$$= \frac{1}{1+e^{-s\pi}}\left[\frac{-\pi}{s}e^{-s\pi} - \frac{1}{s^2}e^{-s\pi} + \frac{1}{s^2}\right]$$

$$= \frac{1-(1+s\pi)e^{-s\pi}}{s^2(1+e^{-s\pi})}$$

## Exercise 2

1) For a square wave function

$$f(t) = -1 \text{ for } 0 < t < \frac{\pi}{2}$$

$$= -1 \text{ for } \frac{\pi}{2} < t < \pi$$

show that the transform of this function is $\frac{1}{s}\tan\left(\frac{as}{\pi}\right)$

2) If $f(t)$ is the staircase function such that

$$f(t) = \begin{cases} b & \text{when } 0 < t < a \\ 2b & \text{when } a < t < 2a \end{cases}$$ and so forth show that

$$L\{f(t)\} = \frac{b}{s(1-e^{-\pi s})}$$

**Section 4:**

**Inverse Laplace Transform**

Let us find the inverse laplace transform of functions of $s$ using the above properties.In many cases partial fractions method will enable us to find the inverse Laplace transform easily. We also use the properties of inverse Laplace Transform to determine the inverse Laplace transform in some other cases.



**Inverse Laplace Transform of Some Functions**

| | $\bar{f}(s)$ | $f(t)$ |
|---|---|---|
| | $\frac{1}{s}$ | 1 |
| | $\frac{1}{s^{n+1}}, n = 0,1,1.........$ | $\frac{t^n}{n!}$ |
| | $\frac{1}{s+a}$ | $e^{-at}$ |
| | $\frac{1}{s^2+a^2}$ | $\frac{\sin at}{a}$ |
| 5. | $\frac{s}{s^2+a^2}$ | $\cos at$ |
| 6. | $\frac{1}{s^2-a^2}$ | $\frac{\sinh at}{a}$ |
| 7. | $\frac{s}{s^2-a^2}$ | $\cosh at$ |

**Properties of Inverse Laplace Transform**

| | | |
|---|---|---|
| 1. | $L^{-1}c_1\bar{f}(s)+c_2\bar{f}_2(s)$ | $c_1f_1(t)+c_2f_2(t)$ |
| 2. | $L^{-1}[\bar{f}(s-a)$ | $e^{-at}f(t)$ |
| 3. | $L^{-1}[\bar{f}(as)]$ | $\frac{1}{a}f\left(\frac{t}{a}\right)$ |
| 4. | $L^{-1}[\bar{f}^n(s)$ | $(-1)^n t^n f(t)$ |
| 5. | $L^{-1}\int_s^\infty\{\bar{f}(x)dx\}$ | $\frac{f(t)}{t}$ |
| 6. | $L^{-1}[s\bar{f}(s)$ | $f'(t)$ |
| 7. | $L^{-1}\left[\frac{\bar{f}(s)}{s}\right]$ | $\int_0^t f(x)dx$ |

**Example 1:**

$$\text{Find } L^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s+3)}\right\}$$

**Solution:**

$$\text{Let } \left\{\frac{7s-1}{(s+1)(s+2)(s+3)}\right\} = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}$$

$$7s-1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

Put $s=-1,\quad -8 = A(1)(2) \Rightarrow A = -4$

Put $s=-2,\quad -15 = B(-1)(1) \Rightarrow B = 15$

Put $s=-3,\quad -22 = C(-2)(-1) \Rightarrow C = -11$

$$L^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s+3)}\right\} = L^{-1}\left\{-\frac{4}{s+1}+\frac{15}{s+2}-\frac{11}{s+3}\right\}$$

$$= -4L^{-1}\left\{\frac{1}{s+1}\right\}+15L^{-1}\left\{\frac{1}{s+2}\right\}-11L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= -4e^{-t}+15e^{-2t}-11e^{-3t}$$

**Example 2:**

$$\text{Find } L^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+2)}\right\}$$

**Soultion**

$$\text{Let } \frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$$

$$s^2+9s+2 = A(s-1)^2 + B(s-1)(s+2) + C(s+2)$$

Put $s = 1$, ; $\quad C = 4$.

Put $s = -2$, $4-18+2 = A(-3)^2$

$-12 = 9A$, $A = -\frac{4}{3}$

Equating the co-efficients of $s^2$,

$$A+B = 1$$

$$B = 1+\frac{4}{3} = \frac{7}{3}$$

$$\frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{-4}{3(s+2)}+\frac{7}{3(s-1)}+\frac{4}{(s-1)^2}$$



$$L^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+2)}\right\} = \frac{-4}{3}L^{-1}\left\{\frac{1}{s+2}\right\}+\frac{7}{3}L^{-1}\left\{\frac{1}{s-1}\right\}+4L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= \frac{-4}{3}e^{-2t}+\frac{7}{3}e^{t}+4e^{t}t$$

**Example 3:**

$$\text{Find } L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$$

**Solution:**

$$\frac{s^2}{(s^2+4)(s^2+9)} = \frac{A}{(s^2+4)}+\frac{B}{s^2+9}.$$

$$s^2 = A(s^2+9)+B(s^2+4)$$

Put $s^2 = -4; -4 = 5A$

$$A = -\frac{4}{5}.$$

Put $s^2 = -9, -9 = -5B \Rightarrow B = \frac{9}{5}$

$$\frac{s^2}{(s^2+4)(S^2+9)} = \frac{-4}{5(s^2+4)}+\frac{9}{5(s^2+4)}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = -\frac{4}{5}L^{-1}\left\{\frac{1}{s^2+4}\right\}+\frac{9}{5}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= -\frac{4}{5}\frac{\sin 2t}{2}+\frac{9}{5}\frac{\sin 3t}{3}$$

$$= -\frac{2}{5}\sin 2t+\frac{9}{15}\sin 3t$$

**Example 4:**

$$\text{Find } L^{-1}\left\{\frac{s+2}{(s-4)(s^2+1)}\right\}$$

**Solution:**

$$\text{Let } \frac{s+2}{(s-4)(s^2+1)} = \frac{A}{s-4}+\frac{Bs+C}{s^2+1}$$

$$s+2 = A(s^2+1)+(Bs+C)(s-4)$$

Put $s = 4; 6 = 17A \Rightarrow A = \frac{6}{17}.$

Equating the co-efficient of $s^2$, $A + B = 0$

$$\Rightarrow B = -\frac{6}{17}$$

Equating the constant term, $A - 4C = 2$

$$\Rightarrow 4C = A - 2 = \frac{6}{17} - 2 = -\frac{28}{17}$$

$$C = -\frac{7}{17}.$$

$$\frac{s+1}{(s-4)(s^2+1)} = \frac{\frac{6}{17}}{s-4} - \frac{\frac{6}{7}s + \frac{7}{17}}{s^2+14}$$

$$L^{-1}\left\{\frac{s+2}{(s-4)(s^2+1)}\right\} = \frac{6}{17} L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{6}{7} L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$- \frac{7}{17} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{6}{17} e^{4t} - \frac{6}{7} \cos t - \frac{7}{17} \sin t$$

**Example 5:**

Find $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

**Solution**

Let $\frac{1}{s(s^2+a^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+a^2}$

$$1 = A(s^2+a^2) + (Bs+C)s$$

Put $s = 0$; $A = \frac{1}{a^2}$

Equating the co-efficient of $s^2$, $A + B = 0$

$$B = -\frac{1}{a^2}$$

Equating the co-efficient of $s$, $C = 0$

$$\frac{1}{s(s^2+a^2)} = \frac{1}{a^2}\frac{1}{s} - \frac{\frac{1}{a^2}s}{s^2+a^2}$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2} L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{a^2} L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$$



$$= \frac{1}{a^2}.1 - \frac{\cos at}{a^2}$$

$$= \frac{1 - \cos at}{a^2}$$

$2(s+1)[s+1+5]$

**Example 6:**

Find $L^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$

**Solution:**

$$L^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} = L^{-1}\left\{\frac{2s^2+10s}{[(s-1)^2+4][(s-1)-2]}\right\}$$

$$= e^t L^{-1}\left\{\frac{2(s+1)^2 + 10(s+1)}{(s^2+4)(s+2)}\right\}$$

$2(s+1)[s+1+5]$

$$= e^t L^{-1}\left\{\frac{2(s+1)(s+6)}{(s+2)(s^2+4)}\right\}$$

Let $\frac{2(s+1)(s+6)}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$

$$2(s+1)(s+6) = A(s^2+4) + (s+2)(Bs+C)$$

Put $s = -2$; $2(-1)4 = A(8)$

$$\Rightarrow A = -1$$

Equating the co-efficient of $s^2$, $A + B = 2$

$$\therefore B = 3.$$

Equating constant term, $4A + 2C = 12$; $C = 8$

$$L^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} = e^t\left[L^{-1}\left\{\frac{-1}{s+2}\right\} + L^{-1}\left\{\frac{3s+8}{s^2+4}\right\}\right]$$

$$= e^t\left[L^{1}\left\{\frac{-1}{s+2}\right\} + 3L^{-1}\left\{\frac{s}{s^2+4}\right\} + 8L^{-1}\left\{\frac{1}{s^2+4}\right\}\right]$$

$$= e^t\left[-e^{-2t} + 3\cos 2t + 4\sin 2t\right]$$

$$= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}$$

**Example 7:**

Find the Laplace inverse of $\frac{10}{(s+2)^6}$.

**Solution :**

$$L^{-1}\left[\frac{10}{(s+2)^6}\right] = 10L^{-1}\left[\frac{1}{(s+2)^6}\right]$$

$$= 10e^{-2t}L^{-1}\left[\frac{1}{s^6}\right]$$

$$= 10e^{-2t}\frac{t^5}{5!}$$

$$= e^{-2t}\frac{t^5}{12}$$

**Example 8:**

Find the inverse Laplace transform of $\frac{2(s+1)}{(s^2+2s+2)^2}$

**Solution:**

$$L^{-1}\left\{\frac{2(s+1)}{(s^2+2s+2)^2}\right\} = L^{-1}\left\{\frac{2(s+1)}{[(s+1)^2+1]^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

Let $\bar{f}(s) = \frac{1}{s^2+1}$ $\qquad f(t) = \sin t$

$$\frac{d}{ds}\bar{f}(s) = \frac{-1}{(s^2+1)^2}2s$$

$$-\frac{d}{ds}\bar{f}(s) = \frac{2s}{(s^2+1)^2}$$

But $L^{-1}\left\{-\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$

$$L^{-1}\left\{\frac{2(s+1)}{s^2+2s+2)^2}\right\} = e^{-t}\,tf(t)$$

$$= te^{-t}\sin t$$



**...ple 9:**

Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)^2}$

**...tion:**

$$\frac{s^2}{(s^2+a^2)^2} = \frac{s^2+a^2-a^2}{(s^2+a^2)^2}$$

$$= \frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - a^2L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= \frac{\sin at}{a} - a^2L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$\bar{f}(s) = \frac{1}{s^2+a^2};\ f(t) = \frac{\sin at}{a}$$

$$\frac{d}{ds}\bar{f}(s) = \frac{-2s}{(s^2+a^2)^2}$$

$$\frac{s}{(s^2+a^2)^2} = \frac{-1}{2}\frac{d}{ds}\bar{f}(s)$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2}L^{-1}\left\{\frac{-d}{ds}\bar{f}(s)\right\}$$

$$= \frac{1}{2}t\,f(t)$$

$$= \frac{1}{2}\frac{t\sin at}{a}$$

Now $\frac{1}{(s^2+a^2)^2} = \frac{1}{s}\frac{s}{(s^2+a^2)^2}$

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right\}$$

$$= \int_0^t \frac{t\sin at}{2a}\,dt$$

$$= \left[-\frac{t\cos at}{2a^2}\right]_0^t + \int_0^t \frac{\cos at}{2a^2}\,dt$$

$$= \frac{-t\cos at}{2a^2} + \frac{\sin at}{2a^3}$$

$$= \frac{\sin at - at\cos at}{2a^3}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{\sin at}{a} - \frac{\sin at - at\cos at}{2a}$$

$$= \frac{\sin at + at\cos at}{2a}$$

**Example 10:**

Find $L^{-1}\left\{\dfrac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}\right\}$

**Solution:**

Let $$\frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2}$$

$$7s^3 - 2s^2 - 3s + 6 = As^2(s-2) + Bs(s-2) + C(s-2) + Ds^3$$

Put $s = 0$, $-2C = 6 \Rightarrow C = -3$

Put $s = 2$, $56 - 8 - 6 + 6 = 8D \Rightarrow D = 6$

Equating the co-efficient of $s^3$, $A + D = 7 \Rightarrow A = 1$

Equating the co-efficient of $s^2$, $-2A + B = -2$

$B = 0$

$$L^{-1}\left\{\frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^3} + \frac{6}{s-2}\right\}$$

$$= 1 - \frac{3t^2}{2} + 6e^{2t}$$

**Example 11 :**

Find the inverse Laplace transform of $\bar{f}(s)$ if

$$s^2\bar{f}(s) + s\bar{f}(s) - 6\bar{f}(s) = \frac{s^2+4}{s^2+s}$$

**Solution :**

$$\bar{f}(s)\,[s^2 + s - 6] = \frac{s^2+4}{s(s+1)}$$



$$\bar{f}(s) = \frac{s^2+4}{s(s+1)(s^2+s-6)}$$

$$= \frac{s^2+4}{s(s+1)(s+3)(s-2)}$$

$$\frac{s^2+4}{s(s+1)(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{D}{s-2}$$

$$s^2 + 4 = A(s+1)(s+3)(s-2) + Bs(s+3)(s-2) + Cs(s+1)(s-2) + Ds(s+1)(s+3)$$

Put $s = 0$, $\quad -6A = 4$ or $A = -\frac{2}{3}$

Put $s = -1$, $\quad -B(2)(-3) = 5 \Rightarrow B = \frac{5}{6}$

Put $s = 2$, $\quad D\,2.3.5 = 8 \Rightarrow D = \frac{4}{15}$

Put $s = -3$, $C(-3)(-2)(-5) = 13 \Rightarrow C = -\frac{13}{30}$

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

$$= L^{-1}\left\{\frac{-2/3}{s} + \frac{5/6}{s+1} - \frac{13/30}{s+3} + \frac{4/15}{s-2}\right\}$$

$$= -\frac{2}{3} + \frac{5}{6}e^{-t} + \frac{13}{30}e^{-3t} + \frac{4}{15}e^{2t}$$

**Example 13 .**

Find the inverse Laplace transform of $\dfrac{s-1}{2s^2+s+6}$

**Solution:**

$$L^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\} = L^{-1}\left\{\frac{s-1}{2\left(s^2+\frac{s}{2}+3\right)}\right\}$$

$$= L^{-1}\left\{\frac{s-1}{2\left[\left(s+\frac{1}{4}\right)^2 + 3 - \frac{1}{16}\right]}\right\}$$

$$= L^{-1}\left\{\frac{s-1}{2\left[\left(s+\frac{1}{4}\right)^2 + \frac{47}{16}\right]}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{s+\frac{1}{4}-\frac{5}{4}}{(s+\frac{1}{4})^2+\frac{47}{16}}\right\}$$

$$= \frac{1}{2} e^{\frac{-1}{4}t} L^{-1}\left\{\frac{s-\frac{5}{4}}{s^2+\frac{47}{16}}\right\}$$

$$= \frac{e^{-\frac{1t}{4}}}{2}\left[L^{-1}\left\{\frac{s}{s^2+\frac{47}{16}}\right\} - \frac{5}{4}L^{-1}\left\{\frac{1}{s^2+\frac{47}{16}}\right\}\right]$$

$$= \frac{e^{-\frac{1t}{4}}}{2}\left[\cos\frac{\sqrt{47}}{4}t - \frac{5}{\sqrt{47}}\sin\frac{\sqrt{47}}{4}t\right]$$

**Example 14 :**

Find the Laplace inverse of $\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

**Solution:**

$$L^{-1}\left\{\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right\}$$

$$= L^{-1}\left\{\frac{7s^2+23s+30}{(s-2)\left[(s+1)^2+4\right]}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{7(s-1)^2+23(s-1)+30}{(s-3)(s^2+4)}\right\}$$

$$\left\{\frac{7(s-1)^2+23(s-1)+30}{(s-3)(s^2+4)}\right\} = \frac{A}{s-3}+\frac{Bs+C}{s^2+4}$$

$$7(s-1)^2+23(s-1)+30 = A(s^2+4)+(s-3)(Bs+C)$$

Put $s = 3$, $28+46+30 = 13A$

$\Rightarrow A = 8$

Equating the co-efficient of $s^2$; $7 = A + B$

$\Rightarrow B = -1$

Equating the constant term, $4A - 3C = 7-23+30$

$\Rightarrow C = 6.$



$$L^{-1}\left\{\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right\} = e^{-t}\left[L^{-1}\left\{\frac{8}{s-3}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{6}{s^2+4}\right\}\right]$$

$$= e^{-t}\left[8e^{3t} - \cos 2t + \frac{6}{2}\sin 2t\right]$$

$$= 8e^{2t} - e^{-t}\cos 2t + + 3e^{-t}\sin 2t$$

**Example 15 :**

Find the inverse laplace transform of $\log\left(\frac{1+s}{s}\right)$

**Solution:**

$$\bar{f}(s) = \log\left(\frac{1+s}{s}\right)$$

$$= \log(1+s) - \log s$$

$$\frac{d}{ds}\bar{f}(s) = \frac{1}{s+1} - \frac{1}{s}$$

$$L^{-1}\left\{-\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$t f(t) = 1 - e^{-t}$$

$$\therefore f(t) = \frac{1-e^{-t}}{t}$$

**Example 16 :**

Find the inverse laplace transform of $\log\left(\frac{s^2+9}{s^2+1}\right)$

**Solution:**

$$\text{Let } \bar{f}(s) = \log\left(\frac{s^2+9}{s^2+1}\right)$$

$$= \log(s^2+9) - \log(s^2+1)$$

$$\frac{d}{ds}\bar{f}(s) = \frac{2s}{s^2+9} - \frac{2s}{s^2+1}$$

$$L^{-1}\left\{-\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{2s}{s^2+1} - \frac{2s}{s^2+9}\right\}$$

i.e. $tf(t) = 2\cos t - 2\cos 3t$

$$\therefore f(t) = 2\frac{(\cos t - \cos 3t)}{t}$$

**Example 17 :**

Find $L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\} = L^{-1}\left\{\frac{s+2-5}{(s+2)^2+9}\right\}$

**Solution:**

$$= e^{-2t} L^{-1}\left\{\frac{s-5}{s^2+9}\right\}$$

$$= e^{-2t}\left[L^{-1}\left\{\frac{s}{s^2+9}\right\} - 5L^{-1}\left\{\frac{1}{s^2+9}\right\}\right]$$

$$= e^{-2t}\left[\cos 3t - \frac{5\sin 3t}{3}\right]$$

$$= \frac{e^{-2t}}{3}\left(3\cos 3t - 5\sin 3t\right)$$

**Example 18 :**

Find $L^{-1}\left\{\frac{s+2}{(s-2)^7}\right\}$

**Solution:**

$$L^{-1}\left\{\frac{s+2}{(s-2)^7}\right\} = L^{-1}\left\{\frac{s-2+4}{(s-2)^7}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{s+4}{s^7}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{1}{s^6}+\frac{4}{s^7}\right\}$$

$$= e^{2t}\left[\frac{t^5}{\lfloor 5} + 4\cdot\frac{t^6}{\lfloor 6}\right]$$

$$= \frac{e^{2t}}{720}\left[6t^5 + 4t^6\right]$$

**Example 19 :**

Find $L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\}$



...n:

$$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = L^{-1}\left\{\frac{s+3}{((s+3)^2+4)^2}\right\}$$

$$= e^{-3t} L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

Let $\bar{f}(s) = \frac{1}{s^2+4}$ $\therefore f(t) = \frac{\sin 2t}{2}$ ... (1)

$$\frac{d}{ds}\bar{f}(s) = -\frac{1}{(s^2+4)^2}\cdot 2s$$

$$-\frac{d}{ds}\bar{f}(s) = 2\cdot\frac{s}{(s^2+4)^2}$$

$$L^{-1}\left\{-\frac{d}{ds}\bar{f}(s)\right\} = 2L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

$$\therefore L^{-1}\frac{s}{(s^2+4)^2} = \frac{1}{2}t f(t)$$

from (1)

$$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = \frac{e^{-3t}t\sin 2t}{4}$$

**...ple 20 :**

Find $L^{-1}\left\{\frac{a^2}{s^3+a^3}\right\}$ $(s^3+a^3) = (s+a)(s^2-as+a^2)$

**...tion:**

$$\frac{a^2}{s^3+a^3} = \frac{A}{s+a} + \frac{Bs+c}{s^2-as+a^2}$$

$$a^2 = A(s^2-as+a^2) + (s+a)(Bs+C)$$

Put $s = -a$, $a^2 = A(3a^2) \Rightarrow A = \frac{1}{3}$

...ting the co-efficient of $s^2$, $A+B=0$ $B = -\frac{1}{3}$;

Equating the constant term,

$$a^2 = Aa^2 + Ca$$

$$a^2 = \frac{a^2}{3} + Ca$$

$$\Rightarrow C = \frac{2a}{3}$$

$$L^{-1}\left\{\frac{a^2}{s^3+a^3}\right\} = L^{-1}\left\{\frac{\frac{1}{3}}{s+a} + \frac{-\frac{1}{3}s + \frac{2a}{3}}{s^2 - as + a^2}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s+a}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s-2a}{\left(s-\frac{a}{2}\right)^2 + \frac{3a^2}{4}}\right\}$$

$\sin at = \frac{a}{s^2+a^2}$

$\cos at = \frac{s}{s^2+a^2}$

$$= \frac{1}{3}e^{-at} - \frac{1}{3}L^{-1}\left\{\frac{s - \frac{a}{2} - \frac{3a}{2}}{\left(s-\frac{a}{2}\right)^2 + \frac{3a^2}{4}}\right\}$$

$\frac{s - \frac{3a}{2}}{s^2 + \frac{3a^2}{4}}$

$$= \frac{1}{3}e^{-at} - \frac{1}{3}\left[e^{\frac{at}{2}}\left(\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right)\right]$$

$$= \frac{1}{3}e^{-at} - \frac{1}{3}e^{\frac{at}{2}}\cos\frac{\sqrt{3}}{2}at + \frac{1}{\sqrt{3}}e^{\frac{at}{2}}\sin\frac{\sqrt{3}}{2}at$$

**Example 21 :**

Find $L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}$

**Solution:**

$$\frac{s^2}{s^4+4a^4} = \frac{s^2}{(s^2+2a^2)^2 - 4a^2s^2}$$

$(a^2 - b^2) = (a-b)(a+b)$

$$= \frac{s^2}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

$$= \frac{1}{4a}\left[\frac{s}{s^2-2as+2a^2} - \frac{s}{s^2+2as+2a^2}\right]$$

$$L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \frac{1}{4a}\left[L^{-1}\left\{\frac{s-a+a}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{s+a-a}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}e^{at}L^{-1}\left\{\frac{s+a}{(s^2+a^2)}\right\} - \frac{1}{4a}e^{-at}L^{-1}\left\{\frac{s-a}{s^2+a^2}\right\}$$





$$= \frac{1}{4a}\left[e^{at}(\cos at + \sin at) - e^{-at}(\cos at - \sin at)\right]$$

$$= \frac{1}{2a}\left[\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at + \frac{e^{at}-e^{-at}}{2}\cos at\right]$$

$$= \frac{1}{2a}(\cosh at \sin at + \sinh at \cos at)$$

## Exercise 3

Find the inverse laplace transform of the following functions:

(1) $\frac{6}{(s+2)^4}$ (2) $\frac{6}{(s-1)^4}$ (3) $\frac{4}{s^2+9}$ (4) $\frac{3}{(2s+5)^3}$

(5) $\frac{1}{s^2+4s+1}$ (6) $\frac{3s-15}{2s^2-4s+16}$ (7) $\frac{3}{2s^2+8s+10}$

(8) $\frac{s-1}{s^2-2s+5}$ (9) $\frac{3}{s^4}$

Find the inverse laplace transform of the following rational function

(1) $\frac{7s-1}{(s+1)(s+2)(s+3)}$ (2) $\frac{-8s^2-5s+9}{(s+1)(s-1)(s-2)}$

(3) $\frac{3s+5}{s(s-2)(s+3)}$ (4) $\frac{6s^2-13s+2}{s(s-1)(s-2)}$

(5) $\frac{5s^2+34s+53}{(s+3)^2(s+4)}$ (6) $\frac{7s^3-2s^2-3s+6}{s^3(s-2)}$

(7) $\frac{s^2+9s+2}{(s-1)^2(s+3)}$ (8) $\frac{2s^2+10s}{(s^2-2s+5)(s+1)}$

(9) $\frac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$ (10) $\frac{1}{(s+1)(s^2+2s+2)}$

(11) $\frac{1}{s(s^2-2s+3)}$ (12) $\frac{5s+3}{(s-1)(s^2+2s+5)}$

(13) $\frac{1}{s^2(s^2+1)(s^2+4)}$ (14) $\frac{1}{s^4-1}$

(15) $\frac{s}{(s^2+1)(s^2+4)}$ (16) $\frac{s+2}{(s-1)(s^2+1)}$

**(III)**

(1) $\log\left(\dfrac{s}{s^2+1}\right)$ (2) $\log\left(\dfrac{s+2}{s-5}\right)$ (3) $\log\left(\dfrac{s-4}{s-5}\right)$

(4) $\log\left(\dfrac{s^2+9}{s^2+1}\right)$ (5) $\tan^{-1}\left(\dfrac{1}{s}\right)$

(6) $s\bar{f}(s) - 2\bar{f}(s) = \dfrac{10s^2+12s+14}{s^2-2s+2}$

(7) $s^2\bar{f}(s) + s\bar{f}(s) - 6\bar{f}(s) = \dfrac{s^2+4}{s^2+s}$

(8) $\dfrac{s+a}{(s^2+a^2)^2}$ (9) $\dfrac{s^2-1}{(s^2+1)^2}$

$L[f(t)] = \bar{f}(s)$

$L[f'(t)] = s\bar{f}(s) - f(0)$

$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$

## Section 5

**Solving Differential Equations Using Laplac transform**

**Example 1:**

Solve $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 0$ given $y(0) = -2, y'(0) = 5$

**Solution:**

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

Taking Laplace transform both sides

$$L\left\{\frac{d^2y}{dt^2}\right\} - L\left\{\frac{dy}{dt}\right\} - 2L\{y\} = 0$$

$$s^2L(y) - sy(0) - y'(0) - \left[sL(y) - y(0)\right] - 2L(y) = 0$$

$$L(y)\left[s^2 - s - 2\right] - sy(0) - y'(0) + y(0) = 0$$

$$y(0) = -2, y'(0) = 5$$

$$\therefore (s^2 - s - 2)L(y) + 2s - 5 - 2 = 0$$

$$\therefore (s^2 - s - 2)L(y) = 7 - 2s$$

$$L(y) = \frac{7-2s}{s^2-s-2}$$

$$y = L^{-1}\left[\frac{7-2s}{(s-2)(s+1)}\right]$$

Let $\dfrac{7-2s}{(s-2)(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{s+1}$

$$7 - 2s = A(s+1) + B(s-2)$$



Put $s = -1;\ 9 = -3B \Rightarrow B = -3$

$s = 2\quad 3 = 3A \Rightarrow A = 1$

$$y = L^{-1}\left\{\frac{1}{s-2} - \frac{3}{s+1}\right\}$$

$$= e^{2t} - 3e^{-t}$$

**Example 2:**

Solve $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 5y = 4e^{3t}$ given $y(0) = 2, y'(0) = 7.$

**Solution:**

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y + 4e^{3t}$$

Taking Laplace transform on both sides

$$L\left\{\frac{d^2y}{dt^2}\right\} - 4\left\{\frac{dy}{dt}\right\} + 5L\{y\} = 4L\{e^{3t}\}$$

$$s^2L(y) - sy(0) - y'(0) - 4[L(y) - y(0)] + 5L(y) = \frac{4}{s-3}$$

$$(s^2 - 4s + 5)L(y) = \frac{4}{s-3} + 2s - 1$$

$$L(y) = \frac{4 + (2s-1)(s-3)}{(s^2-4s+5)(s-3)}$$

Let $\dfrac{4 + (2s-1)(s-3)}{(s^2-4s+5)(s-3)} = \dfrac{A}{s-3} + \dfrac{Bs+C}{s^2-4s+5}$

$$4 + (2s-1)(s-3) = A(s^2-4s+5) + (s-3)(Bs+C)$$

Put $s = 3,\ 4 = A(9 - 12 + 5)$

$4 = 2A \Rightarrow A = 2.$

Equating the co-efficient of $s^2$, $A + B = 2$

$\Rightarrow B = 0$

Equating the constant term, $7 = 5A - 3C$

$3C = 3$

$\Rightarrow C = 1$

$$\therefore L(y) = \frac{2}{s-3} + \frac{1}{s^2-4s+5}$$

$$y = L^{-1}\left\{\frac{2}{s-3}\right\} + L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$$

$$y = 2e^{3t} + L^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$$

$$y = 2e^{3t} + e^{2t}\sin t.$$

**Example 3:**

Solve the differential equation $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = te^t$ given $y(0) = 0; y'(0) = 0$

**Solution:**

Taking Laplace transform we get

$$L\left\{\frac{d^2y}{dt^2}\right\} - 4L\left\{\frac{dy}{dt}\right\} - 5L[y] = \{te^t\}$$

$$s^2L(y) - sy(0) - y'(0) - 4[sL(y) - y(0)] - 5L(y) = \frac{1}{(s-1)^2}$$

$$(s^2 - 4s - 5)L(y) = \frac{1}{(s-1)^2} \quad (\text{since } y(0) = y'(0) = 0)$$

$$\therefore L(y) = \frac{1}{(s^2-4s-5)(s-1)^2}$$

$$= \frac{1}{(s-5)(s+1)(s-1)^2}$$

Let $\frac{1}{(s-5)(s+1)(s-1)^2} = \frac{A}{s-5} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$

$$1 = A(s+1)(s-1)^2 + B(s-5)(s-1)^2 + C(s-5)(s+1)(s-1) + D(s-5)(s+1)$$

Put $s = 1$, $1 = -8D \quad \Rightarrow D = -\frac{1}{8}$

Put $s = -1, 1 = B(-6)(-2)^2 \Rightarrow B = -\frac{1}{24}$

Put $s = 5$; $1 = A(6)4^2 \therefore A = \frac{1}{96}$

Equating the co-efficient of $s^3, A + B + C = 0$

$\frac{1}{96} - \frac{1}{24} + C = 0$





$$\therefore C = \frac{1}{32}$$

$$\therefore L(y) = \frac{1}{96}\cdot\frac{1}{s-5} - \frac{1}{24}\cdot\frac{1}{s+1} + \frac{1}{32}\cdot\frac{1}{s-1} - \frac{1}{8}\cdot\frac{1}{(s-1)^2}$$

$$y = \frac{1}{96}L^{-1}\left\{\frac{1}{s-5}\right\} - \frac{1}{24}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{32}\frac{1}{s-1} - \frac{1}{8}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$y = \frac{1}{96}e^{5t} - \frac{1}{24}e^{-t} + \frac{1}{32}e^t - \frac{1}{8}te^t$$

**Example 4:** H-W

Solve $y'' + 4y' + 5y = 4e^{3t}$ given $y(0) = 2$; $y'(0) = 7$

**Solution:**

Taking Laplace transform on both sides,

$$L(y'') + 4L(y') + 5L(y) = 4L(e^{3t})$$

$$s^2L(y) - sy(0) - y'(0) + 4[sL(y) - y(0)] + 5L(y) = \frac{4}{s-3}$$

$$L(y)\cdot[s^2 + 4s + 5] - 2s - 7 - 8 = \frac{4}{s-3}$$

$$\therefore (s^2 + 4s + 5)L(y) = \frac{4}{s-3} + 2s + 15$$

$$L(y) = \frac{4 + (2s+15)(s-3)}{(s^2+4s+5)(s-3)}$$

Let $\frac{4 + (2s+15)(s-3)}{(s^2+4s+5)(s-3)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4s+5}$

$$4 + (2s+15)(s-3) = A(s^2+4s+5) + (s-3)(Bs+C)$$

Put $s = 3$, $4 = A(9 + 12 + 5)$

$$A = \frac{4}{26} = \frac{2}{13}$$

Equating the co-efficient of $s^2$,

$$2 = A + B$$

$$\therefore B = 2 - \frac{2}{13} = \frac{24}{13}$$

Equating the constant term,

$$-41 = 5A - 3C$$

$$3C = \frac{10}{13} + 41$$

$$= \frac{10 + 533}{13} = \frac{543}{13}$$

$$C = \frac{181}{13}$$

$$\therefore L(y) = \frac{2}{13(s-3)} + \frac{\frac{24s}{13} + \frac{181}{13}}{s^2 + 4s + 5}$$

$$y = \frac{2}{13} L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{24}{13} L^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$$

$$+ \frac{181}{13} L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \frac{2}{13} e^{3t} + \frac{24}{13} e^{-2t} \cos t + \frac{181}{13} e^{-2t} \sin t$$

$$= \frac{1}{13}\left[2e^{3t} + 24e^{-2t} \cos t + \frac{181}{\cancel{11}} e^{-2t} \sin t\right]$$

**Example 5:**

Solve $y'' - 3y' + 2y = \sin t$ given $y(0) = 0, y'(0) = -1.$

**Solution:**

Taking laplace transform on both sides,

$$L(y'') - 3L(y') + 2L(y) = L(\sin t)$$

$$\left[s^2 L(y) - sy(0) - y'(0)\right] - 3\left[sL(y) - y(0)\right] + 2L(y) = L(\sin t)$$

$$(s^2 - 3s + 2) L(y) - 1 = \frac{1}{s^2+1}$$

$$(s^2 - 3s + 2) L(y) = 1 + \frac{1}{s^2+1}$$

$$= \frac{s^2+2}{s^2+1}$$

$$L(y) = \frac{s^2+2}{(s^2 - 3s + 2)(s^2+1)}$$

$$y = L^{-1}\left\{\frac{s^2+2}{(s-1)(s-2)(s^2+1)}\right\}$$

$$\frac{s^2+2}{(s-1)(s-2)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{(Cs+D)}{s^2+1}$$







$$\ldots = A(s-2)(s^2+1) + B(s-1)(s^2+1)$$

$$+ (Cs + D)(s-1)(s-2)$$

Put $s = 1,\ 3 = -2A \Rightarrow A = -\frac{3}{2}$

Put $s = 2, 6 = B(5) \Rightarrow B = \frac{6}{5}$

...ing the co-efficient of $s^3, A + B + C = 0$

$$C = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}$$

...ing the constant term

$$2 = -2A - B + 2D$$

$$2D = 2 + 2A + B = 2 - 3 + \frac{6}{5}$$

$$\Rightarrow D = \frac{1}{10}$$

$$y = L^{-1}\left\{-\frac{3}{2}\cdot\frac{1}{s-1} + \frac{6}{5}\cdot\frac{1}{s-2} + \frac{\frac{3}{10}s}{s^2+1} + \frac{1}{10(s^2+1)}\right\}$$

$$= -\frac{3}{2}e^{t} + \frac{6}{5}e^{2t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t$$

$$y = \frac{1}{10}\left\{\sin t + 3\cos t + 12e^{2t} - 15e^{t}\right\}$$

**...mple 6:**

Solve $u''(t) - 2u'(t) + 5u(t) = -8e^{\pi - t}$ given
$u(\pi) = 2, u'(\pi) = 12.$

**...lution:**

To solve this equation using Laplace transform let us first make the ...ial condition $t = 0$.

For this, give transformation $y(t) = u(t + \pi)$

then $y'(t) = u'(t + \pi)$

$y''(t) = u''(t + \pi)$

Replacing $t$ by $t + \pi$ in the given differential equation, we have

$$u''(t+\pi) - 2u'(t+\pi) + 5u(t+\pi)$$

$$= -8e^{\pi - (t + \pi)}$$

$$= -8e^{-t}$$

Now substituting $y(t) = u(t + \pi)$ in the above equation the equation becomes

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t} \text{ given } y(0) = 2 \text{ and } y'(0) = 12.$$

Taking laplace transform on both sides

$$L\{y''(t)\} - 2L\{y'(t)\} + 5L\{y(t)\} = -8L\{e^{-t}\}$$

$$s^2L(y) - sy(0) - y'(0) - 2[L(y) - y(0)] + 5L(y) = -\frac{8}{s+1}$$

$$(s^2 - 2s + 5)L(y) = 2s + 8 - \frac{8}{s+1}$$

$$= \frac{2s^2 + 10s}{s+1}$$

$$L(y) = \frac{2s(s+5)}{(s+1)(s^2 - 2s + 5)}$$

$$= \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 5}$$

$$2s(s+5) = A(s^2 - 2s + 5) + (s+1)(Bs + C)$$

Put $s = -1$, $-8 = A(1 + 2 + 5)$

$$A = -1.$$

Equating the co-efficients of $s^2$, $A + B = 2$

$$\therefore B = 3$$

Equating the constant term, $5A + C = 0$

$$C = 5$$

$$\therefore L(y) = -\frac{1}{s+1} + \frac{3s+5}{s^2 - 2s + 5}$$

$$= -\frac{1}{s+1} + \frac{3(s-1)+8}{(s-1)^2 + 4}$$

$$y = -e^{-t} + 3e^{t}\cos 2t + 4e^{t}\sin 2t$$

Since $u(t + \pi) = y(t)$

$$u(t) = y(t - \pi)$$

Hence replacing $t$ by $t - \pi$ we get

$$u = 3e^{t-\pi}\cos(2t - 2\pi) + 4e^{t-\pi}\sin(2t - 2\pi) - e^{-(t-\pi)}$$

$$u = 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi - t}$$