27/8/19

Unit - II.

Integrals.

Derivative of functions $w(t)$.

Consider, the derivatives of complex-valued functions w of a real variable t we write

$$w(t) = u(t) + iv(t) \quad -① $$

where the function u and v are real-valued function of t.

The derivative $w'(t)$ (or) $\frac{d}{dt} w(t)$ of the function ① at a point t is defined as

$$w'(t) = u'(t) + iv'(t) \quad -②,$$

provided each of the derivaties u' and v' exists at t.

For every complex constant $z_0 = x_0 + iy_0$.

From definition ② it follows that

$$\frac{d}{dt}[z_0(w(t))] = [(x_0 + iy_0)(u(t) + iv(t))]'$$

$$= [(x_0 u + i x_0 v + i y_0 u - y_0 v)]'$$

$$= [(x_0 u - y_0 v) + i [x_0 v + y_0 u]]'$$

$$= (x_0 u - y_0 v)' + i\,(x_0 v + y_0 u)'$$

$$-1 = +i^2$$

$$= (x_0 u' - y_0 v') + i\,(y_0 u' + x_0 v').$$

$$= u'(x_0 + i y_0) + i\,(x_0 + i y_0) v'$$

$$= (u' + i v')(x_0 + i y_0)$$

$$\frac{d}{dt}[z_0 w(t)] = z_0 \cdot w'(t) \quad \text{— ③.}$$

Also,

$$\frac{d}{dt}[e^{z_0 t}] = z_0 \cdot e^{z_0 t} \quad \text{— ④}$$

To prove this,

$$\text{consider, } e^{z_0 t} = e^{x_0 t} \cdot e^{i y_0 t}$$

$$\therefore e^{z_0 t} = e^{x_0 t}(\cos y_0 t + i \sin y_0 t)$$

$$= e^{x_0 t}\cos y_0 t + i e^{x_0 t} \sin y_0 t.$$

From defintion ②,

$$\frac{d}{dt}[e^{z_0 t}] = (e^{x_0 t} \cos y_0 t)' + i\,(e^{x_0 t} \sin y_0 t)'.$$

$$= (-e^{x_0 t} \sin y_0 t \cdot y_0) + (x_0 e^{x_0 t} \cos y_0 t)$$

$$+ i\,(e^{x_0 t} \cos y_0 t \cdot y_0)$$

$$+ i\,(x_0 e^{x_0 t} \cancel{\cos}\ \sin y_0 t).$$

$$= x_0 (e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t)$$

$$+ i y_0 (e^{x_0 t} \cos y_0 t + i \sin y_0 t \cdot e^{x_0 t}$$

$$= e^{x_0 t} \cdot e^{i y_0 t} (x_0 + i y_0)$$

$$\frac{d}{dt}\left[e^{z_0 t}\right] = z_0 \cdot e^{z_0 t}.$$

Example:

consider the function $w(t) = e^{it}$ on the interval $0 \leq t \leq 2\pi$.

Solu

$$w(t) = e^{it}$$

$$\therefore w'(t) = ie^{it}$$

$$\Rightarrow |w'(t)| = 1.$$

$\therefore w'(t) \neq 0$. [$w'(t)$ never belongs to 0]

But $w(2\pi) - w(0) = 0$

Here, $w(t)$ is continuous in $0 \leq t \leq 2\pi$ and $w'(t)$ exist in $0 < t < 2\pi$.

But $w'(t) \neq \dfrac{w(2\pi) - w(0)}{2\pi}$ for any $t \in (0, 2\pi)$.

$\therefore$ Mean-valued theorem ~~is the~~ for derivatives is not satisfied.

Hence, mean-value theorem for derivatives is not neccesarily true for complex-value function.

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Definite Integrals of Functions w(t)

Consider, the complex-valued function of a real variable t and is written as

$$w(t) = u(t) + i\,v(t) \quad \text{--- (1)}$$

where u and v are real-valued functions.

The definite integral of $w(t)$ in the interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \quad \text{--- (2)},$$

Provided the individual integrals on the right exists. $\left[\int_a^b u(t)dt \text{ exist and } \int_a^b v(t)dt \text{ exist}\right]$

Thus,

$$\operatorname{Re}\left[\int_a^b w(t)\,dt\right] = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and}$$

$$\operatorname{Im}\left[\int_a^b w(t)\,dt\right] = \int_a^b \operatorname{Im}[w(t)]\,dt \quad \text{--- (3)}$$

Example 1:

Prove that, $\int_0^1 (1+it)^2\,dt = \frac{2}{3} + i.$

Soln:

Let $w(t) = (1+it)^2 = 1 - t^2 + 2it.$

W.K.T $\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$

where $u(t) = 1 - t^2$ and $v(t) = 2t$

$$\therefore \int_0^1 w(t)\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^1 + i\left[\frac{2t^2}{2}\right]_0^1$$

$$= \left[1 - \frac{1}{3} - 0 + 0\right] + i\,[1-0]$$

$$= \left[\frac{3-1}{3}\right] + i\,(1)$$

$$\therefore \int_0^1 (1+it)^2\,dt = \frac{2}{3} + i.$$

Hence, proved.

Notes: Fundamental theorem of calculus.

Let $f(x)$ and $F(x)$ are continuous function on the interval $a \le x \le b$ and $F'(x) = f(x)$.

$$\Rightarrow \int_a^b f(x) = \int_a^b F'(x) = F(x)\Big]_a^b$$

$$= F(b) - F(a).$$

Mean-Value theorem.

Let $f(x)$ be the function continuous on the interval $a \le x \le b$ and $f'(x)$ exist on the interval $a < x < b$.

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$

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Result 1:

$$\Rightarrow \int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$$

Result 2:

Fundamental theorem of calculus.

Suppose that $w(t) = u(t) + i\,v(t)$ and $W(t) = U(t) + iV(t)$ are continuous on the interval $a \le t \le b$.

If $W'(t) = w(t)$ when $a \le t \le b$.

Then $U'(t) = u(t)$ and $V'(t) = v(t)$.

Hence, By definition ②,

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$$= \int_a^b U'(t)\,dt + i\int_a^b V'(t)\,dt$$

$$= [U(t)]_a^b + i\,[V(t)]_a^b$$

$$= U(b) - U(a) + i\,V(b) - i\,V(a)$$

$$= [U(b) + iV(b)] - [U(a) - iV(a)]$$

$$= W(b) - W(a).$$

$$\int_a^b w(t)\,dt = [W(t)]_a^b$$

Hence, fundamental theorem of calculus is true for complex-valued function.

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Example 2:

ⓐ prove that $\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = e^{it}$.

$\left\langle \frac{d}{dt}(e^{z_0 t}) = z_0 \cdot e^{z_0 t}\right.$

Solu.

$$\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{\cancel{i}}(\cancel{i} \cdot e^{it})$$

$$\therefore \frac{d}{dt}\left(\frac{e^{it}}{i}\right) = e^{it}$$

ⓑ Find $\int_0^{\pi/4} e^{it}\,dt$

Solu.

we know that, $e^{it} = \frac{d}{dt}\left(\frac{e^{it}}{i}\right)$.

using Fundamental theorem.

$$\int_0^{\pi/4} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/4}$$

$$= \left[\frac{\cos t + i\sin t}{i}\right]_0^{\pi/4}$$

$$= \frac{\cos\frac{\pi}{4} + i\sin\pi/4}{i} - \frac{\cos 0 + i\sin 0}{i}$$

$$= \frac{1}{i}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} - 1 + i(0)\right]$$

$$= \frac{1}{i}\left(\frac{1}{\sqrt{2}}\right) + \frac{\cancel{i}}{\cancel{i}}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{i}.$$

$\left\langle \frac{1}{i} = -i\right.$

$$= \frac{1}{\sqrt{2}} + \frac{1}{i}\left[\frac{1}{\sqrt{2}} - 1\right]$$

$$\int_0^{\pi/4} e^{it}\,dt = \frac{1}{2} + i\left[1 - \frac{1}{\sqrt{2}}\right]$$

Example 3:

Let $w(t)$ be a continuous complex-valued function of t defined on an interval $a \le t \le b$. In order to show that its not necessarily true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)\,(b-a).$$

For example.

$a = 0$; $b = 2\pi$ and consider the function, $w(t) = e^{it}$ $(0 \le t \le 2\pi)$.

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt$$

$$= \left[\frac{e^{it}}{i}\right]_0^{2\pi} = 0$$

$$\therefore \int_0^{2\pi} e^{it}\,dt = 0.$$

But, for any c such that $0 < c < 2\pi$,

$$|w(c)\,(b-a)| = |w(c)\,(2\pi - 0)|$$

$$= |e^{ic}.(2\pi)|$$

$$= |e^{ic}| \cdot 2\pi = 1(2\pi)$$

$$\therefore |w(c)\,(b-a)| = 2\pi$$

Hence, $|w(c)\,(2\pi - 0)|$ is never zero

$\therefore$ The mean value theorem for integral is not necessarily true for complex.

Exercise:

1) Use rules in calculus to establish the following rules when

$w(t) = u(t) + iv(t)$ is a complex-valued function of a real variable t and $w'(t)$ exists:

(a) $\frac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

(b) $\frac{d}{dt} [w(t)]^2 = 2w(t)\, w'(t)$.

Sol:

we know that, $w(t) = u(t) + iv(t)$ and

$$w'(t) = u'(t) + iv'(t).$$

where $u(t)$ and $v(t)$ are real valued function of a real variable.

(a) $w(-t) = u(-t) + iv(-t)$.

$\therefore \frac{d}{dt} u(-t) = -u'(-t)$ and $\frac{d}{dt} v(-t) = -v'(-t)$.

$$\therefore \frac{d}{dt} w(-t) = -u'(-t) - iv'(-t)$$
$$= -(u'(-t) + iv'(-t))$$
$$= -(w'(-t)).$$

Hence, $\frac{d}{dt} w(-t) = -w'(-t)$.

(b) $\frac{d}{dt}[w(t)]^2$

$$[w(t)]^2 = (u(t)+iv(t))^2$$

$$= (u(t))^2 + i^2 (v(t))^2 + 2i\, v(t)\, u(t).$$

$$= (u(t))^2 - (v(t))^2 + 2i\, v(t)\, u(t)$$

$$= u^2 - v^2 + 2iuv$$

$$\frac{d}{dt}[w(t)]^2 = (u^2-v^2)' + 2i\,(uv)'.$$

$$= 2uu' - 2vv' + 2i\, u'v + 2iv'u.$$

$$= 2u'(u+iv) + 2iv'(u+iv)$$

$$= 2(u'+iv')(u+iv)$$

$$\frac{d}{dt}[w(t)]^2 = 2w(t)\, w'(t)$$. Hence, proved.

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2) Evaluate the following integrals:

a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

Solu

$$\left(\frac{1}{t} - i\right)^2 = \left(\frac{1}{t}\right)^2 - (i)^2 - 2i\left(\frac{1}{t}\right)$$

$$= \frac{1}{t^2} + 1 - 2i\left(\frac{1}{t}\right).$$

Let $w(t) = \left(\frac{1}{t} - i\right)^2$ and $u(t) = \left(\frac{1}{t^2}+1\right)$; $v(t) = \frac{-2}{t}$.

we know that $\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$

$$\therefore \int_1^2 (1/t - i)^2\, dt = \int_1^2 \left(\frac{1}{t^2}+1\right) dt + i\int_1^2 \frac{-2}{t}\, dt.$$

$$= \left[t - \frac{1}{t}\right]_1^2 - 2i\left[\log t\right]_1^2$$

$$= \left[2 - \frac{1}{2} - 1 + 1\right] - 2i\,(\log 2 - \log 1)$$

$$= \left(2 - \frac{1}{2}\right) - 2i \log 2$$

$$= \frac{3}{2} - i \log 2^2$$

$$\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \frac{3}{2} - i \log 4.$$

b) $\displaystyle\int_0^{\pi/6} e^{i2t}\, dt.$

Soln

$$e^{i2t} = \cos 2t + i \sin 2t$$

we know that $\displaystyle\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$

where $w(t) = e^{i2t}$; $u(t) = \cos 2t$; $v(t) = \sin 2t$.

$$\int_0^{\pi/6} e^{i2t}\, dt = \int_0^{\pi/6} \cos 2t\, dt + i\int_0^{\pi/6} \sin 2t\, dt.$$

$$= \left[\sin 2t\right]_0^{\pi/6} + i\left[-\cos 2t\right]_0^{\pi/6}$$

$$= \left(\sin 2(\pi/6) - \sin 0\right) + i\left[-\cos 2(\pi/6) + \cos 0\right]$$

$$= \frac{\sqrt{3}}{2} + i\left(-\frac{1}{2} + 1\right)$$

$$\therefore \int_0^{\pi/6} e^{i2t}\, dt = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

(c) $\int_0^{\infty} e^{-zt}\, dt \quad (\mathrm{Re}\, z>0).$

Solu

$$\int_0^{\infty} e^{-zt}\, dt = \left[\frac{-e^{-zt}}{z}\right]_0^{\infty} \qquad \left\langle \frac{d}{dt}\left(\frac{e^{it}}{i}\right) = e^{it}\right\rangle$$

$$= \frac{-1}{z}\left[\cos zt - i\sin zt\right]_0^{\infty}$$

$$= \frac{-1}{z}\left[e^{-z(\infty)} - e^{-z(0)}\right] = \frac{-1}{z}[0-1]$$

$$\int_0^{\infty} e^{-zt}\, dt = \frac{1}{z}.$$

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3) Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} \cdot e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when } m \neq n \\ 2\pi & \text{when } m = n \end{cases}$$

Solu

when $m \neq n$.

$$\int_0^{2\pi} e^{im\theta - in\theta}\, d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta$$

$$= \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left[\cos(m-n)\theta + i\sin(m-n)\theta\right]_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left[\cos(m-n)(2\pi) + i\sin(m-n)2\pi - \cos 0 - i\sin 0\right]$$

$$\int_0^{2\pi} e^{im\theta - in\theta}\, d\theta = \frac{1}{i(m-n)}[1-1] = 0$$

when $m = n$.

$$\int_0^{2\pi} e^{im\theta - in\theta}\, d\theta = \int_0^{2\pi} e^{0}\, d\theta = [\theta]_0^{2\pi}$$

$$\therefore \int_0^{2\pi} e^{im\theta - in\theta}\, d\theta = 2\pi.$$

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5) b) Show that if $w(t)$ is an odd function, that is $w(-t) = -w(t)$ for each point t in the given interval then $\int_{-a}^{a} w(t)\,dt = 0$.

Soln

$w(t)$ is odd. function.

where $w(-t) = -w(t)$.

Given $w(t) = u(t) + iv(t)$ [$u(t), v(t)$ are real numbers]

$w(-t) = u(-t) + iv(-t)$ — ①

$w(-t) = -w(+t) = -u(t) - iv(t)$. — ②

Equating the real and imaginary axis of ① and ②,

$u(-t) = -u(t)$ and $v(-t) = -v(t)$.

where t is odd, we know that

$\int_{-a}^{a} u(t)\,dt = 0$ and $\int_{-a}^{a} v(t)\,dt = 0$.

Hence, $\int_{-a}^{a} w(t)\,dt = \int_{-a}^{a} u(t)\,dt + i\int_{-a}^{a} v(t)\,dt$

$= 0$.

Thus, proved.

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Contours:

The set of points $z = (x, y)$ in the complex plane is said to be an arc if

$x = x(t)$; $y = y(t)$ $(a \le t \le b)$ — ①

where $x(t)$ and $y(t)$ are continuous.

(ie) An arc is a continuous mapping of an interval $a \le t \le b$ into the xy (or) z plane.

Note: We describe the points of an arc C by the equation $z = z(t)$ $a \le t \le b$ — ②

where $z(t) = x(t) + iy(t)$ — ③

Simple arc:

An arc C is called simple arc (or) Jordan arc if it does not cross itself.

(ie) C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.

Simple closed curve:

Let $C: z = z(t)$ $a \leq t \leq b$ be an arc. If C is simple except for the fact that $z(b) = z(a)$. we say that,

C is a simple closed curve (or) a Jordan curve.

positively oriented simple closed curve:

A simple closed curve (or) a Jordan curve is positively oriented if it in the counter clockwise direction.

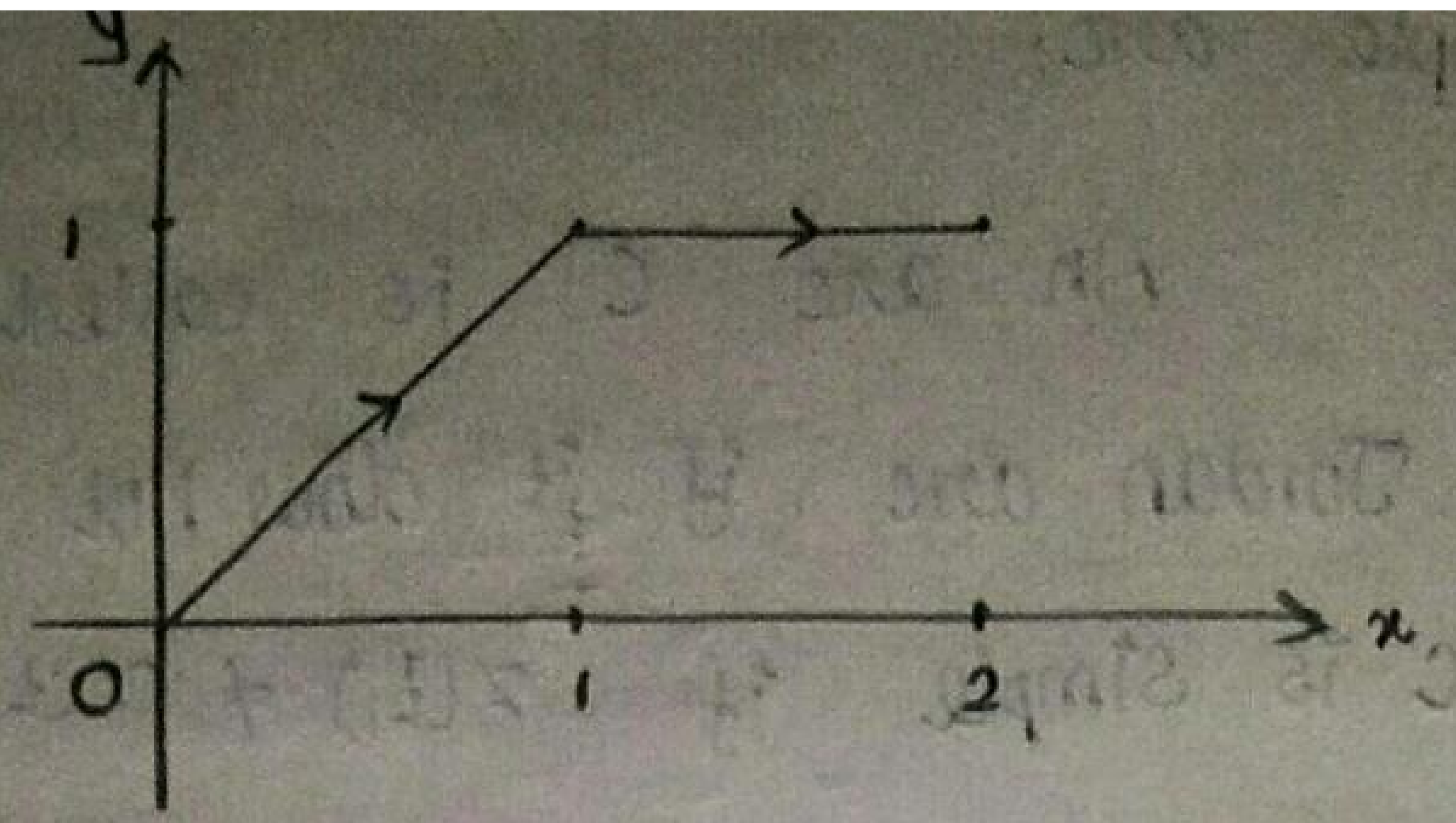
Example 1:

consider the polygonal line, define by the equation $z = \begin{cases} x+ix & \text{when } 0 \leq x \leq 1 \\ x+i & \text{when } 1 \leq x \leq 2 \end{cases}$ is a simple arc.

Soln.

If $0 \leq t \leq 1$, $x(t) = y(t)$ (x, x)

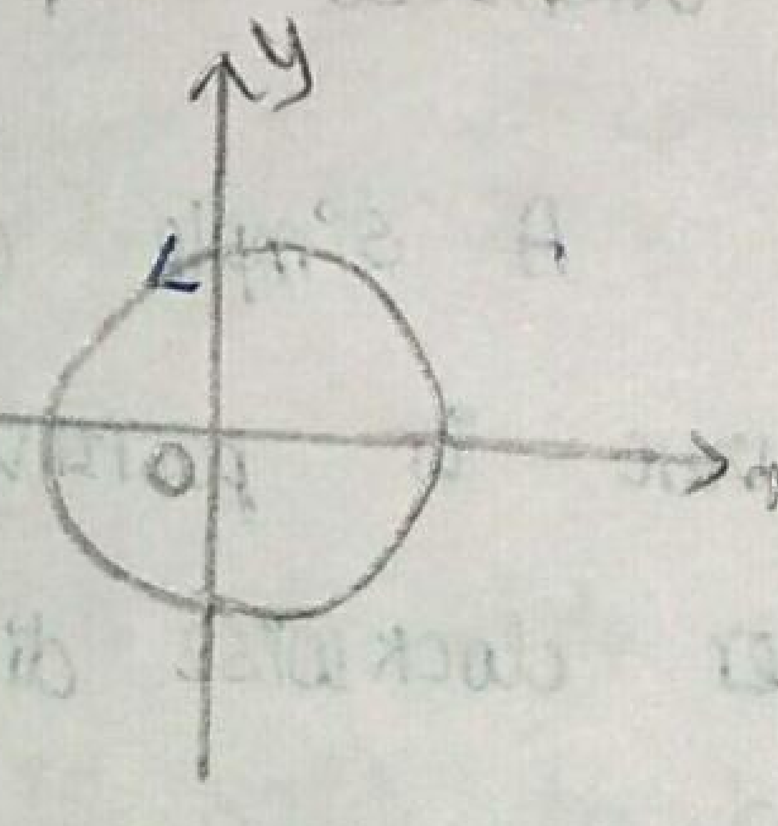
If $1 \leq t \leq 2$, $y(t) = 1; x = x(t)$ $(x, 1)$.



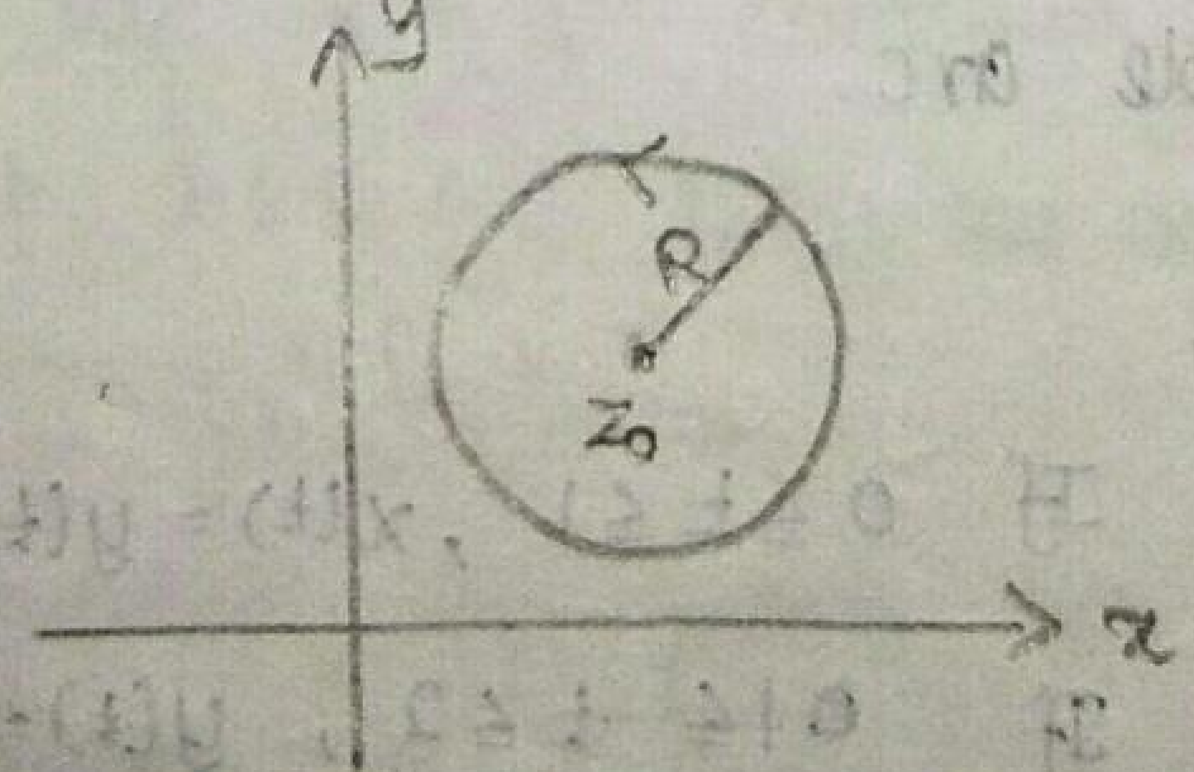
This arc consisting of the line segment from 0 to $1+i$ followed by $1+i$ to $2+i$.

Example 2: The unit circle.

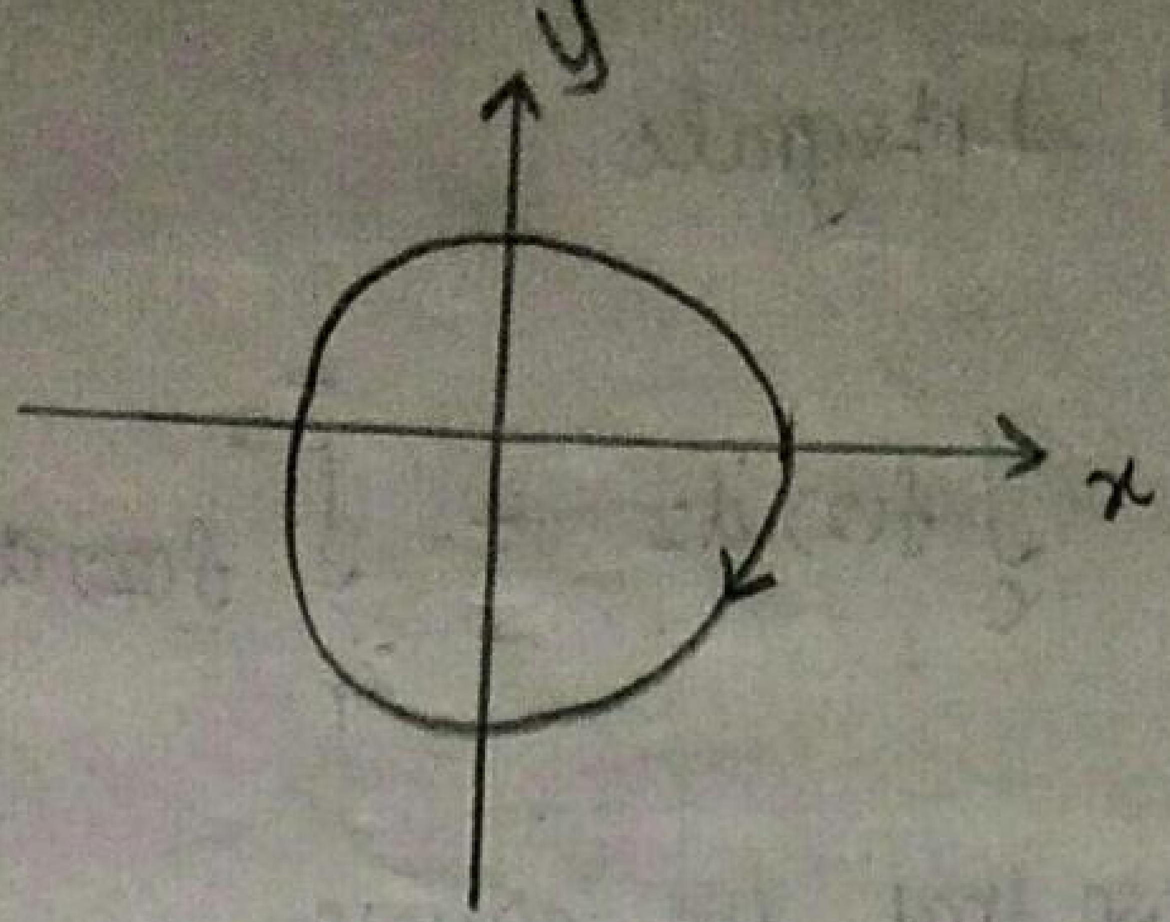
1) $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ about the origin is a simple closed curve (positively oriented).



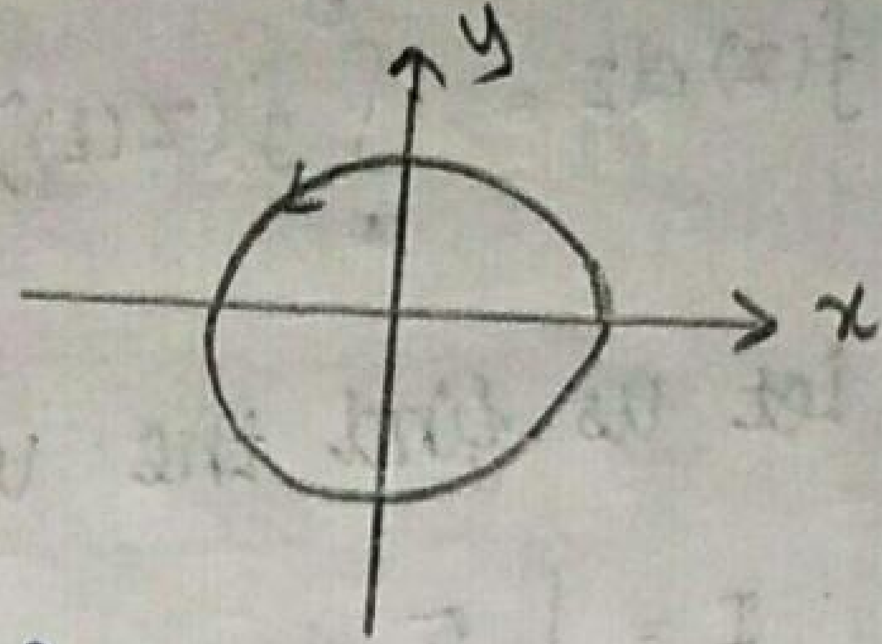
2) $z = z_0 + Re^{i\theta}$ $(0 \le \theta \le 2\pi)$ where z_0 is the centre point and R is radius and is a simple closed curve.



Example 3: $z = e^{-i\theta}$ $(0 \le \theta \le 2\pi)$ is a simple closed curve (negatively oriented)



Example 4: $z = e^{i2\theta}$ $(0 \le \theta \le 2\pi)$.



$z = e^{i2\theta}$ $(0 \le \theta \le 2\pi)$ is an arc consisting the circle ~~tran~~versed twice in the counter clockwise direction.

Differentiable arc:

Consider the arc 'c' such that

$$z = z(t) \quad a \le t \le b \quad -①$$

where $z(t) = x(t) + iy(t)$ $-②$, $x(t)$ and $y(t)$ are continuous function of real parameter t.

Then, $z'(t) = x'(t) + iy'(t)$ $-③$

If the components $x'(t)$ and $y'(t)$ are continuous on the entire interval $a \le t \le b$, then the arc 'c' is called a differentiable arc.

$$\int_c f(z)\, dz$$

change to Real axis $\int_a^b f(z(t)) \cdot z'(t)\, dt$.

Contour Integrals.

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz.$$

Suppose that the equation $z = z(t)$ $(a \leq t < b)$.

then, $\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt.$

Example 1: Let us find the value of the integral

$$I = \int_C \bar{z}\,dz.$$

when c is the right-hand half

$$z = 2e^{i\theta} \quad (-\pi/2 \leq \theta \leq \pi/2)$$ of the

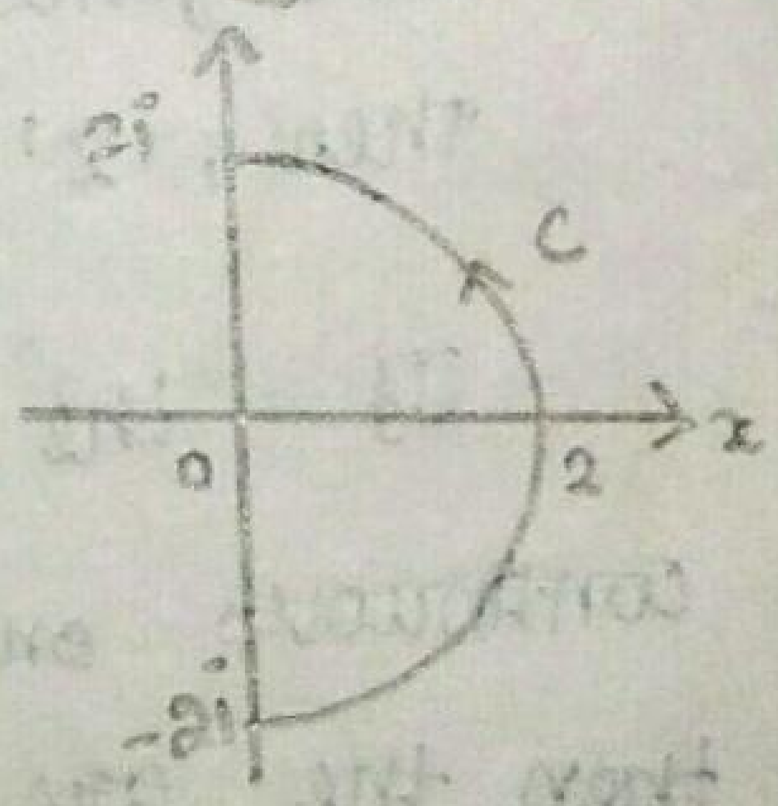
circle $|z| = 2$ from $z = -2i$ to $z = 2i$. According to definition (2).

Soln:

$z = 2e^{i\theta}$

$\bar{z} = 2e^{-i\theta}$

$dz = 2e^{i\theta}.\, i\,d\theta.$



$\therefore I = \int_C f(z)\,dz$

$= \int_a^b f(z(\theta))\, z'(\theta)\,d\theta.$

$= \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} \cdot 2e^{+i\theta} \cdot i\,d\theta.$

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$$= 4i \int_{-\pi/2}^{\pi/2} e^{-i\theta + i\theta} \, d\theta$$

$$= 4i \left[\theta\right]_{-\pi/2}^{\pi/2} = 4i \left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= 4i \left[\frac{\cancel{2}\pi}{\cancel{2}}\right] = 4i\pi.$$

$$\therefore I = 4i\pi.$$

Note: when z is a point on the Semicircle c

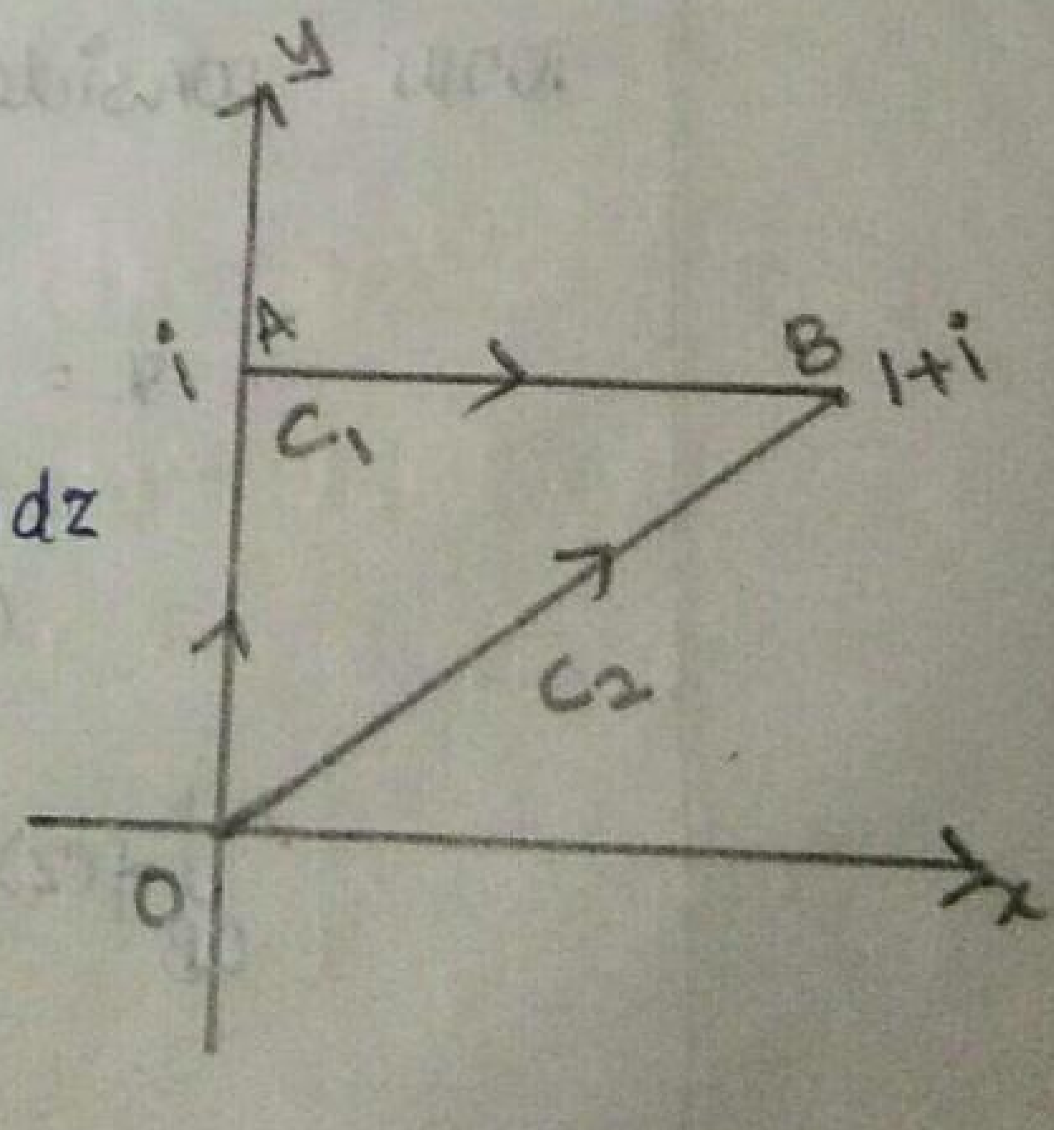
$$z\bar{z} = |z|^2 = 4 \Rightarrow \frac{\bar{z}}{4} = \frac{1}{z}$$

$$\therefore \int_c \frac{\bar{z}}{4} dz = \pi i \Rightarrow \int_c \frac{1}{z} dz = \pi i.$$

Example 2:

Evaluate the integral $\int_{c_1} f(z)\, dz$ where c_1 is the polygonal line OAB shown in figure and $f(z) = y - x - i\,3x^2$ $(z = x+iy)$.

Soln:



$$\text{Let } I = \int_{c_1} f(z)\, dz$$

$$= \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

Now, consider $\int_{OA} f(z)\, dz$.

Let $x = 0$; $y = y(t)$, $0 \le y \le 1$.

$$\therefore z = 0 + iy \quad 0 \le y \le 1.$$

$$dz = i\,dy.$$

$$\int_{OA} f(z)dz = \int_0^1 y \,(i\,dy) = i\left[\frac{y^2}{2}\right]_0^1 = \frac{i}{2}.$$

on AB,

$$x = x(t) \quad 0 \le x \le 1.$$

$$y = 1.$$

$$z = x + i \quad 0 \le x \le 1.$$

$$dz = dx$$

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - x - 3ix^2)\,dx$$

$$= \int_0^1 (1-x)\,dx - 3i\int_0^1 x^2\,dx.$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 - 3i\left[\frac{x^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{2} - \frac{\cancel{3}i}{\cancel{3}} = \frac{1}{2} - i.$$

$$\therefore I = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz$$

$$= \frac{i}{2} + \frac{1}{2} - i = \frac{1-i}{2}.$$

Note:

Now consider $\int_{OB} f(z)\,dz$.

$$x = y \Rightarrow z = x + ix \quad 0 \le x \le 1.$$

$$dz = (1+i)\,dx.$$

$$\therefore \int_{OB} f(z)\,dz = \int_0^1 (x - x - 3ix^2)(1+i)\,dx$$

$$= -3i\int_0^1 x^2\,(1+i)\,dx.$$

$$= -\cancel{3}i\,(1+i)\left[\frac{x^3}{\cancel{3}}\right]_0^1$$

$$\int_{OB} f(z)\,dz = (-i+1)(1) = 1 - i.$$

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on closed contour OABO

$$\int_{OABO} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz - \int_{OB} f(z)\,dz.$$

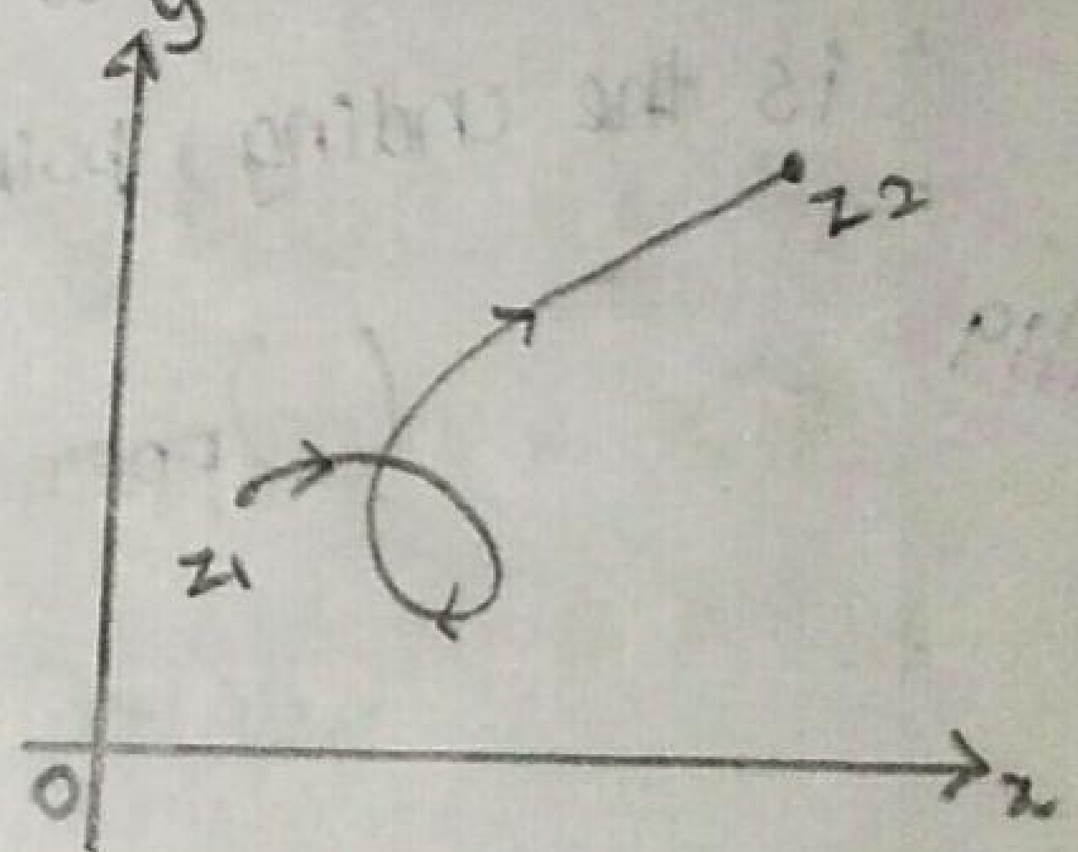
$$= \frac{1-i}{2} - (1-i)$$

$$= -\frac{1+i}{2} = \frac{i-1}{2} \neq 0.$$

Example 3:

Evaluate $\int_C z\,dz$ where z is an arbitrary smooth arc. $z = z(t)$. $[a \le t \le b]$ from a fixed point z_1 to a fixed point z_2.

<u>Solu</u>.



$$\int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt.$$

$$= \int_a^b z(t)\;d(z(t))$$

$$= \left[\frac{[z(t)]^2}{2}\right]_a^b.$$

$$= \frac{(z(b))^2 - (z(a))^2}{2}.$$

$$= \frac{z_2^{\,2} - z_1^{\,2}}{2}.$$

$$\therefore \int_C z\,dz = \frac{z_2^{\,2} - z_1^{\,2}}{2}.$$

Note: If C is the contour consisting of a finite number of smooth arc C_k $(k = 1, 2, \dots n)$, Joint end to end.

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Suppose that each C_k extends from z_k to z_{k+1}.

$$\int_C z\,dz = \sum_{k=1}^{n} \int_{C_k} z\,dz$$

$$= \sum_{k=1}^{n} \frac{(z_{k+1})^2 - (z_k)^2}{2}$$

$$= \frac{(z_{n+1})^2 - \cancel{(z_n)^2} + \cancel{(z_n)^2} - \cancel{(z_{n-1})^2} \ldots\ldots + \cancel{(z_2)^2} - (z_1)^2}{2}$$

$$\int z\,dz = \frac{(z_{n+1})^2 - (z_1)^2}{2}$$

where z_1 is the starting point of C and z_{n+1} is the ending point of C.

12/9/19

Upper Bounds for Moduli of Coutour Integrals.

Lemma:

If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt.$$

Proof:

If $\int_a^b w(t)\,dt = 0$, then the result is trivial.

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$\therefore$ Assume that $\int_a w(t)\,dt \neq 0$.

Let

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \quad \text{— ①}$$

$$\Rightarrow r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt \quad \text{— ②}$$

Since r_0 is a real number, so $\int_a^b e^{-i\theta_0} w(t)\,dt$ is also a real number.

Since the real part of a real number is the number itself, we have

$$r_0 = Re \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$\Rightarrow r_0 = \int_a^b Re\left[e^{-i\theta_0} w(t)\right]dt \quad \text{— ③}$$

but;

$Re\, z \leq |z|$ implies

$$Re\left[e^{-i\theta_0}(w(t))\right] \leq \left|e^{-i\theta_0} w(t)\right| = |w(t)|$$

$$\therefore ③ \Rightarrow r_0 \leq \int_a^b |w(t)|\,dt \quad \text{— ④}$$

$$① \Rightarrow \left|\int_a^b w(t)\,dt\right| = |r_0 e^{i\theta_0}| = r_0.$$

Hence, $\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$.

Thus proved.

Theorem:

Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non negative constant such that

$$|f(z)| \leq M \quad \text{(bound)}$$ for all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z)\,dz \right| \leq ML$$

Proof:

Let $z = z(t)$, $a \leq t \leq b$, be a parametric representation of the contour C.

Then by the lemma,

" If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \leq t \leq b$ then

$$\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt \text{ ''}.$$

we have,

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\; z'(t)\,dt \right|$$

$$\leq \int_a^b | f(z(t)) \cdot z'(t) |\,dt.$$

But $\left| f(z(t))\, z'(t) \right| \le M \left| z'(t) \right|$, when

$a \le t \le b$.

$$\therefore \left| \int_C f(z)\, dz \right| \le M \int_a^b \left| z'(t) \right| dt$$

But, the length of the arc C is

$$L = \int_a^b \left| z'(t) \right| dt$$

$$\therefore \left| \int_C f(z)\, dz \right| \le ML.$$

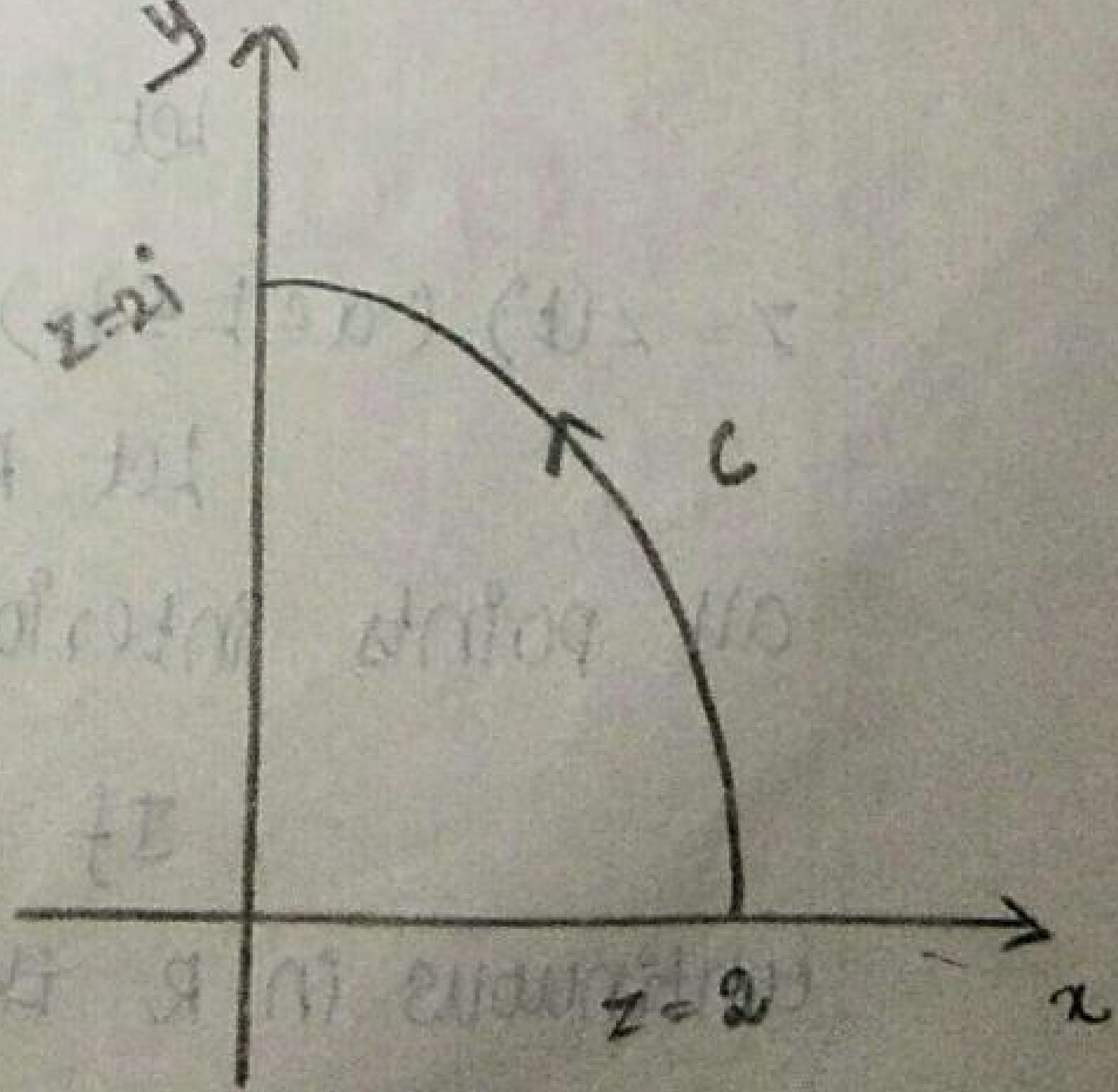
Hence, the proof.

Example 1:

Let C be the arc of the circle $|z| = 2$. from $z = 2$ to $z = 2i$ that lies in the first quadrant. Inequality can be used to show that

$$\left| \int_C \frac{z+4}{z^3-1}\, dz \right| \le \frac{6\pi}{7}.$$

Solution,



Let $f(z) = \dfrac{z+4}{z^3-1}$

on C.

$$|z| = 2$$

$$\therefore |z+4| \le |z| + 4 = 6.$$

$$|z^3-1| \geq ||z^3|-1| = |8-1| \Rightarrow 7.$$

$$\frac{1}{z^3-1} < \frac{1}{7}.$$

$$\therefore \left|\frac{z+4}{z^3-1}\right| \leq \frac{6}{7}.$$

$|f(z)| \leq 6/7$ on C.

Also length of $C = \pi$.

$\therefore \left|\int_C f(z)\,dz\right| \leq ML$, we have

$$\left|\int_C f(z)\,dz\right| \leq \frac{6}{7}\pi.$$

Hence the proof.

12/9/19. Cauchy - Goursat Theorem.

(Cauchy's theorem) Theorem. (Fundamental)

Let C be a simple closed contour $z = z(t)$ $(a \leq t \leq b)$ described in the positive sense.

Let R be the closed region consisting of all points interior to and on C.

If f is analytic in R and f' is continuous in R then,

$$\int_C f(z)\,dz = 0.$$

Proof:

Given that $f(z)$ is analytic in R.

Then the integral of f along c in terms of the parameter t is

$$\int_c f(z)dz = \int_a^b f(z(t))\, z'(t) \cdot dt \quad — ①$$

Let $f(z) = u(x,y) + iv(x,y)$ and

$z(t) = x(t) + iy(t)$.

$$\therefore f(z(t)) \cdot z'(t) = \left[u(x(t), y(t)) + i\, v(x(t), y(t)) \right] (x'(t) + iy'(t))$$

$$= ux' + iuy' + ivx' - vy'$$

$$= (ux' - vy') + i(vx' + uy').$$

$$\therefore \int_c f(z)dz = \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')\, dt \quad — ②$$

$\langle x'(t)dt = d(x(t))$

$$\Rightarrow \int_c f(z)dz = \int_c udx - vdy + i\int_c vdx + udy \quad — ③.$$

Since f is analytic in R, then it is continuous in R. Hence the function u and v are also continuous in R. Since f' is continuous in R, the first order partial derivatives of u and v are all continuous in R.

By Greens theorem

" Suppose that two real valued function $p(x,y)$ and $Q(x,y)$ and their first-order partial derivative are continuous throughout the closed region R consisting of all point inferior to and on the simple closed contour C.

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy.$$

$$③ \rightarrow \int_C f(z)\,dz = \iint_R (-V_x - U_y)\,dx\,dy + i\iint_R (U_x - V_y)\,dx\,dy \quad -④.$$

Since $f(z)$ is analytic, the first order partial derivatives of u and v satisfy the Cauchy-Riemann equation.

$$U_x = V_y \text{ and } U_y = -V_x \quad -⑤$$

use ⑤ in ④,

$$\int_C f(z)\,dz = 0.$$ Hence the proof.

Example:

Evaluate $\int_C \exp(z^3)\,dz$ (or) prove that $\int_C \exp((z)^3)\,dz = 0$.

<u>Soln</u>:

where c is any simple closed contour.

$$f(z) = e^{z^3}$$

$$f'(z) = e^{z^3} \cdot 3z^2$$

$$= 3e^{z^3} z^2$$

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$\therefore f'(z)$ is continuous everywhere. Hence, by the cauchy's fundamental theorem

$$"\int_C f(z)dz = 0"$$

We get,

$$\int_C e^{z^3} dz = 0.$$ Hence proved.

19/9/19. Cauchy-Goursat theorem:

Theorem:

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0.$$

Proof:

Cauchy-Goursat Lemma:

Statement:

Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour c together with the points on c itself. For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$, such that in each one there is a fixed point z_j for which the inequality

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \text{ is } \quad \text{—— ①}$$

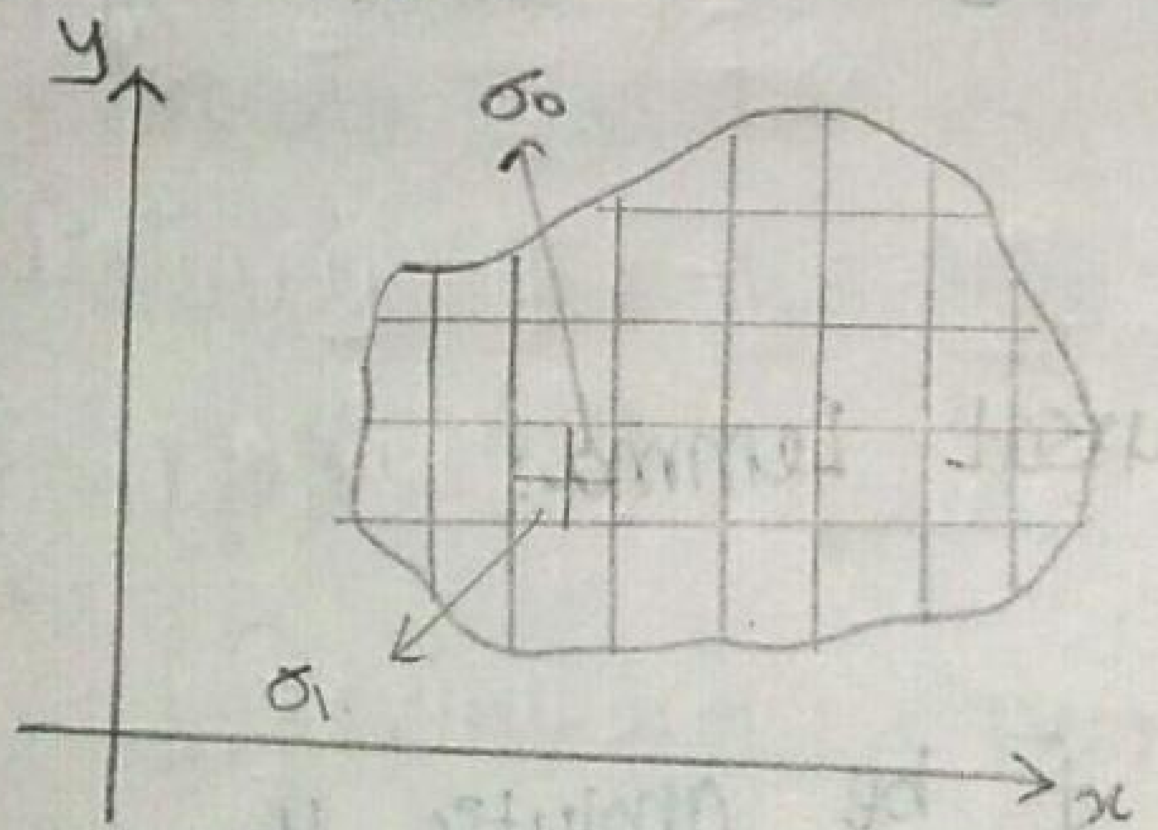
Statisfied by all points other than z_j in that square or partial square.

proof:

First we draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines.

Thus, we covered the region R with a finite number of squares and partial squares.

~~First we cover the region R with a finite number of squares and partial squares~~ as shown in figure.



In this cover, suppose that there is some ~~&~~ subregion (squares or partial squares) in which no point z_j exist. such that ① holds for all other points z in it.

Let σ_0 denote one of that subregions, if it is a square.

If it is a partial square, Let σ_0 denote the entire square of which it is a part.

After we subdivide σ_0, atleast one of the four smaller squares denoted by σ_1 must contains points of R but no appropriate z_j.

We than subdivide σ_1 and continue in this manner.

After a square σ_{k-1} $(k=1,2,\ldots)$ has been subdivided, if more than one of the four smaller squares be constructed, Take σ_k to be the one lowest and then furthest to the left.

In this manner the nested infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots$ of square is constructed —(2)

Let, $I_k = [a_k, b_k]$, $k=0,1,2,\ldots$ be the projection of the squares σ_k on the real axis.

~~σ_k~~ $(k=0,1,2,\ldots)$ on the real axis.

clearly, $I_0 \supset I_1 \supset I_2 \ldots\ldots$ and

$b_k - a_k =$ Length of $I_k = \dfrac{b_0-a_0}{2^k}$, $k=0,1,2,\ldots$

$$\therefore \lim_{k\to\infty}(b_k-a_k) = \lim_{k\to\infty}\left(\frac{b_0-a_0}{2^k}\right) = 0$$

Since, $I_{k-1} \supset I_k$

$$a_{k-1} \le a_k \le b_k \le b_{k-1}.$$

$\therefore \{a_k\}_{k=0}^{\infty}$ is non decreasing and $\{b_k\}_{k=0}^{\infty}$ is non increasing.

since all the terms of these sequence lie in I_0, They are both bounded.

Hence, $\lim_{k\to\infty} a_k = A$ (say) and

$$\lim_{k\to\infty} b_k = B \text{ (say)}.$$

$$B-A = \lim_{k\to\infty} b_k - \lim_{k\to\infty} a_k = \lim_{k\to\infty} (b_k - a_k) = 0$$

$$\Rightarrow B = A = x_0 \text{ (say)}.$$

Since $a_k \leq A \leq B \leq b_k$, $k = 0, 1, 2, \ldots$

$$x_0 \in \bigcap_{k=0}^{\infty} I_k.$$

Similarly if $T_k = [c_k, d_k]$, $k=0,1,2\ldots$ be the projection of the squares σ_k, $k=0,1,2,\ldots$ on the imaginary axis. Then $y_0 \in \bigcap_{k=0}^{\infty} T_k$.

Let $z_0 = x_0 + iy_0$. Then $z_0 \in \bigcap_{k=0}^{\infty} \sigma_k$.

Also each of these squares σ_k contains points of R other than z_0.

For any $\delta > 0$, the δ-neighborhood $|z - z_0| < \delta$ of z_0 contains such squares σ_k, If length of the diagonal of σ_k

$$\sqrt{2}\,\frac{(b_0 - a_0)}{2^k} < \delta \quad (\sqrt{2}\,(b_k - a_k) < \delta)$$

$\therefore$ Every δ-neighborhood $|z-z_0| < \delta$ contains points of R distinct from z_0. This means that z_0 is an accumulation point of R.

Since R is a closed set, it follows that $z_0 \in R$.

Since $f(z)$ is analytic throughout R, and is particular at z_0. $f'(z_0)$ exists,

Hence for each positive number ε, there is a δ-neighborhood $|z-z_0| < \delta$. such that the inequality,

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| < \varepsilon \quad \text{is}$$

Satisfied by all points $z = z_0$ in that neighborhood.

But, the neighbourhood contains the sequence σ_k, when k is large enough that

$$\sqrt{2}(b_k - a_k) < \delta.$$

consequantly, z_0 serves as the point z_j in ① for the subregion consisting of the sequence σ_k or a part of σ_k.

This contradicts the formation of σ_k, in which no z_j such that ① holds for all other points in it.

we thus arrive at a contradiction and the proof of the lemma is complete.

Proof of the theorem:

Given an arbitrary positive number ε, we consider the covering of R in the statement of the (above) lemma.

We define a function $\delta_j(z)$ on the j^{th} square (or) partial square as follows: $\delta_j(z_j) = 0$.

where z_j is the fixed point in inequality ① and

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad \text{when } z \neq z_j \quad \text{—③}$$

By the inequality ①, $|\delta_j(z)| < \varepsilon$ —④

At all points in the subregion on which $\delta_j(z)$ is defined. Also, the function $\delta_j(z)$ is continuous throughout the subregion.

Since $f(z)$ is continuous in that subregion and $\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$ —⑤

Next, we let C_j $(j = 1, 2, \ldots n)$ denote the positively oriented boundaries of the above squares (or) partial squares covering R.

From the definition of $\delta_j(z)$ in (4) at a point z on any particular C_j, f can be written as,

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j) z + (z - z_j)\,\delta_j(z).$$

This means that,

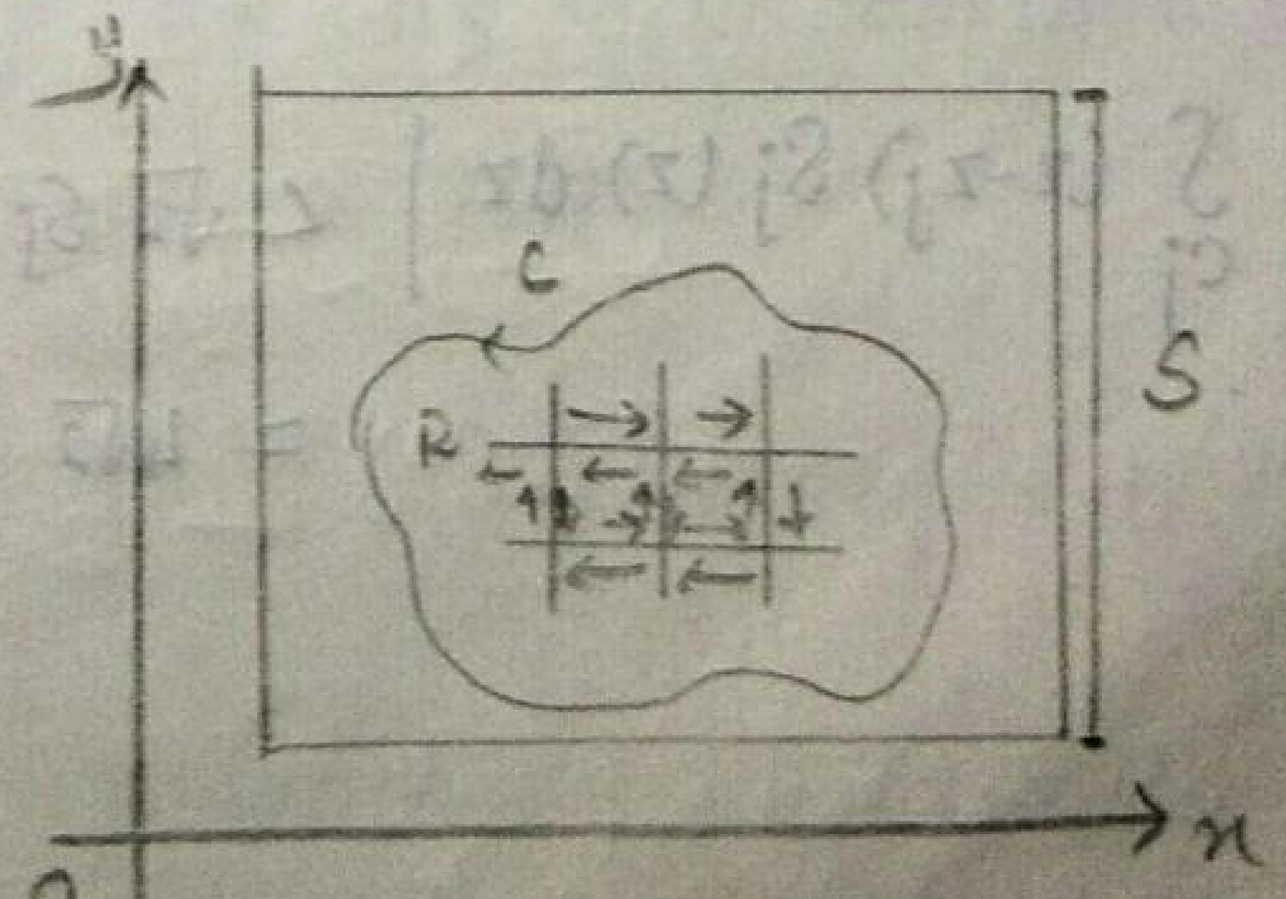
$$\int_{C_j} f(z)\,dz = \underbrace{[f(z_j) - z_j f'(z_j)] \int_{C_j} dz}_{A} + \underbrace{f'(z_j) \int_{C_j} z\,dz}_{B} + \int_{C_j} (z - z_j)\,\delta_j(z)\,dz \quad \text{--- (6).}$$

Since the function (1) (A) and (2) (B) posses anti-derivatives everywhere in the fixed plane and C_j is a closed contour. we have,

$$\int_{C_j} dz = 0 \quad \text{and} \quad \int_{C_j} z\,dz = 0.$$

$$\therefore (6) \Rightarrow \int_{C_j} f(z)\,dz = \int_{C_j} (z - z_j)\,\delta_j(z)\,dz \quad (j = 1, 2, \dots n) \quad \text{--- (7)}$$

Since the sum of the integrals along the common sides is 0 as shown in figure.



$$\sum_{j=1}^{n} \int_{C_j} f(z)\,dz = \int_C f(z)\,dz$$

$$\therefore (7) \Rightarrow \int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\,\delta_j(z)\,dz.$$

$$\left| \int_C f(z)\,dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} (z-z_j)\,\delta_j(z)\,dz \right| \quad — (8)$$

Let S_j denote the length of a side of the either square (or) partial square in the j^{th} integral. Since, both the variable z and the point z_j lie in that Square.

$$|z-z_j| \leq \sqrt{2}\, S_j$$

From (4), $|\delta_j(z)| < \varepsilon$

Hence, $|(z-z_j)\,\delta_j(z)| = |z-z_j| \cdot |\delta_j(z)|$

$$< \sqrt{2}\, S_j \cdot \varepsilon \quad — (9)$$

Let A_j denote the area of the ~~full~~ Square of sides. If C_j is the boundary of the full square, then the length of $C_j = 4 S_j$.

$\therefore$ By the upper bound for the moduli of the contour integrals,

$$\left| \int_{C_j} (z-z_j)\,\delta_j(z)\,dz \right| < \sqrt{2}\, S_j\, \varepsilon \times 4 S_j$$

$$= 4\sqrt{2}\, A_j \cdot \varepsilon \quad — (10)$$

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If C_j is the boundary of the partial square,

length of $C_j < 4s_j + L_j$, where L_j the length of the part of C_j which is also a part of C.

$$\therefore \left| \int_{C_j} (z - z_j)\, \delta_j(z)\, dz \right| < \sqrt{2}\, s_j\, \epsilon\, (4s_j + L_j)$$

$$= 4\sqrt{2}\, A_j\, \epsilon + \sqrt{2}\, s_j\, L_j\, \epsilon$$

$$< 4\sqrt{2}\, A_j\, \epsilon + \sqrt{2}\, S L_j\, \epsilon \quad \text{— (11)}$$

where S is the length of a side of some square that encloses the entire contour C as well as the squares used in the covering R.

Note that $\sum_{j=1}^{n} A_j \leq S^2$.

If L denotes the length of C, then from the inequalities (8), (10) and (11),

$$\left| \int_C f(z)\, dz \right| \leq \sum_{j=1}^{n} (4\sqrt{2}\, A_j\, \epsilon + \sqrt{2}\, S L_j\, \epsilon)$$

$$< (4\sqrt{2}\, S^2 + \sqrt{2}\, S L)\, \epsilon \quad \text{— (12)}$$

Since ϵ is arbitrary, we can choose it so that the R.H.S of (12) is as small as possible.

$\therefore$ The LHS of (12) which is independent of ϵ, must be equal to zero.

(ie) $\left| \int_C f(z)\, dz \right| = 0.$

Hence, $\int_C f(z)\, dz = 0$. This complete the Cauchy-Goursat theorem.

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Exercises (sec. 38).

4) According to $\int_a^b w(t)dt = \int_a^b u(t)\,dt + i\int_a^b v(t)dt$, of definite integrals of complex-valued functions of a real Variables,

$$\int_0^{\pi} e^{(1+i)x}dx = \int_0^{\pi} e^{x}\cos x\,dx + i\int_0^{\pi} e^{x}\sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integrals on the left and then using the real and imaginary parts of the value found.

<u>Solu,</u>

Given that,

$$\int_0^{\pi} e^{(1+i)x}dx = \int_0^{\pi} e^{x}\cos x\,dx + i\int_0^{\pi} e^{x}\sin x\,dx. \quad —①$$

First we consider. $RHS = \int_0^{\pi} e^{x}\cos x\,dx + i\int_0^{\pi} e^{x}\sin x\,dx$

we know that,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$$

$$\therefore \int_0^{\pi} e^{x}\cos x\,dx = \left[\frac{e^{x}}{2}(\cos x + \sin x)\right]_0^{\pi}$$

$$= \frac{e^{\pi}}{2}(\cos\pi + i\sin\pi) - \frac{e^{0}}{2}(\cos 0 + i\sin 0)$$

$$= \frac{e^{\pi}}{2}(-1) - \frac{1}{2}(1) = \frac{-e^{\pi}}{2} - \frac{1}{2}$$

$$\therefore \int_0^{\pi} e^{x}\cos x\,dx = \frac{-1}{2}(1+e^{\pi}) \quad — ②$$

$$\int_0^{\pi} e^{x}\sin x\,dx = \left[\frac{e^{x}}{2}(\sin x - \cos x)\right]_0^{\pi}$$

$$= \frac{e^{\pi}}{2}[\sin\pi - \cos\pi] - \frac{e^{0}}{2}[\sin 0 - \cos 0]$$

$$= \frac{e^{\pi}}{2}(+1) - \frac{1}{2}(-1)$$

$$\therefore \int_0^{\pi} e^{x}\sin x\,dx = \frac{e^{\pi}}{2} + \frac{1}{2} = \frac{1}{2}(1+e^{\pi}) \quad — ③$$

$\therefore$ From ② and ③ we get.

$$RHS = \frac{-1}{2}(1+e^{\pi}) + i\,\frac{1}{2}(1+e^{\pi}).$$

Hence, $\displaystyle\int_0^{\pi} e^{(1+i)x}\,dx = \frac{-1}{2}(1+e^{\pi}) + \frac{i}{2}(1+e^{\pi})$

5) Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

a) Suppose that $w(t)$ is even; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^{a} w(t)\,dt.$$

Soln.

when $w(t)$ is even, $w(-t) = w(t)$

$$w(t) = u(t) + iv(t)$$
$$w(-t) = u(-t) + iv(-t)$$

comparing the real and imaginary parts, we get

$$u(-t) = u(t) \quad \text{and} \quad v(-t) = v(t)$$

we know that,

$$\int_{-a}^{a} w(t)\,dt = \int_{-a}^{a} u(t)\,dt + i\int_{-a}^{a} v(t)\,dt \quad — ①$$

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when t is even, we know that

$$\int_{-a}^{a} u(t)\,dt = 2\int_0^a u(t)\,dt \text{ and}$$

$$\int_{-a}^{a} v(t)\,dt = 2\int_0^a v(t)\,dt.$$

use these 2 equation in ① we get,

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^a u(t)\,dt + i\,2\int_0^a v(t)\,dt.$$

$$= 2\int_0^a (u(t) + iv(t))\,dt$$

$$\therefore \int_{-a}^{a} w(t)\,dt = 2\int_0^a w(t)\,dt.$$

Hence, proved.

28/09/19.

Simply Connected domain

A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D.

Example: * The set of points interior to a simple closed contour and

* $|z| < 5$ is simply connected domain.

Theorem: If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0 \text{ for every closed}$$

contour c lying in D.

Proof:

* If c is a simple closed contour in simply connected domain D,

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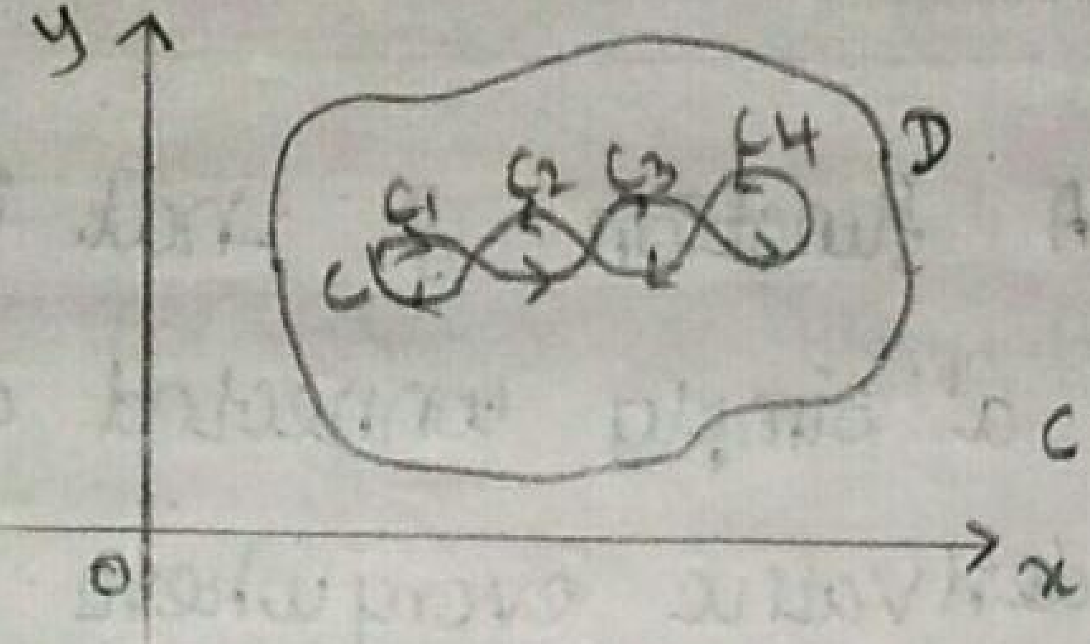
then $f(z)$ is analytic at each point interior to and on C.

Then by Cauchy - Goursat theorem,

$$\int_C f(z)\,dz = 0.$$

* If C is a closed contour but intersect itself a finite number of times, then it consists of a finite number of simple closed contour C_k, $k = 1, 2, \ldots n$ as shown in figure.



$C = C_1 + C_2 + C_3 + C_4$.

By Cauch-Goursat theorem, $\int_{C_k} f(z)\,dz = 0$ $(k = 1, 2, \ldots n)$

$$\therefore \int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

Hence, $\int_C f(z)\,dz = 0$. This theorem applies for closed contours having an infinite number of self intersection points.

Example: If C denotes any closed contour lying in the open disk $|z| < 2$, then

$$\int_C \frac{ze^2}{(z^2+9)^5}\,dz = 0.$$

Solu:

Let D be $|z| < 2$. Here, $f(z) = \dfrac{ze^2}{(z^2+9)^5}$

$\therefore z^2+9=0 \Rightarrow z=\pm 3i$.

Thus, $f(z)$ is not analytic at $\pm 3i$ but $z=\pm 3i$ lie outside D. Hence, $f(z)$ is analytic throughout D (simply connected domain).

Therefore, $\int_C f(z)dz = 0$ for any closed contour C lying in D.

Hence, $\int_C \frac{ze^2}{(z^2+9)^5} dz = 0$.

Corollary: A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Proof:

Given that, f is analytic throughout a simply connected domain D.

$\therefore \int_C f(z)dz=0$ for any closed contour C lying in D.

By the theorem,

" Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statement is true, then so are the others:

(a) $f(z)$ has an antiderivatives $F(z)$ throughout D.

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely

$$\int_{z_1}^{z_2} f(z)\,dz = \Big[F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero."

we get,

the function f has an anti derivatives througout D.

Hence proved the theorem.

Multiple connected domain:

A domain which is not simply connected is said to be multiple connected domain.

Theorem:

Suppose that

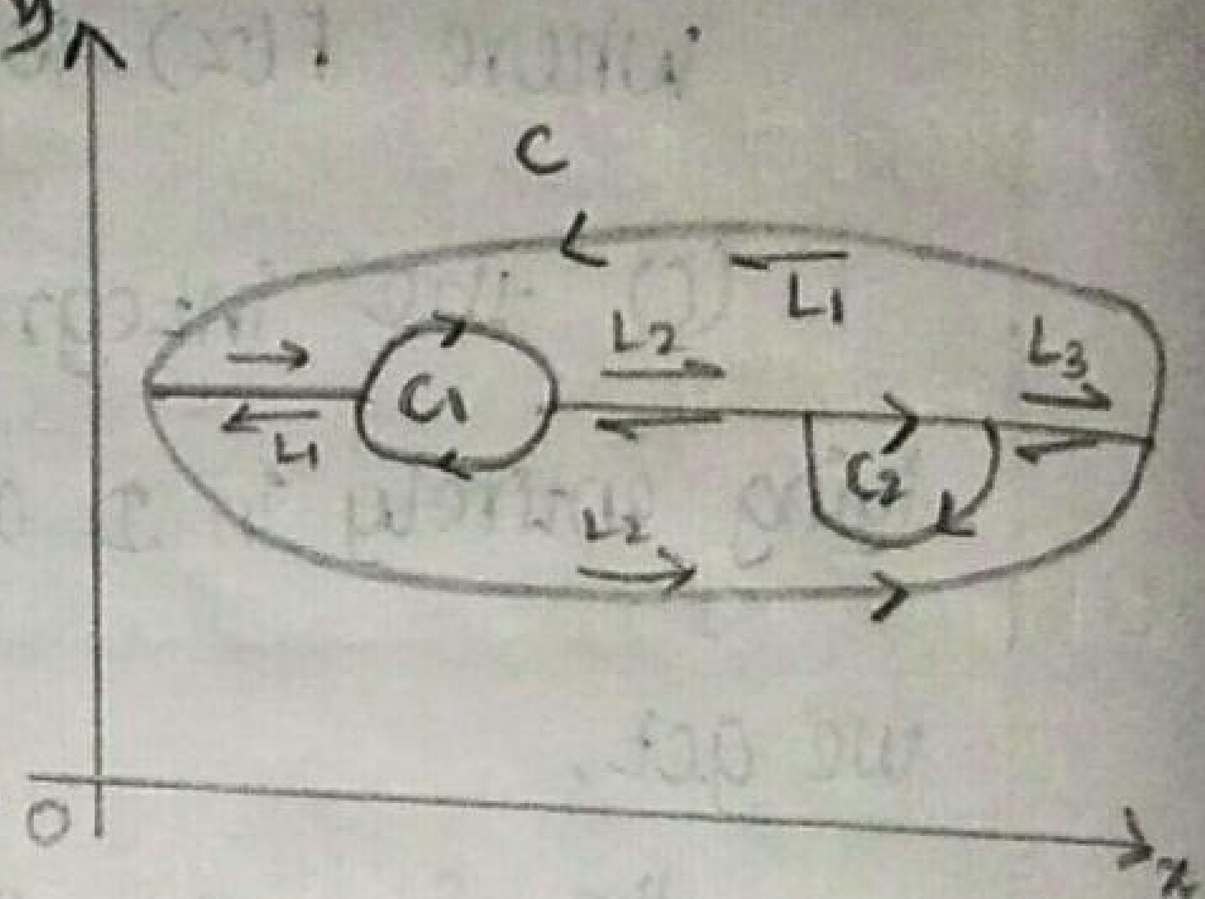
(a) C is a simple closed contour, described in the counterclockwise direction.

(b) C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interiors to C, all described the clockwise direction, that are disjoint and whose interiors have no points common.

If a function f is analytic on all of these contours and throughout the multiple connected domain consisting of the points inside C and exterior to each ~~other~~ C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Proof:



Let L_1 be a polygonal path consisting of a finite number of line segments joint end to end to connect the outer contour C to the inner contour C_1. Let L_2 be the polygonal path which connects to C_1, C_2 and are continue in this manner, L_{n+1} is the polygonal path connecting C_n to C. As shown in figure.

Let Γ_1 and Γ_2 be two simple closed contour consisting of polygonal paths L_k on ~~L_k~~ L_k and pieces of C and C_k and each described in that direction that the points enclosed by them, lie to the left as shown in figure.

clearly $f(z)$ is analytic at all points interior to and on Γ_1 and Γ_2.

$\therefore$ By Cauchy-Goursat theorem,

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0.$$

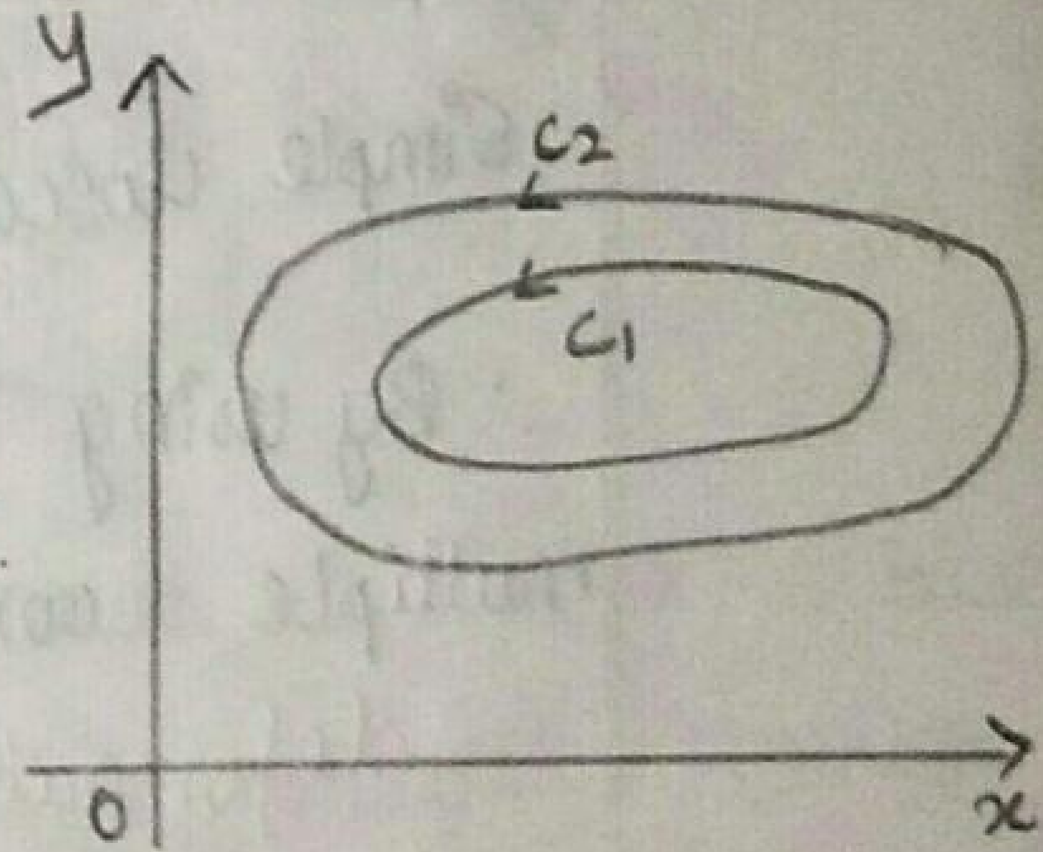
Since the integrals in the opposite direction along each path L_k cancel, only the integrals along C and C_k remain.

Hence, $\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$.

Corollary: [Principle of deformations of paths].

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them. then,

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz.$$



Proof:

By Cauchy - Goursat theorem for multiple connected domain, we have

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0.$$

$$\Rightarrow \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz = 0$$

$$\therefore \int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz.$$

Hence, proved the theorem.

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Example: when C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i$$

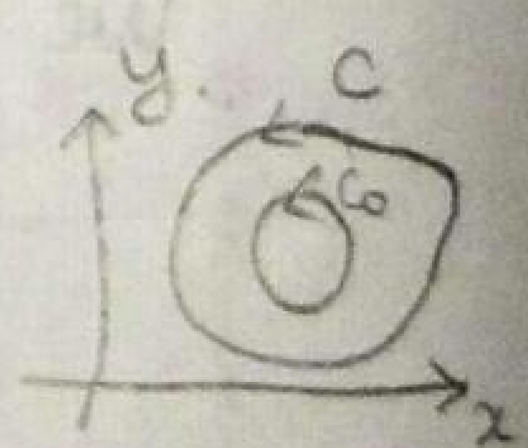
Solu:

Let C_0 be a positively oriented simple closed contour and given that C is any positively oriented simple closed contour surrounding the origin.

$$\text{Here, } f(z) = \frac{1}{z}$$

$\therefore$ $f(z)$ is analytic throughout the region expect $z=0$. where C & C_0 is positively oriented simple closed contour surrounding the origin.

$\therefore$ By using the Cauchy-Goursat theorem for multiple connected domain, (Principle of deformation of path)



$$\int_C f(z)\,dz = \int_{C_0} f(z)\,dz$$

$$= \int_{C_0} \frac{1}{z}\,dz$$

Let C_0 be a circle. $\therefore$ The limit of C_0 is 0 to 2π and let $z = re^{i\theta}$.

$$\therefore \int_C f(z)\,dz = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta$$

$$\therefore \int_C f(z)\,dz = \int_C \frac{dz}{z} = 2\pi i.$$

$$\therefore \int_C \frac{dz}{z} = 2\pi i. \quad \text{Hence, proved.}$$

Exercise [Sec 49].

1) Apply the Cauchy - Goursat theorem to show that

$$\int_C f(z)\,dz = 0.$$

When the contour C is the unit circle $|z|=1$, in either direction and when

a) $f(z) = \frac{z^2}{z-3}$.

Soln.

where $|z|=1$ is a simply connected domain D

$\therefore$ C is simple closed contour with in it encloses only points of D.

Here $f(z) = \frac{z^2}{z-3}$

$$\therefore z-3=0 \Rightarrow z=3.$$

Thus, $f(z)$ is not analytic at $z=3$ but $z=3$ lie outside D. Hence, $f(z)$ is analytic through-out D (simply connected domain).

$\therefore \int_C f(z)\,dz = 0$ for any closed contour C lying in D.

$$\therefore \int_C \frac{z^2}{z-3}\,dz = 0. \quad \text{Hence, proved.}$$

b) $f(z) = ze^{-z}$.

Soln.

where $|z|=1$ is a simply connected domain D.

Here $f(z) = ze^{-z} = \frac{z}{e^z}$.

$e^z = 0 \Rightarrow e^{\infty} = 0$

$\therefore z \neq \infty$

Thus, $f(z)$ is not analytic at $z = \infty$ but $z = \infty$ lie outside D. Hence, $f(z)$ analytic throughout D (simply connected domain).

$\therefore \int_C f(z)\,dz = 0$ for any closed contour C lying in D.

Hence, $\int_C ze^{-z}\,dz = 0$.

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c) $f(z) = \frac{1}{z^2+2z+2}$.

Soln.

Where $|z| = 1$ is a simply connected domains

Here, $f(z) = \frac{1}{z^2+2z+2}$

$\therefore z^2+2z+2 = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4-8}}{2}$

$\therefore z = \frac{-2 \pm 2i}{2} = -1 \pm i$.

Thus $f(z)$ is not analytic at $-1 \pm i$ but $z = -1 \pm i$ lie outside D. Hence $f(z)$ is analytic throughout D (simply connected domain)

$\therefore \int_C f(z)\,dz = 0$ for any closed contour C lying in D.

Hence, $\int_C \left(\frac{1}{z^2+2z+2}\right) dz = 0$.

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