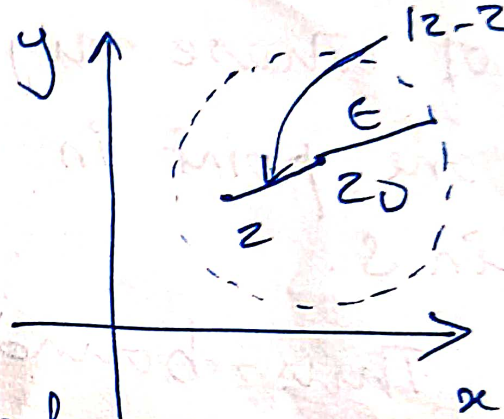
UNIT - I.

Regions in the Complex Plane:

ϵ- neighbourhood:

An ϵ- neighbourhood $|z - z_0| < \epsilon$ of a given point z_0 consists of all points z lying inside but not on a circle centered at z_0 and radius ϵ.



Deleted neighbourhood:

A deleted neighbourhood or punctured disk $0 < |z - z_0| < \epsilon$ consisting of all points z in an ϵ- neighbourhood of z_0 except for the point z_0 itself.

Interior Point:-

A point z_0 is said to be an interior point of a set S whenever there is ~~some~~ neighbourhood of z_0 that contains only points of S.

Exterior Point:

A point z_0 is said to be an exterior point of a set S, if there exists a neighbourhood of z_0 containing no points of S.

Boundary point:-

A boundary point of a set S is a point all of whose neighbourhoods contains at least one point in S and at least one point not in S.

Thus boundary point is neither interior nor exterior point.

The collection of all boundary points of a set S is called the boundary of S.

Ex: The circle $|z|=1$, is the boundary of the sets $|z|<1$ and $|z|\leq 1$.

Open Set:-

A set is open if it contains none of its boundary points.

Closed Set:

A set is closed if it contains all of its boundary points.

Closure of a set:

The closure of a set S is the closed set consisting of all points in S together with the boundary of S.

Ex: The closure of $|z|<1$ is $|z|\leq 1$.

Note:-

If a set is not open, then there exists a boundary point that is contained in the set.

(ii) If a set is not closed, there exists a boundary point not contained in the set.

Ex: The punctured disk $0<|z|\leq 1$ is neither open nor closed.

Connected Set:-

An open set S is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end that lies entirely in S. 4

Ex:- The open set $|z|<1$ is connected

The annulus $1<|z|<2$ is open and also connected.

Domain:-

A nonempty open set that is connected is called a domain.

Ex:- Every neighbourhood is a domain.

Region:

A domain together with some, none or all of its boundary points is referred to as a region.

Bounded Set:

A set S is bounded if every point of S lies inside some circle $|z|=R$. Otherwise it is unbounded.

Ex:- $|z|<1$ & $|z|\leq 1$ are bounded sets but $\operatorname{Re} z \geq 0$ is unbounded.

Accumulation Point:

A point z_0 is said to be an accumulation point of a set S if each deleted neighbourhood of z_0 contains at least one point of S.

Note:- If a set S is closed then it contains each of its accumulation points.

For, if z_0 is an accumulation point but $z_0 \notin S$, Then it would be a boundary point.

This is a contradiction, because a closed set contains all of its boundary points.

Conversely,

if a set S contains each of its accumulation points then it must be closed.

For if,

z_0 is a boundary point of S which is not in S. (ie) $z_0 \notin S$).

Then it would be an accumulation point of S.

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This is a contradiction to the fact that S contains all of its accumulation point.

Remark:-

A point z_0 is not an accumulation point of S whenever there exists some deleted neighbourhood of z_0 that does not contain atleast one point of S.

Ex:- The set $z_n = i/n$, $n = 1, 2, 3, \ldots$ has origin is the only accumulation point.

LIMITS

Let a function f be defined at all points z in some deleted neighbourhood of z_0. We say that the limit of $f(z)$ as z approaches z_0 is a number w_0, if for each positive number ϵ, there is a positive number δ such that,

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta. \quad \text{--- (1)}$$

It can be written as $\lim_{z \to z_0} f(z) = w_0$.

When a limit of a function $f(z)$ exists at a point z_0, it is unique:

Proof:-

Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then for each postive number ϵ, there are positive numbers δ_0 & δ_1 such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0 \quad \text{--- (1)}$$

and $$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1. \quad \text{--- (2)}$$

Let $\delta = \min(\delta_0, \delta_1)$.

So, if $0 < |z - z_0| < \delta$,

$$|w_1 - w_0| = |f(z) - w_1 - f(z) - w_0|$$
$$\leq |f(z) - w_0| + |f(z) - w_1|$$
$$< \epsilon + \epsilon = 2\epsilon.$$

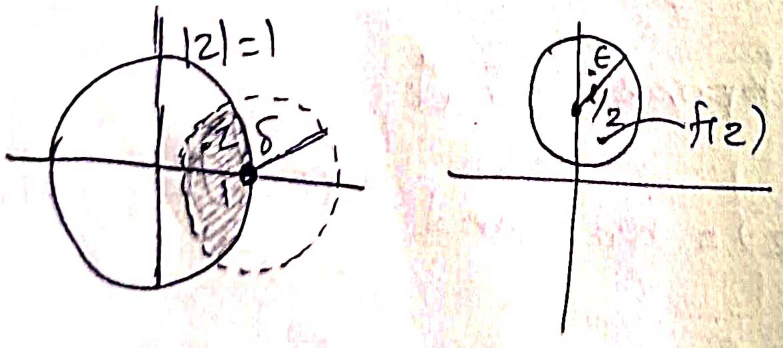
Since $|w_1 - w_0| \geq 0$ and ϵ is arbitrary small, we have:

$$|w_1 - w_0| = 0.$$

$$\Rightarrow w_1 - w_0 = 0 \text{ (or) } \boxed{w_0 = w_1}$$

Ex:1 Show that $\lim\limits_{z \to 1} f(z) = i/2$,

where $f(z) = \frac{i\bar{z}}{2}$ in the open disc $|z| < 1$.



For any positive number ϵ, we must find a positive number $\delta > 0$ such that

$$\left| f(z) - i/2 \right| < \epsilon \text{ whenever } 0 < |z - 1| < \delta \quad \text{——} (1).$$

(i) $\left|\frac{i\bar{z}}{2} - \frac{i}{2}\right| < \epsilon$ whenever $0 < |z-1| < \delta$.

(ii) $\frac{|z-1|}{2} < \epsilon$ 〃 $0 < |z-1| < \delta$.

If we choose $\delta = 2\epsilon$, ① holds.

$\therefore \lim_{z \to 1} \frac{i\bar{z}}{2} = i/2$.

Ex:2 Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Suppose, $\lim_{z \to 0} \frac{z}{\bar{z}} = w_0$.

This means that $z = (x, y)$ approaches $(0,0)$ in any direction, $\frac{z}{\bar{z}}$ approaches w_0.

But if $z = (x, 0)$ is a nonzero point on the real axis,

then $f(z) = \frac{x + i0}{x - i0} = 1$

$\lim_{z \to 0} f(z) = 1$. [along the real axis]

If $z = (0, y)$ is a nonzero point on the imaginary axis,

$f(z) = \frac{0 + iy}{0 - iy} = -1$.

$\therefore \lim_{z \to 0} f(z) = -1$, [along the imaginary axis]

But the limit w_0 is unique.

$\therefore \lim_{z \to 0} f(z)$ does not exist.

THEOREMS ON LIMITS

Theorem: 1

Suppose that

$$f(z) = u(x,y) + iv(x,y) \quad (z = x+iy)$$

and $x_0 + iy_0 = z_0$, $w_0 = u_0 + iv_0$

Then $\lim_{z \to z_0} f(z) = w_0$ if and only if —— (1)

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$ —— (2)

Proof:-

First Assume that

$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

$\therefore$ For each positive number ϵ, there exist positive numbers δ_1 and δ_2 such that

$\therefore$ $|u-u_0| < \epsilon/2$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_1$

and ——— ③

$|v-v_0| < \epsilon/2$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2$

——— ④

Let $\delta = \min(\delta_1, \delta_2)$.

Since

$$|f(z)-w_0| = |(u-u_0)+i(v-v_0)| \le |u-u_0|+|v-v_0|$$

and

$$|z-z_0| = |x+iy-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)|$$
$$= \sqrt{(x-x_0)^2+(y-y_0)^2}.$$

$\therefore$ From ③ and ④,

$$|f(z)-w_0| < \epsilon/2+\epsilon/2 = \epsilon, \text{ whenever } |z-z_0| < \delta.$$

That is $\lim_{z\to z_0} f(z) = w_0$.

Conversely,

Assume that $\lim_{z\to z_0} f(z) = w_0$.

$\therefore$ For each positive ϵ, there is a positive number δ such that

$$|f(z)-w_0| = |u+iv - u_0+iv_0| < \epsilon, \text{ whenever} \quad \text{——— (5)}$$

$$0 < |z-z_0| < \delta \ (\text{i.e.}) \ 0 < |x+iy-(x_0+iy_0)| < \delta \quad \text{——— (6)}$$

But

$$|u-u_0| \leq |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|$$
$$|v-v_0| \leq |u-u_0+i(v-v_0)| = |u+iv-(u_0+iv_0)|$$

and

$$|z-z_0| = |(x+iy)-(x_0+iy_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

Hence from (5) & (6),

$$|u-u_0| < \epsilon \text{ and } |v-v_0| < \epsilon,$$

whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$.

This implies

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \ \& \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Hence the theorem.

Theorem : 2

Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0. \quad \text{——— (1)}$$

Then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0. \quad \text{——— (2)}$$

$$\lim_{z \to z_0} [f(z) F(z)] = w_0 W_0. \quad \text{——— (3)}$$

and if $W_0 \neq 0$.

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \quad \text{——— (4)}$$

Proof:-

Let $f(z) = u + iv$, $F(z) = U + iV$,

$w_0 = u_0 + iv_0$, $W_0 = U_0 + iV_0$, $z = x + iy$, $z_0 = x_0 + iy_0$.

Given that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$

∴ By the theorem

" Theorem-1. " we have ——— *

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0, \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

$\lim_{(x,y)\to(x_0,y_0)} U(x,y) = U_0$, $\lim_{(x,y)\to(x_0,y_0)} V(x,y) = V_0$ 14

To prove (3)

Now $f(z)F(z) = (u+iv)(U+iV)$
$= (uU - vV) + i(uV + vU)$

Since u, v, U & V are real valued functions of two variables,

$$\lim_{(x,y)\to(x_0,y_0)} uU - vV = u_0U_0 - v_0V_0 \quad \text{and}$$

$$\lim_{(x,y)\to(x_0,y_0)} uV + vU = u_0V_0 + v_0U_0$$

Again by the theorem $*$,

$$\lim_{z\to z_0} f(z)F(z) = (u_0U_0 - v_0V_0) + i(u_0V_0 + v_0U_0).$$

$$= w_0W_0.$$

Hence (3).

To prove (2),

consider $f(z) + F(z) = (u+U) + i(v+V)$

Since

$$\lim_{(x,y)\to(x_0,y_0)} u+U = u_0+U_0 \text{ and}$$

$$\lim_{(x,y)\to(x_0,y_0)} v+V = v_0+V_0,$$

Again by the theorem ✱,

$$\lim_{z\to z_0} f(z)+F(z) = (u_0+U_0)+i(v_0+V_0)$$
$$= w_0+W_0.$$

Hence (2).

<u>To prove (8)</u>

$$\frac{f(z)}{F(z)} = \frac{u+iv}{U+iV} = \frac{u}{U^2+V^2} + i\frac{v}{U^2+V^2}.$$

Since $W_0 \neq 0$, U_0 & V_0 both not equal to zero.
(ie) $U_0^2+V_0^2 \neq 0$.

$$\therefore \lim_{(x,y)\to(x_0,y_0)} \frac{u}{U^2+V^2} = \frac{u_0}{U_0^2+V_0^2}$$

$$\text{and } \lim_{(x,y)\to(x_0,y_0)} \frac{v}{U^2+V^2} = \frac{v_0}{U_0^2+V_0^2}$$

Again by the theorem ✱.

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{u_0}{U_0^2 + V_0^2} + i \frac{v_0}{U_0^2 + V_0^2}$$

$$= \frac{w_0}{W_0} \text{ provided } W_0 \neq 0.$$

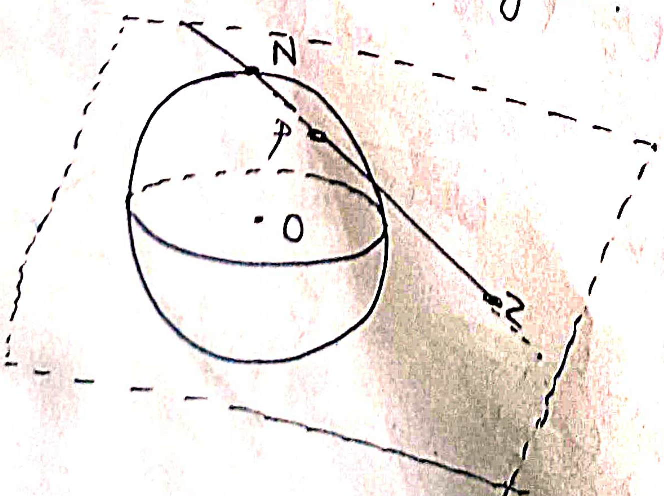
Hence (4).

LIMIT INVOLVING THE POINT AT INFINITY.

The complex plane together with the point at infinity denoted by ∞ is called the extended complex plane.

Stereographic Projection.

Consider the complex plane passing through the equator of a unit sphere centered at the origin.



To each point z in the plane there corresponds exactly one point P on the surface of the sphere.

The point P is the point where the line through z and the north pole N intersects the sphere.

Similarly, to each point P on the surface of the sphere, other than the north pole, there corresponds exactly one point z in the plane.

By letting the point N of the sphere correspond to the point at infinity, we obtain a 1-1 correspondence b/w the points of the sphere and points of the extended complex plane.

The sphere is known as the Riemann sphere and the correspondence is called a stereographic projection.

For each positive number ϵ, the set $|z| > 1/\epsilon$ is called an ϵ-neighbor-hood of ∞.

Defn: (1) The statement $\lim_{z \to z_0} f(z) = \infty$, means that

For each positive number $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$|f(z)| > 1/\epsilon$ whenever $0 < |z - z_0| < \delta$.

(2) The statement $\lim_{z \to \infty} f(z) = w_0$ means that

For each positive number $\epsilon > 0$, $\exists \delta > 0$ $\ni$

$|f(z) - w_0| < \epsilon$ whenever $|z| > 1/\delta$.

(3) The statement $\lim_{z \to \infty} f(z) = \infty$ means that For each $\epsilon > 0$, $\exists$ $\delta > 0$ $\ni$

$|f(z)| > 1/\epsilon$ whenever $|z| > 1/\delta$.

Theorem:- If z_0 and w_0 are points in the z plane and w plane respectively

Then $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$ ——— ①

$\lim_{z \to \infty} f(z) = w_0$ iff $\lim_{z \to 0} f(1/z) = w_0$. ——— ②

$\lim_{z \to \infty} f(z) = \infty$ iff $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$ ——— ③.

Proof:-

(1) consider $\lim_{z \to z_0} f(z) = \infty$

$\Leftrightarrow$ For each positive $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z)| > \frac{1}{\epsilon} \text{ whenever } 0 < |z - z_0| < \delta.$$

$\Leftrightarrow$ $\frac{1}{|f(z)|} < \epsilon$ whenever $0 < |z - z_0| < \delta$

$\Leftrightarrow$ $\left| \frac{1}{f(z)} - 0 \right| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

$\Leftrightarrow$ $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

(2) Consider

$$\lim_{z\to\infty} f(z) = w_0$$

$\Leftrightarrow$ For each positive number $\epsilon > 0$, $\exists$ a positive number δ such that,

$$|f(z) - w_0| < \epsilon \text{ whenever } |z| > 1/\delta.$$

$\Leftrightarrow$ $|f(1/z) - w_0| < \epsilon$ whenever $0 < |z - 0| < \delta$.

$\Leftrightarrow$ $$\lim_{z\to 0} f(1/z) = w_0.$$

(3) Consider

$$\lim_{z\to\infty} f(z) = \infty.$$

$\Leftrightarrow$ For each $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(z)| > 1/\epsilon \text{ whenever } |z| > 1/\delta.$$

$\Leftrightarrow$ $\dfrac{1}{|f(z)|} < \epsilon$ whenever $\dfrac{1}{|z|} < \delta$.

$\Leftrightarrow$ $\dfrac{1}{|f(1/z)|} < \epsilon$ whenever $0 < |z - 0| < \delta$

$\Leftrightarrow$ $\left|\dfrac{1}{f(1/z)} - 0\right| < \epsilon$ whenever $0 < |z - 0| < \delta$.

$$\Leftrightarrow \lim_{z\to 0} \frac{1}{f(1/z)} = 0.$$

Hence the theorem.

Ex:-

1. P.T $\displaystyle\lim_{z\to -1} \frac{iz+3}{z+1} = \infty.$

W.K.T $\displaystyle\lim_{z\to z_0} f(z) = \infty \Leftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0.$

Consider $\displaystyle\lim_{z\to -1} \frac{z+1}{iz+3} = 0.$

$$\therefore \lim_{z\to -1} \frac{iz+3}{z+1} = \infty.$$

2. P.T $\displaystyle\lim_{z\to\infty} \frac{2z+i}{z+1} = 2.$

W.K.T $\displaystyle\lim_{z\to\infty} f(z) = w_0 \Leftrightarrow \lim_{z\to 0} f(1/z) = w_0.$

$\therefore$ Consider $\displaystyle\lim_{z\to 0} \frac{2/z+i}{1/z+1}$

$$= \lim_{z\to 0}\left(\frac{2+iz}{1+z}\right) = 2.$$

$$\therefore \lim_{z\to\infty} \frac{2z+i}{z+1} = 2.$$

3. P.T $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$.

W.K.T $\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f(1/z)} = 0$

Consider

$$\lim_{z\to 0} \frac{1/z^2+1}{2/z^3-1}$$

$$= \lim_{z\to 0} \frac{(1+z^2)z}{2-z^3}$$

$$= \frac{0}{2} = 0.$$

$$\therefore \lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty.$$

CONTINUITY

A function f is continuous at a point z_0 if all three of the following conditions are satisfied.

(1) $\lim_{z\to z_0} f(z)$ exists

(2) $f(z_0)$ exists

(3) $\lim_{z\to z_0} f(z) = f(z_0)$

Note :-

The statement (3) actually contains (1) & (2)

The statement $\lim_{z \to z_0} f(z) = f(z_0)$ means that for each positive number ϵ, there is a positive number δ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

A function of a complex variable is said to be continuous in a Region R if it is cts at each point in R.

Theorem :- A composition of continuous functions is itself continuous.

Proof :-

Let $w = f(z)$ be a function that is defined for all z in a nbd $|z - z_0| < \delta$ of a point z_0.

Let $W = g(w)$ be a function whose domain of definition contains the image of the nbd under f.

Then the composition $W = g(f(z))$ is defined for all z in the nbd $|z - z_0| < \delta$.

Suppose that f is cts at z_0 and g is cts at $f(z_0)$ in the w plane. Then

We shall prove that the composition $g[f(z)]$ is continuous at z_0.

Since g is cts at $f(z_0)$,

for each positive number ϵ, there is a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \epsilon \text{ whenever } |f(z) - f(z_0)| < \gamma \quad \text{——— (1)}$$

Since f is cts at z_0, for given positive number γ, there is a positive number δ such that,

$$|f(z) - f(z_0)| < \gamma \text{ whenever } |z - z_0| < \delta \quad \text{——— (2)}$$

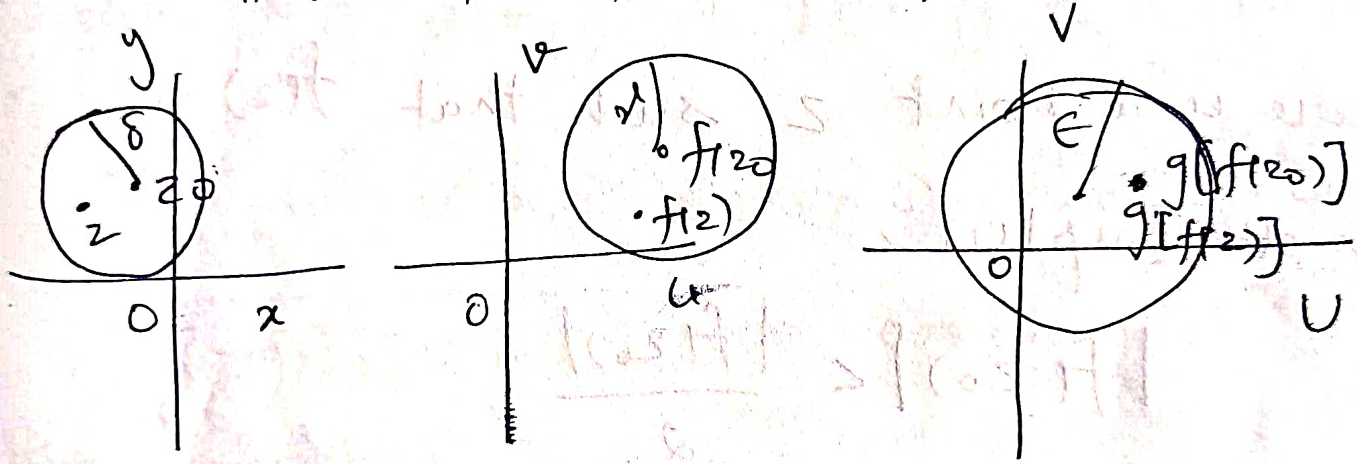
Hence for each positive number ϵ, there is a $\delta > 0$ such that

$|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \gamma$ and

$|f(z) - f(z_0)| < \gamma$ implies $|g[f(z)] - g[f(z_0)]| < \epsilon$.

(ii) $(g \circ f)(z)$ is continuous at z_0.

Hence the theorem.



Theorem:- If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point.

Proof:-

Assume that $f(z)$ is continuous at the point z_0 and $f(z_0) \neq 0$.

Let $\epsilon = \dfrac{|f(z_0)|}{2}$.

For this positive ϵ, there exists a positive number $\delta > 0$ such that,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta. \quad \text{——— (1)}$$

~~Suppose~~ In the neighbourhood $|z - z_0| < \delta$,
If there is a point z such that $f(z) = 0$,
then (1) implies,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

This is a contradiction.

Hence in the neighbourhood $|z - z_0| < \delta$, $f(z) \neq 0$.

Hence the theorem.

<u>Theorem:-</u> If a function f is continuous throughout a region R that is both closed and bounded, there exists a

nonnegative real number M such that

$$|f(z)| \leq M \text{ for all points } z \text{ in } R,$$

where the equality holds for atleast one such z.

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Proof:

Let $f(z) = u(x,y) + i v(x,y)$.

Since $f(z)$ is continuous throughout the region R, $u(x,y)$ & $v(x,y)$ are continuous in R.

Therefore $u^2(x,y)$ & $v^2(x,y)$ and hence

$\sqrt{[u(x,y)]^2 + [v(x,y)]^2}$ are continuous throughout R.

$\therefore |f(z)| = \sqrt{[u(x,y)]^2 + [v(x,y)]^2}$ attains a maximum value M at some point in R.

Hence $|f(z)| \leq M$.

DERIVATIVES

Defn:-

Let f be a function whose domain of definition contains a neighbourhood $|z-z_0|<\epsilon$ of a point z_0.

The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{---} (1)$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

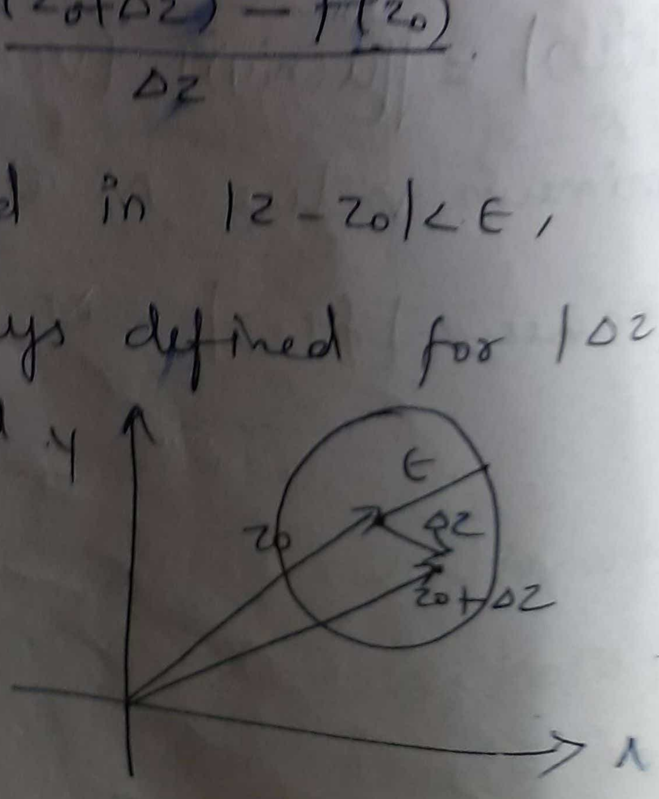
Remark:

Let $\Delta z = z - z_0 \; (z \neq z_0)$

Then (1) $\Rightarrow$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

Since f is defined in $|z-z_0|<\epsilon$, $f(z_0+\Delta z)$ is always defined for $|\Delta z|$ sufficiently small.



Let $\Delta w = f(z+\Delta z) - f(z)$.

Then $\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$.

Ex: 1.3. The existence of the derivative of a function at a point implies the continuity of the function at that point.

Proof:-

Assume that $f'(z_0)$ exists.

consider

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0)$$

$$\Rightarrow \lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0.$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

$\therefore$ f is continuous at z_0.

The continuity of a function at a point does not imply the existence of a derivative at that point.

For example, consider the function

$$f(z) = |z|^2$$

(ii) $f(z) = u + iv = |z|^2 = \sqrt{x^2+y^2}$

$\therefore u(x,y) = \sqrt{x^2+y^2}$ & $v(x,y) = 0$.

Since $u(x,y)$ and $v(x,y)$ are continuous at each point, $f(z) \overset{=|z|^2}{\text{is}}$ continuous at each point.

Consider $\dfrac{\Delta w}{\Delta z} = \dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z}$

$= \dfrac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$

$= \bar{z} + \overline{\Delta z} + z \cdot \dfrac{\overline{\Delta z}}{\Delta z}$

When $\Delta z = (\Delta x, 0)$, $\dfrac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z$

and when $\Delta z = (0, \Delta y)$,

$\dfrac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z$

If $\lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z}$ exists, the uniqueness of limits implies

$\cancel{\bar{z} + \Delta z + z = \bar{z} - \Delta z} - z$

$$\bar{z} + z = \bar{z} - z$$

$$\Rightarrow z = 0.$$

Hence $\frac{dw}{dz}$ cannot exist at $z \neq 0$.

At $z = 0$, $\frac{dw}{dz} = \lim\limits_{\Delta z \to 0} \Delta \bar{z} = 0$.

$\therefore f'(0) = 0$.

Hence $f(z) = |z|^2$ is continuous at each point but f is not differentiable at each point except at $z = 0$.

DIFFERENTIATION FORMULAS

The derivative of a function f at a point z is denoted by either $\frac{d f(z)}{dz}$ (or) $f'(z)$.

If c is a complex constant and f is a function whose derivative exists at a point z. Then S.T

(1) $\frac{d}{dz} c = 0$ (2) $\frac{d}{dz} z = 1$ (3) $\frac{d}{dz}[c f(z)] = c f'(z)$.

Soln:-

$f(z) = c$.

$f(z + \Delta z) = c$

$$\Delta w = f(z+\Delta z) - f(z) = c - c = 0$$

$$\therefore \frac{\Delta w}{\Delta z} = 0.$$

$$\therefore \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = 0 \Rightarrow \frac{d}{dz} c = 0.$$

(2) $\Delta w = \cancel{z} + \Delta z - \cancel{z} = \Delta z$

$$\therefore \frac{\Delta w}{\Delta z} = 1.$$

$$\frac{d}{dz} z = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = 1.$$

(3). $\Delta w = (cf)(z+\Delta z) - (cf)(z)$

$$= c\left[f(z+\Delta z) - f(z)\right]$$

$$\therefore \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{d}{dz}(cf(z)) = c. \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f}{\Delta z.}$$

$$= c\, f'(z).$$

If n is a positive integer,

S.T $\frac{d}{dz} z^n = n z^{n-1}.$

$f(z) = z^n$, $f(z+\Delta z) = (z+\Delta z)^n$.

$$(z+\Delta z)^n = z^n + nC_1 z^{n-1} \cdot \Delta z + nC_2 z^{n-2} \Delta z^2 + \cdots$$
$$\cdots + \Delta z^n .$$

$$f(z+\Delta z) - f(z) = \Delta z \left(n z^{n-1} + nC_2 z^{n-2} \Delta z + \cdots + \Delta z^{n-1} \right)$$

$$\therefore \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = n z^{n-1} .$$

$$\boxed{\frac{d}{dz} z^n = n z^{n-1}}$$

<u>Note:</u> This formula remains valid If n is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions f and g exist at a point z, then

P.T (1) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$

(2) $\frac{d}{dz}[f(z)\, g(z)] = f(z) g'(z) + g(z) f'(z)$

(3) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z) f'(z) - f(z) g'(z)}{[g(z)]^2}$

provided that $g(z) \neq 0$.

(1) Let $w = f(z) + g(z)$.

Then $\Delta w = f(z+\Delta z) + g(z+\Delta z) - f(z) - g(z)$

$= f(z+\Delta z) - f(z) + g(z+\Delta z) - g(z)$

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} + \frac{g(z+\Delta z) - g(z)}{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$$

$$\Rightarrow \frac{d}{dz}\left[f(z) + g(z)\right] = f'(z) + g'(z).$$

(2) Let $w = f(z)g(z)$.

$\Delta w = f(z+\Delta z)\, g(z+\Delta z) - f(z)g(z)$

$= f(z)\, g(z+\Delta z) + f(z+\Delta z) g(z+\Delta z) - f(z)g(z) - f(z)\, g(z+\Delta z)$.

$= f(z)\left[g(z+\Delta z) - g(z)\right] + g(z+\Delta z)\left[f(z+\Delta z) - f(z)\right]$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\, \frac{g(z+\Delta z) - g(z)}{\Delta z} + g(z+\Delta z) \frac{\left[f(z+\Delta z) - f(z)\right]}{\Delta z}$$

letting Δz tend to zero, we get $\lim_{\Delta z \to 0} g(z+\Delta z)$

$$\frac{d}{dz}[f(z)\,g(z)] = f(z)\,g'(z) + f'(z)\,\cancel{g(z)}.$$

It is clear that $\lim_{\Delta z \to 0} g(z+\Delta z) = \lim_{p \to z} g(p)$.

~~Since g is continuous because g'(z) exists~~

Since $g'(z)$ exists, g is continuous at z.

$$\therefore \lim_{p \to z} g(p) = g(z)$$

Thus $$\frac{d}{dz}[f(z)\,g(z)] = f(z)\,g'(z) + f'(z)\cdot g(z).$$

(3) $$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{[g(z)]^2}.$$

Let $w = \dfrac{f(z)}{g(z)}$

$$\Delta w = \frac{f(z+\Delta z)}{g(z+\Delta z)} - \frac{f(z)}{g(z)}$$

$$= \frac{g(z)\,f(z+\Delta z) - f(z)\,g(z+\Delta z)}{g(z)\,g(z+\Delta z)}.$$

$$= \frac{g(z)\,f(z+\Delta z) - f(z)\,g(z) + f(z)\,g(z) - f(z)\,g(z+\Delta z)}{g(z)\,g(z+\Delta z)}.$$

$$\Delta w = g(z)\,\frac{f(z+\Delta z) - f(z)}{g(z)\,g(z+\Delta z)} - f(z)\,\frac{g(z+\Delta z) - g(z)}{g(z)\,g(z+\Delta z)}$$

$$\frac{\Delta w}{\Delta z} = g(z)\cdot\frac{f(z+\Delta z) - f(z)}{\Delta z} \times \frac{1}{g(z)\,g(z+\Delta z)}$$

$$f(z)\,\frac{g(z+\Delta z) - g(z)}{\Delta z} \times \frac{1}{g(z)\,g(z+\Delta z)}$$

Since g is continuous,

$$\lim_{\Delta z \to 0} g(z+\Delta z) = \lim_{P \to z} g(P) = g(z).$$

$\therefore$ letting $\Delta z \to 0$,

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{-f(z)\cdot g'(z) + g(z)\,f'(z)}{[g(z)]^2}$$

$$= \frac{g(z)\,f'(z) - f(z)\,g'(z)}{[g(z)]^2}$$

* **Chain Rule:-**

Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the function

$F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]\, f'(z_0). \quad \text{——} ①.$$

[If we write $w = f(z)$ $W = g(w)$, so that $W = F(z)$, then the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}.]$$

Proof:-

Given that f has a derivative at z_0 and g has a derivative at $f(z_0)$.

Let $w_0 = f(z_0)$.

Then $f'(z_0)$ and $g'(w_0)$ exist.

That is $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ and —— (1)

$$g'(w_0) = \lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0}. \quad \text{——} (2)$$

Define a function Φ in some ϵ-neighbourhood $|w - w_0| < \epsilon$ of w_0 by

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when } w \neq w_0$$

—— (3)

$$= 0 \quad \text{when } w = w_0.$$

From (2),

$$\lim_{\omega \to \omega_0} \Phi(\omega) = g'(\omega_0) - g'(\omega_0) = 0 = \Phi(\omega_0) \quad \text{(4)}$$

$\therefore \Phi$ is continuous at ω_0.

Now (3) can be written in the form,

$$g(\omega) - g(\omega_0) = [g'(\omega_0) + \Phi(\omega)](\omega - \omega_0) \text{ in } |\omega - \omega_0| < \epsilon. \quad \text{(5)}$$

This expression is valid even when $\omega = \omega_0$.

Since $f'(z_0)$ exists, f is continuous at z_0.

$\therefore$ For the given positive number ϵ, we can choose a positive number δ such that

$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

(i.e.) $|\omega - \omega_0| < \epsilon$ whenever $|z - z_0| < \delta$. (6)

$\therefore$ For $0 < |z - z_0| < \delta$, (5) becomes,

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \left\{ g'[f(z_0)] + \Phi[f(z)] \right\} \frac{f(z) - f(z_0)}{z - z_0}$$

Since f is continuous at z_0 and Φ is continuous at $f(z_0) = w_0$, the composition $\Phi[f(z)]$ is continuous at z_0.

$$\therefore \lim_{z \to z_0} \Phi[f(z)] = \Phi[f(z_0)] = \Phi(w_0) = 0.$$

$$\text{(ie)} \lim_{z \to z_0} \Phi[f(z_0)] = 0. \quad \text{——— (8)}$$

$\therefore$ (7) implies,

$$\lim_{z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \left[g'[f(z_0)] + 0\right] f'(z_0).$$

$$\text{(ie)} \quad F'(z_0) = g'[f(z_0)]\, f'(z_0)$$

Hence the proof.

CAUCHY-RIEMANN EQUATIONS.

Theorem:-

Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy Riemann equations

$$u_x = v_y, \quad u_y = -v_x \quad \text{——— (I)}$$

there.

Also $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x \quad \text{———} \quad (P)$$

where these partial derivatives are to be evaluated at (x_0, y_0).

<u>Proof :-</u>

Given that

$$f(z) = u(x,y) + i\, v(x,y) \quad \text{———} \quad (1)$$

$z_0 = x_0 + iy_0$ and $f'(z_0)$ exists.

Let $\Delta z = \Delta x + i\Delta y$ and $w = f(z)$.

Then $\Delta w = f(z_0 + \Delta z) - f(z_0)$

$$= \left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)\right] + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right].$$

and $$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad \text{———} \quad (2).$$

Then by the theorem

" If $f(z) = u(x,y) + iv(x,y)$ and $z = x+iy$, $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \ \& \ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

We have,

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Re} \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Im} \frac{\Delta w}{\Delta z} \right) \quad \text{---(3)}.$$

It is clear that (3) is valid as $(\Delta x, \Delta y) \to (0,0)$ in any manner.

Let $(\Delta x, \Delta y) \to (0,0)$ horizontally through the points $(\Delta x, 0)$.

Then $$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Thus

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Re} \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$= u_x(x_0, y_0).$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\text{Im} \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= v_x(x_0, y_0).$$

Where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first order partial derivatives w.r.to x of the functions u and v respectively at (x_0, y_0)

$\therefore$ (3) $\Rightarrow$,

$$\boxed{f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)}$$

Let $(\Delta x, \Delta y) \to (0,0)$ vertically through the points $(0, \Delta y)$,

Then
$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$= \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y}$$

Thus,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} Re\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y}$$

$$= v_y(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(Im \frac{\Delta w}{\Delta z}\right) = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y}$$

$$= -u_y(x_0, y_0)$$

Hence (3) implies,

$$f'(z_0) = v_y(x_0, y_0) - i\,u_y(x_0, y_0) \longrightarrow (5).$$

where u_y & v_y are the partial derivatives w.r.to y.

Since $f'(z_0)$ exists, we must have,

$$u_x(x_0, y_0) + i\,v_x(x_0, y_0) = v_y(x_0, y_0) - i\,u_y(x_0, y_0)$$

Equating the real & Imaginary parts,

$$\left.\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \text{ and} \\ v_x(x_0, y_0) &= -u_y(x_0, y_0) \end{aligned}\right\} \longrightarrow (6).$$

These equations are called Cauchy-Riemann equations.

Note:-

(1) (5) can be written as

$$\boxed{f'(z_0) = -i\left[u_y + i v_y\right]}$$

Thus $f'(z_0) = u_x + i v_x = -i\left[u_y + i v_y\right]$ at (x_0, y_0).

(2) The above theorem is the neccessary condition for differentiabily of $f(z)$ at z_0.

Eg:1 S.T $f(z) = z^2$ is diff.ble everywhere and $f'(z) = 2z$ and verify the C-R equations are satisfied everywhere.

Soln:-

We proved already f is diff.ble everywhere and $f'(z) = 2z$.

Now $f(z) = z^2 = x^2 - y^2 + i2xy$.

$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

$u_x = 2x$ $\qquad v_x = 2y$

$u_y = -2y$ $\qquad v_y = 2x$.

$\therefore u_x = v_y = 2x$.

$u_y = -v_x = -2y$.

Also $f'(z) = u_x + iv_x$

$= 2x + i2y = 2(x+iy) = 2z$.

Eg.2 $f(z) = |z|^2$.

here $u(x,y) = \cancel{\sqrt{\ }}\ x^2 + y^2$

$v(x,y) = 0$.

W.K.T f is not differentiable at any nonzero point.

$u_x = 2x \qquad u_y = 2y$

$v_x = 0 \qquad v_y = 0.$

C.R equations are satisfied only at $(0,0)$.

SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY.

Satisfaction of Cauchy - Riemann equations at a point z_0 is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point.

Theorem: Let the function $f(z) = u(x,y) + i\,v(x,y)$ be throughout some ϵ-neighbourhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) The first order partial derivatives of the functions u & v w.r.to x and y exist everywhere in the neighbourhood;

(b) Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x \text{ at } (x_0, y_0).$$

Then $f'(z_0)$ exists and its value being

$$f'(z_0) = u_x + i v_x \text{ at } (x_0, y_0).$$

Proof:-

Assume the conditions (a) & (b) in its hypothesis are satisfied.

Let $\Delta z = \Delta x + i \Delta y$ where $0 < |\Delta z| < \epsilon$.

Then $\Delta w = f(z_0 + \Delta z) - f(z_0)$

Thus,
$$\Delta w = \Delta u + i \Delta v \quad \text{———} (1)$$

where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

and
$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

Consider
$$\Delta u = [u(x_0 + \Delta x, y_0) - u(x_0, y_0)] + [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 + \Delta x, y_0)] \quad \text{———} (2)$$

Since u_x exists, By Mean value theorem

$$\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_1, y_0),$$

where $x_0 < x_1 < x_0 + \Delta x$.

Since u_x is continuous at (x_0, y_0),

The difference $\epsilon_1 = u_x(x_1, y_0) - u_x(x_0, y_0)$

approaches 0 as $\Delta x \to 0$.

$\therefore u(x_0 + \Delta x, y_0) - u(x_0, y_0) = u_x(x_0, y_0)\Delta x + \epsilon_1 \Delta x$. ——— (3)

Similarly, since u_y exists and continuous at (x_0, y_0), we have

$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 + \Delta x, y_0) = u_y(x_0, y_0)\Delta y + \epsilon_2 \Delta y$ ——— (4)

where $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

From (3) & (4), (2) implies

$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ ——— (5)

Since v_x, v_y exists and continuous at (x_0, y_0), the same argument follows,

$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$ ——— (6)

where $\epsilon_2, \epsilon_3 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ in the

Δz plane.

Substituting ⑤ & ⑥ in ①, we get

$$\Delta w = [u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y] + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y] \quad \text{——— ⑦}$$

By the assumption (b), C-R equations are satisfied at (x_0, y_0).

That is $u_x(x_0, y_0) = v_y(x_0, y_0)$ and

$u_y(x_0, y_0) = -v_x(x_0, y_0)$

Therefore ⑦ implies,

$$\Delta w = (u_x(x_0, y_0) + i v_x(x_0, y_0))\Delta x$$
$$(u_x(x_0, y_0) + i v_x(x_0, y_0)) i\Delta y$$
$$+ \Delta x(\epsilon_1 + i\epsilon_3) + \Delta y(\epsilon_2 + i\epsilon_4)$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} \quad \text{——— ⑧}$$

But $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$, according to $Re\, z \le |z|$ and $Im\, z \le |z|$.

So $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$.

Consequently,

$$\left|(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\epsilon_1 + i\epsilon_3| \le |\epsilon_1| + |\epsilon_3|$$

and $$\left|(\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\epsilon_2 + i\epsilon_4| \le |\epsilon_1| + |\epsilon_4|$$

[Taking ~~lim~~ $\Delta z \to 0$ ~~on~~ both ~~sides~~ ~~of~~ (8),

$\lim_{\Delta z \to 0}$]

When $\Delta z = \Delta x + i\Delta y$ approaches zero, these two terms tends to zero.

It follows from (8),

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

(ie) $f'(z_0)$ exists and

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

Hence the theorem.

Ex 2 If, $f(z) = |z|^2$, Prove that $f'(z)$ does not exist at any non zero point.

Sol'n:- $f(z) = |z|^2 = x^2 + y^2$

$u = x^2 + y^2$, $v = 0$.

$u_x = 2x$ $\quad v_x = 0$

$u_y = 2y$ $\quad v_y = 0$.

At $z \neq 0$, C-R equations are not satisfied

$\therefore f'(z)$ does not exist at $z \neq 0$.

At $z = (x, y) = (0, 0)$,

$u_x = 0 = v_y$, $u_y = 0 = -v_x$.

C-R-equations are satisfied and u_x, u_y, v_x, v_y are continuous.

Hence $f'(0)$ exists and $f'(0) = u_x + i v_x$
$= 0 + i0 = 0$.

Ex: 1 Consider the exponential function $f(z)=e^z$. S.T $f'(z)$ exists for all z and $f'(z)=f(z)$.

Soln:- $f(z) = e^z = e^{x+iy} = e^x . e^{iy}$

$= e^x(\cos y + i\sin y)$ [by Euler formula]

$\therefore u = e^x \cos y \quad v = e^x \sin y$.

$u_x = e^x \cos y \quad v_x = e^x \sin y$

$u_y = -e^x \sin y \quad v_y = e^x \cos y$.

Hence $u_x = v_y$ & $u_y = -v_x$ for all (x,y),

and u_x, u_y, v_x, v_y are continuous.

Thus $f'(z)$ exists everywhere, and

$f'(z) = u_x + i v_x = e^x \cos y + ie^x \sin y$

$= e^x . (\cos y + i \sin y)$

$= e^x . e^{iy} = e^z$

(ie) $f'(z) = f(z)$ for all z.

POLAR COORDINATES.

Assuming that $z_0 \neq 0$,

Consider the transformation

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \text{—— (1)}$$

$$z = x + iy \quad (\text{or}) \quad z = re^{i\theta} \quad (z \neq 0).$$

When $w = f(z) = u(x,y) + iv(x,y)$.

Suppose that u_x, u_y, v_x & v_y exist and are continuous at z_0.

Then by chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \&$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}.$$

$$\Rightarrow u_r = u_x \cos\theta + u_y \sin\theta \qquad \text{—— (2)}$$

$$u_\theta = -u_x r\sin\theta + u_y \cdot r\cos\theta. \qquad \text{—— (3)}$$

$|||^{ly}$

$$v_r = v_x \cos\theta + v_y \sin\theta \qquad \text{—— (4)}$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta \qquad \text{—— (5)}$$

If ~~the~~ the partial derivatives of u & v satisfies C.R equations, then

$$u_x = v_y \ \& \ u_y = -v_x \quad \text{——— (6) at}$$

z_0, Then ④ & ⑤ becomes.

$$\left.\begin{aligned} v_r &= -u_y\cos\theta + u_x\sin\theta, \\ v_\theta &= u_y r\sin\theta + u_x r\cos\theta \end{aligned}\right\} \longrightarrow ⑦ \text{ at the point } z_0.$$

It is clear from ②, ③ & ⑥,

$$\boxed{r u_r = v_\theta \text{ and } u_\theta = -r v_r}$$

These equations are called C-R equations in Polar form.

Also, From ② & ④,

$$u_r + i v_r = \cos\theta(u_x + i v_x) + \sin\theta(u_y + i v_y)$$

$$\begin{aligned} r(u_r + i v_r) &= r\cos\theta(u_x + i v_x) + r\sin\theta(u_y + i v_y) \\ &= x(u_x + i v_x) + y(u_y + i v_y) \\ &= x f'(z) + i y(-i)(u_y + i v_y) \\ &= f'(z)(x + iy) \\ &= f'(z)\, z. \end{aligned}$$

$\therefore f'(z) = \frac{r}{z}(u_r + i v_r)$

$$\boxed{f'(z) = e^{-i\theta}\left[u_r + i v_r\right]} \quad \left\{ \because z = re^{i\theta} \right.$$

<u>Ex:-</u> 1.

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

$\therefore u(r,\theta) = \frac{\cos\theta}{r}$ $\qquad v(r,\theta) = -\frac{\sin\theta}{r}$

$u_r = -\frac{\cos\theta}{r^2}$ $\qquad v_r = +\frac{\sin\theta}{r^2}$

$u_\theta = -\frac{\sin\theta}{r}$ $\qquad v_\theta = -\frac{\cos\theta}{r}$

The first order partial derivatives u_r, u_θ, v_r, v_θ exists and are continuous.

Also, the CR equations

$$u_r = \frac{v_\theta}{r} \quad \text{and} \quad u_\theta = -r v_r$$

are satisfied at each $z \neq 0$.

Hence $f'(z)$ exists when $z \neq 0$

Also $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(\frac{-\cos\theta}{r^2} + i\,\frac{\sin\theta}{r^2}\right)$$

$$= \frac{-e^{-i\theta}}{r^2}\left(\cos\theta - i\sin\theta\right) = \frac{-e^{-i\theta}}{r^2}\cdot e^{-i\theta}$$

$$= \frac{-e^{-2i\theta}}{r^2} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}.$$

$$\boxed{f'(z) = -1/z^2.}$$

Example.2.

P.T $f(z) = \sqrt[3]{r}\,e^{i\theta/3}$, $r>0$, $\alpha<\theta<\alpha+2\pi$, where α is a fixed real number, has a derivative everywhere in its domain of definition.

Soln:-

$$\cancel{f(z) = \left(re^{i\theta/2}\right)^{1/3}}$$

$$\cancel{= r^{3/2}\, e^{i\theta 3/4}}$$

$$u(r,\theta) = r\,.\cos\cancel{\frac{3\theta}{4}}$$

$$f(z) = r^{1/3}\cdot e^{i\theta/3}$$

$$\therefore u(r,\theta) = r^{1/3}\cos\theta/3 \qquad v(r,\theta) = r^{1/3}\sin\theta/3.$$

$$u_r = 1/3\, r^{-2/3}\cos\theta/3 \qquad v_r = 1/3\, r^{-2/3}\sin\theta/3$$

$$u_\theta = -r^{1/3}.\sin\theta/3\cdot 1/3 \qquad v_\theta = 1/3\, r^{1/3}\cos\theta/3$$

Here $u_r = \frac{v_\theta}{r}$ and

$u_\theta = -r . v_r$. Hence C-R equations are satisfied.

Also $u_r, u_\theta, v_r, v_\theta$ are continuous.

$\therefore f'(z)$ exists at each point of z

where $f(z)$ is defined.

Also $f'(z) = e^{-i\theta}[u_r + i v_r]$

$= e^{-i\theta}\left[\frac{1}{3} r^{-2/3} \cos\theta/3 + i \frac{1}{3} r^{-2/3} \sin\theta/3\right]$

$= e^{-i\theta} . \frac{1}{3} r^{-2/3} [\cos\theta/3 + i \sin\theta/3]$

$= \frac{1}{3} e^{-i\theta + i\theta/3} . \frac{1}{3} r^{-2/3}$

$= \frac{1}{3} r^{-2/3} . e^{-2i\theta/3} = \frac{1}{3}\left(r^{-1/3} e^{-i\theta/3}\right)^2$

$= \frac{1}{3[f(z)]^2}$

$$\boxed{\therefore \frac{d}{dz}\left[z^{1/3}\right] = \frac{1}{3[z^{1/3}]^2}}$$

ANALYTIC FUNCTIONS

Defn: A function f of the complex variable z is **analytic** at a point z_0 if it has a derivative at each point in some neighborhood of z_0.

Note:

1. If f is analytic at a point z_0, then it must be analytic at each point in some neighbourhood of z_0.

2. A function f is analytic in an open set if it has a derivative everywhere in that set.

3. A function f is analytic in a set S which is not open, if it is analytic in an open set containing S.

Ex:-

1. $f(z) = 1/z$ is analytic at each nonzero z in the finite plane.

2. $f(z) = |z|^2$ is not analytic at any point, since its derivative exists only at $z=0$, not throughout any neighborhood.

Defn:- An entire function is a function that is analytic at each point in the entire finite plane.

Ex:- Every polynomial is an entire function because the derivative of a polynomial exists everywhere.

Defn: If a function f is not analytic at a point z_0, but is analytic at some point in every neighborhood of z_0, then z_0 is called a singular point or singularity of f

Ex: 1. The point $z=0$ is a singular point of the function $f(z) = \frac{1}{z}$.

2. $f(z) = |z|^2$ has no singular points, since it is nowhere analytic.

Necessary condition for Analyticity:-

1. A function f is analytic in a domain D, then f is continuous in D.

2. A function f is analytic in a domain D,

then f satisfies Cauchy - Riemann equations in each point of D.

Theorem: If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Proof:-

Let $f(z) = u(x,y) + i v(x,y)$.

Assume that $f'(z) = 0$ in D.

We note that $f'(z) = u_x + i v_x$.

$$\therefore u_x + i v_x = 0 \quad \text{———} \quad (1)$$

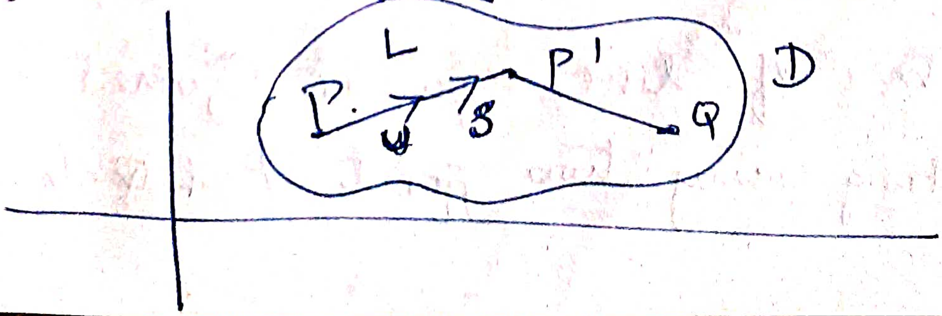
By Cauchy - Riemann equations,

$$v_y - i u_y = 0 \quad \text{———} \quad (2).$$

~~From (1) & (2),~~ $\cancel{(u_x + v_y) + i(v_x - u_y) = 0}$.

From (1) & (2) $u_x = u_y = 0$ and $v_x = v_y = 0$ at each point in D.

Let L be a line segment from a point P to a point P' lying entirely in D.



We shall show that $u(x, y)$ is constant in L.

Let s denote the distance along L from the point P and

let $\hat{v}$ be the unit vector along L in the direction of increasing s.

$\therefore$ The directional derivative $\frac{du}{ds}$ can be written as

$$\frac{du}{ds} = \text{grad } u \cdot \hat{v}, \quad \text{———} \quad (3)$$

where $\text{grad } u = u_x \vec{i} + u_y \vec{j}$ ——— (4).

Since $u_x = u_y = 0$ everywhere in D,

grad $u = \vec{0}$ at all points on L.

Hence from (3), $\frac{du}{ds} = 0$ along L.

This means that u is constant on L.

Since D is a domain, there is always a finite number of line segments joined end to end connecting any two points P & Q in D.

Hence the values of u at P must be the same at Q.

That is, there is a real constant a such that $u(x,y) = a$ throughout D.

Similarly, there is a real constant b such that $v(x,y) = b$ throughout D.

$\therefore f(z) = a + ib$ at each point in D.

(i.e) $f(z)$ is constant in D.

Hence the theorem.

<u>Examples:</u>

$$f(z) = \frac{z^3+4}{(z^2-3)(z^2+1)}.$$

Let $P(z) = z^3+4$, $Q(z) = (z^2-3)(z^2+1)$.

Since $P(z)$ and $Q(z)$ are polynomials, they are analytic through the z-plane.

$\therefore$ The quotient $f(z) = \dfrac{P(z)}{Q(z)}$ is analytic everywhere in the z-plane except at $Q(z)=0$.

$Q(z)=0 \Rightarrow z = \pm\sqrt{3}$ and $z = \pm i$.

That is,

$z = \pm\sqrt{3}$ and $z = \pm i$ are the singular points of $f(z)$.

Example: 2.

$$f(z) = \cosh x \cos y + i \sinh x \sin y.$$

Here $u(x,y) = \cosh x \cos y$ and

$v(x,y) = \sinh x \sin y$

$u_x = \sinh x \cos y$ $\qquad$ $v_x = -\cosh x \sin y$

$u_y = -\cosh x \sin y$ $\qquad$ $v_y = \sinh x \cos y$

Hence $u_x = v_y$ & $u_y = -v_x$.

Hence Cauchy-Riemann equations are satisfied and

u_x, u_y, v_x, v_y exist and are continuous,

$\therefore f$ is diff.ble everywhere.

Hence f is an entire function.

Example: 3

Suppose that $f(z) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)} = u(x,y) - iv(x,y)$ are both analytic in a given domain D. Then S.T, $f(z)$ must be constant throughout D.

Soln:

$$f(z) = u(x,y) + iv(x,y)$$

Let $\overline{f(z)} = U(x,y) + iV(x,y)$,

Then
$$\left.\begin{aligned} U(x,y) &= u(x,y) \ \& \\ V(x,y) &= -v(x,y). \end{aligned}\right\} \quad \text{——— (1)}$$

Since $f(z)$ is analytic in D, by Cauchy-Riemann equations,

$$u_x = v_y \text{ and } u_y = -v_x \quad \text{——— (2)}$$

Also $\overline{f(z)}$ is analytic in D, By Cauchy Riemann equations,

$$U_x = V_y, \quad U_y = -V_x \quad \text{——— (3)}$$

From (1), ~~$u_x = V_x = V_y$~~ $\quad U_x = u_x \;|\; \cancel{V_y} = u_y$

$U_y = u_y$

From ①,

$$U_x = u_x \,,\; U_y = u_y$$

$$V_x = -v_x \,,\; V_y = -v_y .$$

$\therefore$ From ③,

$$\left.\begin{aligned} u_x &= -v_y \\ u_y &= v_x \end{aligned}\right\} \text{———} ④ .$$

From ② & ④,

$$u_x = 0 \text{ and } v_x = 0 .$$

$$\therefore f'(z) = u_x + i v_x = 0 + i0 = 0 \text{ in } D.$$

$\therefore f$ is constant throughout D.

Example: 4

Assume that $f(z)$ is analytic in a domain D. Assume further that $|f(z)|$ is constant throughout D Then $f(z)$ must be constant throughout D.

Soln:

Let $|f(z)| = c$ for all z in D, where c is a real constant.

If $c = 0$, then $f(z) = 0$ everywhere in D.

If $c \neq 0$, then

$$f(z)\overline{f(z)} = |f(z)|^2 = c^2 \neq 0.$$

$\therefore f(z) \neq 0$ for any z in D.

$$\therefore \overline{f(z)} = \frac{c^2}{f(z)} \text{ for all } z \text{ in } D.$$

~~Hence~~ It follows that $\overline{f(z)}$ is analytic everywhere in D.

$\therefore f(z)$ must be constant in D.

HARMONIC FUNCTIONS.

Defn:-

A real valued function H of two real variables x and y is said to be harmonic in a given domain of the xy-plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the P.D.E

$$H_{xx}(x,y) + H_{yy}(x,y) = 0 \quad \text{---(1)}$$

known as Laplace's equation.

Ex:1 Verify the function $T(x,y) = e^{-y}\sin x$ is harmonic in any domain of the xy plane

Soln:-

$$T(x,y) = e^{-y}\sin x.$$

$$T_x = e^{-y}\cos x.$$

$$T_y = -e^{-y}\sin x.$$

$$T_{xx} = -e^{-y}\sin x.$$

$$T_{yy} = e^{-y}\sin x.$$

The first & second order partial derivatives exist and continuous. and

$$T_{xx} + T_{yy} = 0.$$

$\therefore T(x,y)$ is harmonic

Theorem:1

If a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

derivatives of all orders.

Proof :-

Assume that f is analytic in D.
Hence the real & Imaginary components have the partial
∴ The first order partial derivatives of its component functions must satisfy the Cauchy - Riemann equations throughout D.

(i.e) $$u_x = v_y \quad , u_y = -v_x \quad \text{——} (1)$$

Differentiate both sides of these equations w.r.to x

$$u_{xx} = v_{yx} , \; u_{yx} = -v_{xx} \quad \text{——} (2)$$

||ly differentiation w.r.to y yields,

$$u_{xy} = v_{yy} \quad , u_{yy} = -v_{xy} \quad \text{——} (3)$$

Since u_x, u_y, v_x & v_y are continuous,

$$u_{xy} = u_{yx} \quad \& \quad v_{xy} = v_{yx}.$$

∴ (2) & (3) implies,

$$u_{xx} + u_{yy} = 0 \quad \text{and}$$

$$v_{xx} + v_{yy} = 0.$$

Hence u and v are harmonic in D.

Ex:2 The function $f(z) = e^{-y}\sin x - i e^{-y}\cos x$.

$u = e^{-y}\sin x \quad v = -e^{-y}\cos x$

$u_x = e^{-y}\cos x \quad v_x = e^{-y}\sin x$

$u_y = -e^{-y}\sin x \quad v_y = e^{-y}\cos x$.

$u_x = v_y$ & $u_y = -v_x$.

Also u_x, v_y, v_x, v_y are continuous.

Hence $f(z)$ is analytic everywhere.

(i) $f(z)$ is an entire function.

Hence $u(x,y) = e^{-y}\sin x$ is harmonic.

(ii) The temperature function $T(x,y) = e^{-y}\sin x$ is harmonic.

Ex:3

$$f(z) = i/z^2$$

$$f(z) = \frac{P(z)}{Q(z)} = i/z^2$$ is analytic except at $z = 0$.

Since, $\frac{i}{z^2} = \frac{i\cancel{(x-iy)}}{\cancel{x^2+y^2}} \quad \frac{i}{z^2} \times \frac{\bar{z}^2}{\bar{z}^2}$

$$= \frac{i(x-iy)^2}{(|z|^2)^2} = \frac{i(x^2-y^2-2ixy)}{(x^2+y^2)^2}$$

$$\therefore u(x,y) = \frac{2xy}{(x^2+y^2)^2} \qquad v(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Since $f(z)$ is analytic everywhere at $z \neq 0$, u & v are harmonic throughout any domain in the xy plane that doesnot contain the origin.

Theorem 2

Defn:-

If two functions u & v are harmonic in a Domain D and their first order partial derivatives satisfy the C-R equation, $u_x = v_y$, $u_y = -v_x$, then v is said to be a harmonic conjugate of u.

St:

A function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D iff v is a harmonic conjugate of u.

Proof:- Suppose that v is a harmonic conjugate of u.

Then u & v are harmonic, and the first order partial derivatives of u & v are continuous. ~~Also~~ and satisfies ~~harmonic~~ Cauchy-Riemann equations.

Hence $f(z)$ is analytic in D.

Conversely,

If f is analytic in D, Then by the theorem

"If $f(z)$ is analytic in D then the component functions u & v are harmonic in D", we have

$u(x,y)$ & $v(x,y)$ are harmonic in D.

Also ~~u_x &~~ the first order partial derivatives satisfy the C-R equations

$u_x = v_y$ & $u_y = -v_x$.

$\therefore$ v is a harmonic conjugate of u.

Ex: 4

Consider $f(z) = z^2$.

$$f(z) = x^2 - y^2 + i\,2xy.$$

$\therefore u(x,y) = x^2 - y^2 \quad v(x,y) = 2xy.$

W.K.T $f(z) = z^2$ is an entire function.

$\therefore v$ is a harmonic conjugate of u.

<u>To check u is a harmonic conjugate of v or not</u>

Consider $f(z) = 2xy + i(x^2 - y^2)$.

Let $U = 2xy \quad V = x^2 - y^2$.

$U_x = 2y \quad V_x = 2x$

$U_y = 2x \quad V_y = -2y.$

here $U_x \neq V_y$ & $U_y \neq -V_x$.

(i) C-R equation are not satisfied ~~anywhere~~. except at the origin $(0,0)$.

$\therefore f(z)$ is not analytic anywhere.

$\therefore$ Hence u is not a harmonic conjugate of v.

Method of finding harmonic Conjugate function of a given harmonic function.

Consider $u(x, y) = y^3 - 3x^2y$ ——— (1)

$u_x = -6xy$

$u_y = 3y^2 - 3x^2$

$u_{xx} = -6y \qquad u_{yy} = 6y$

$\Rightarrow u_{xx} + u_{yy} = 0$

Hence u is harmonic throughout the entire xy-plane.

To find the harmonic conjugate of u,

Let v be a harmonic conjugate of u.

$\therefore u_x = v_y$ & $u_y = -v_x$ ——— (2)

$\therefore v_y(x, y) = -6xy$

Keeping x fixed and integrate both sides by y, we get

$$v(x,y) = \int -6xy\,dy = -6x\cdot\frac{y^2}{2} = -3xy^2$$

$$v(x,y) = -3xy^2 + \phi(x) \quad \text{——— (3)}$$

$$\Rightarrow v_x = -3y^2 + \phi'(x) \quad \text{——— (4)}$$

$\therefore$ (2) $\Rightarrow$

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)$$

$$\Rightarrow \phi'(x) = 3x^2$$

$$\Rightarrow \phi(x) = x^3 + c$$

Hence $\boxed{v(x,y) = -3xy^2 + x^3 + c}$,

where c is an arbitrary real number.

That is v is a harmonic conjugate of u.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c).$$

Note:

$$f(z) = i(z^3 + c) = i((x+iy)^3 + c)$$

$$= i(x^3 + 3x^2iy - 3xy^2 - iy^3 + c)$$

$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c)$

EXERCISE 23.

1. S.T $f'(z)$ does not exist at any point

(a) $f(z) = \bar{z}$

$f(z) = x - iy$

$u = x \quad v = -y$

$u_x = 1 \quad v_x = 0$

$u_y = 0 \quad v_y = -1$

$\Rightarrow u_x \neq v_y \;\&\; u_y = v_x.$

Hence C.R equations are not satisfied anywhere.

Hence $f'(z)$ does not exist at any point

(b) $f(z) = z - \bar{z}$

$f(z) = \cancel{x} + iy - \cancel{x} + iy$

$= 2iy$

$\therefore u = 0, \quad v = 2y$

$u_x = 0 \quad v_x = 0$

$u_y = 0 \quad v_y = 2$

C.R equations are not satisfied

(c) $f(z) = 2x + ixy^2$

$u = 2x \qquad v = xy^2$

$u_x = 2 \qquad v_x = y^2$

$u_y = 0 \qquad v_y = 2xy$

$u_x \neq v_y \qquad u_y \neq -v_x$.

$\therefore$ CR equations are not satisfied

(d) $f(z) = e^x e^{-iy}$

$f(z) = e^x(\cos y - i\sin y)$

$u = e^x \cos y \qquad v = -e^x \sin y$

$u_x = e^x \cos y \qquad v_x = -e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = -e^x \cos y$

$\therefore u_x \neq v_y$ & $u_y \neq -v_x$.

$\therefore$ CR equations are not satisfied.

Hence $f'(z)$ does not exist anywhere.

② S.T $f'(z)$ and its derivative $f''(z)$ exist every where and find $f''(z)$.

(a) $f(z) = iz + 2$.

$= i(x + iy) + 2$

$u(x,y) = 2 - y \qquad v(x,y) = x$.

$u_x = 0$ $\quad v_x = 1$

$u_y = -1$ $\quad v_y = 0$

* [Hence first order partial derivatives exist and continuous and the C-R equations

$u_x = v_y$ & $u_y = -v_x$.

are satisfied.

Hence $f'(z)$ exists and

$f'(z) = u_x + i v_x] = i$ [constant]

$\therefore f''(z)$ exist and $f''(z) = 0$.

(b) $f(z) = e^{-x} e^{-iy}$

$u = e^{-x} \cos y$

$v = -e^{-x} \sin y$

$u_x = -e^{-x} \cos y$ $\quad v_x = e^{-x} \sin y$

$u_y = -e^{-x} \sin y$ $\quad v_y = -e^{-x} \cos y$

$u_x = v_y$ & $u_y = -v_x$

write *

$\therefore f'(z)$ exist and $f'(z) = -e^{-x} \cos y + i e^{-x} \sin y$

$U = -e^{-x} \cos y$ $\quad V = e^{-x} \sin y$

$U_x = e^{-x}\cos y \qquad V_x = -e^{-x}\sin y$

$U_y = e^{-x}\sin y \qquad V_y = e^{-x}\cos y$

$\therefore U_x = V_y, \quad U_y = -V_x$

$*$ $\therefore f^{11}(z)$ exist and $f^{11}(z) = e^{-x}\cos y - ie^{-x}\sin y$

$= e^{-x}e^{-iy} = f(z)$

(c) $f(z) = z^3$

$f(z) = (x+iy)^3 = x^3 + i3x^2y - 3xy^2 - iy^3$

$\therefore u(x,y) = x^3 - 3xy^2 \qquad v(x,y) = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2 \qquad v_x = 6xy$

$u_y = -6xy \qquad v_y = 3x^2 - 3y^2$

Hence $*$

$\therefore f^1(z) = (3x^2 - 3y^2) + i\,6xy = 3z^2$

Let $U = 3x^2 - 3y^2 \qquad V = 6xy$

$U_x = 6x \qquad V_x = 6y$

$U_y = -6y \qquad V_y = 6x$

$\Rightarrow U_x = V_y$ & $U_y = -V_x$.

Hence ✱,

$$\therefore f''(z) = 6(x+iy) = 6z$$

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

$u = \cos x \cosh y \qquad v = -\sin x \sinh y$

$u_x = -\sin x \cosh y \qquad v_x = -\cos x \sinh y$

$u_y = \cos x \sinh y \qquad v_y = -\sin x \cosh y$

$\Rightarrow u_x = v_y \ \& \ u_y = -v_x$.

Hence ✱,

$$f'(z) = -\sin x \cosh y - i \cos x \sinh y$$

Let $U = -\sin x \cosh y \qquad V = -\cos x \sinh y$

$U_x = -\cos x \cosh y \qquad V_x = \sin x \sinh y$

$U_y = -\sin x \sinh y \qquad V_y = -\cos x \cosh y$

$\therefore U_x = V_y \ \& \ U_y = -V_x$.

Hence ✱.

$$f''(z) = -\cos x \cosh y + i \overset{\sin x \sinh y}{\cancel{\cos x \cosh y}}$$

$$= -(f(z))$$

(3) Determine where $f'(z)$ exist and find its value.

(a) $f(z) = 1/z$.

$$\frac{1}{z} = \frac{x}{(x^2+y^2)} - \frac{iy}{x^2+y^2}$$

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{-y}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)(-1) + y\times 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$u_x = v_y$ & $u_y = -v_x$ at $x^2+y^2 \neq 0$.

(i) $f'(z)$ exists when $z \neq 0$.

Also $f'(z) = u_x + i v_x$

$$= \frac{y^2 - x^2 + i2xy}{(x^2+y^2)^2}$$

$$= \frac{-(x^2 - y^2 - i2xy)}{z^2 \bar{z}^2}$$

$$f'(z) = \frac{-\bar{z}^2}{z^2 \bar{z}^2} = -1/z^2.$$

(b) $f(z) = x^2 + iy^2$