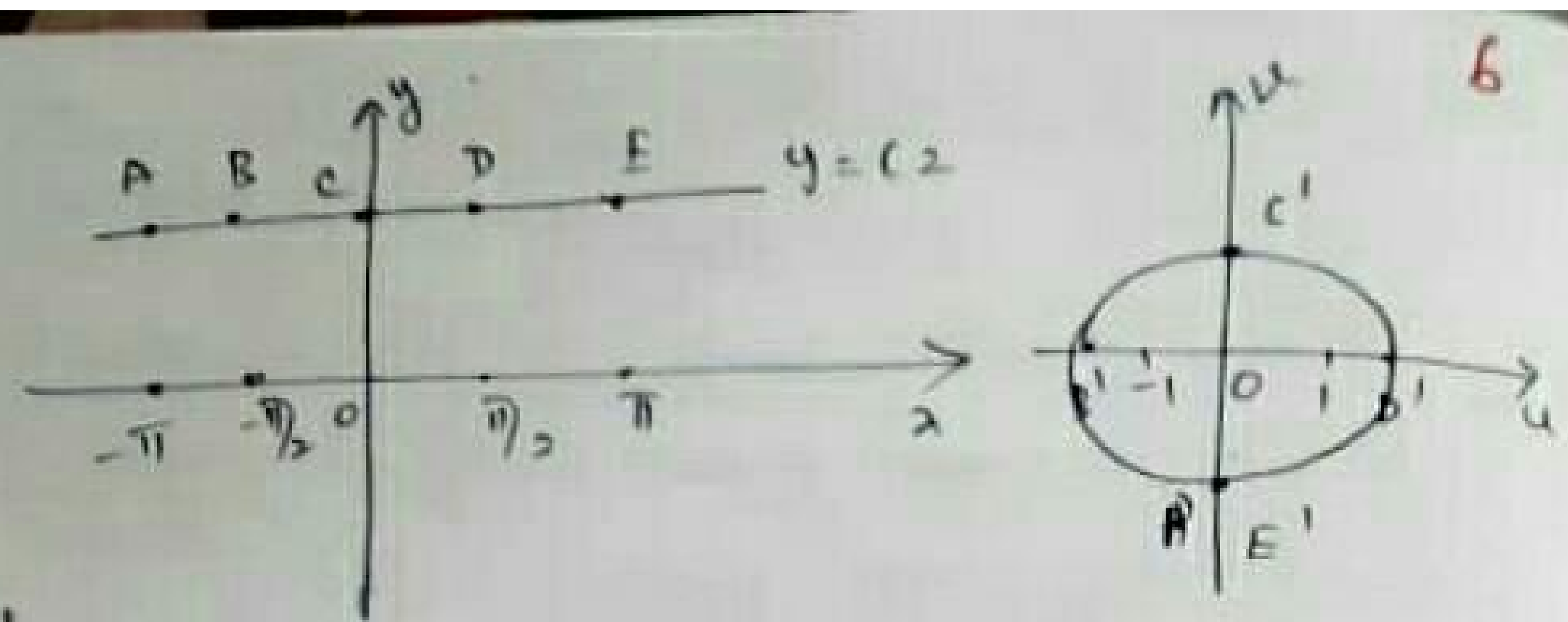
Then (1) $\Rightarrow$

$u = \sin x \cosh c_2$, $v = \cos x \sinh c_2$, $-\pi \leq x \leq \pi$ — (4)

$$\Rightarrow \frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1 \quad \text{—— (5)}$$

which is an ellipse with foci

$w = \pm\sqrt{\cosh^2 c_2 - \sinh^2 c_2} = \pm 1.$



Pf

The point (x, c_2) moves right from A to E, its image ~~...~~ makes one circuit around the ellipse in the clockwise direction.

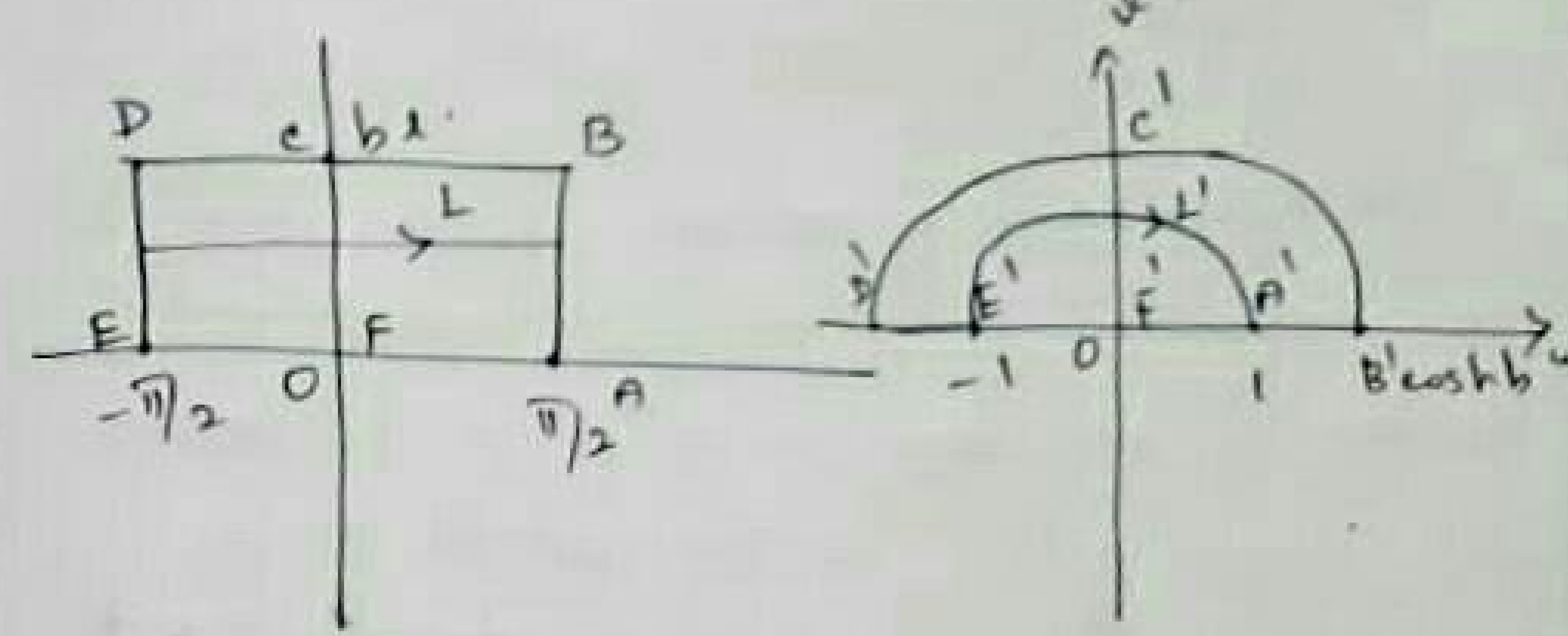
Smaller values of c_2 are taken, the ellipse becomes smaller with the same foci $(\pm 1, 0)$

When $c_2 = 0$,

$u = \sin x,\ v = 0,\ -\pi \le x \le \pi$

(ie) The ~~...~~ interval $[-\pi, \pi]$ of the x-axis is mapped onto the interval $-1 \le u \le 1$ of the u-axis.

The transformation $w = \sin z$ maps the rectangular region $-\pi/2 \le x \le \pi/2$, $0 \le y \le b$ onto the semielliptical region.



If L is a line segment $y = c_2$, $-\pi/2 \le x \le \pi/2$ where $0 < c_2 \le b$, its image L' is the top half of the ellipse,

$$\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1.$$

If c_2 decreases, L moves downward toward the x-axis and the semi ellipse L' also moves downward and tends to become the line segment E'F'A' from $w = -1$ to $w = 1$.

When $c_2 = 0$, $u = \sin x$, $v = 0$, $-\pi/2 \le x \le \pi/2$

(ie) EFA is ma

CONFORMAL MAPPING

Defn:- A transformation $w = f(z)$ is said to be conformal at a point z_0, if f is analytic there and $f'(z_0) \neq 0$.

A transformation $w = f(z)$ is said to be conformal in a domain, if it is conformal at each point in D. That is, the mapping f is conformal in D if f is analytic in D and its derivative f' has no zeros there.

Ex:- The mapping $w = e^z$ is conformal throughout the entire z-plane.

~~let $w = f(z) = e^z$.~~

~~since f(z) is an~~

Since e^z is an entire function, and $(e^z)' = e^z \neq 0$ for each z, $w = e^z$ is conformal.

Defn:- A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an isogonal mapping

Ex:- The transformation $w = \bar{z}$, which is a reflection in the real axis is isogonal but not conformal.

Defn:- Suppose that f is not a constant function and is analytic at a point z_0. If in addition $f'(z_0) = 0$, then z_0 is called a critical point of the transformation $w = f(z)$

Ex:- consider the transformation, $w = f(z)$ where $f(z) = 1 + z^2$.

Then $z = 0$ is a critical point of $w = f(z)$

For, $f(z)$ is analytic at $z = 0$ and $f'(0) = 2 \times 0 = 0$.

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Defn:-

Suppose the transformation $w = f(z)$ is conformal at a point z_0.

Then $|f'(z_0)|$ is called the scale factor of f at z_0 and $\arg[f'(z_0)]$ is called the angle of rotation at z_0.

Ex:-

Find the scale factor and angle of rotation of $f(z) = z^2$ at the point $z = 1+i$.

Soln:-

$f(z) = z^2$

$f'(z) = 2z$.

$z_0 = 1+i$.

$f'(z_0) = 2(1+i)$

Scale factor $= |f'(z_0)| = 2\sqrt{2}$

The angle of rotation is

$\arg[f'(z_0)] = \arg(2(1+i))$

$= \pi/4 + 2n\pi$, $n = 0, \pm1, \pm2, \ldots$

UNIT. I

2. M. Q

1) Define ε-neighborhood, deleted neighborhood, interior point, exterior point, boundary point, open set, closed set, closure of a set, connected set, Domain, Region, bounded set, accumulation point.

2) Define limit of a function,

3) P.T $\lim_{z \to 1} f(z) = i/2$, where $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$.

4) P.T $\lim_{z \to 0} f(z)$ does not exist, where $f(z) = \frac{z}{\bar{z}}$.

5) Define Riemann sphere. Stereographic projection, neighborhood of ∞.

6) Define continuity of a function.