Section 1

Laplace Transf[orm]

Introduction

The Laplace transform was first introduced by Pierre Lapla[ce in] 1779 in his research on probability. G. Doetsch developed the [use of] Laplace transform to solve differential equations. His work in 1930s ... to justify the operational Calculus earlier used by Oliver Heaviside. ... importance of Laplace transform lies in converting a differential equ[ation] into an algebraic equation. We first solve the simple algebraic equ[ation] and using the inverse Laplace transform technique the solution to ... differential equation is easily obtained. In differential equation, we ... seen the methods of solving a given differential equation for its gen[eral] solution and we use the initial conditions to determine the arbi[trary] constants and thereby obtain the specific solution. However the meth[od] of laplace transform leads to the solution of initial value problem with[out] obtaining the general solution. Furthermore the Laplace transf[orm] technique can be used for solving certain linear differential equa[tions] with variable coefficients, a special class of integral equations, system[s of] differential equations and partial differential equations.

In this chapter we define Laplace transform and study certa[in] properties of Laplace transform. Also we study the different methods [of] obtaining the inverse laplace transform $f(t)$ when we are given th[e] transform function $\bar{f}(s)$. That is we seek an inverse mapping for the laplace transform. Then we use the properties of LT to convert a linear differential equation with constant coefficients into an algebraic equation in the laplace transform variable. Using inverse Laplace transform technique we seek the solution for the given DE with initial conditions. We also use LT techniques to solve linear simultaneous differential equations with initial condtions.

Piecewise continuity

A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$ except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be piecewise continuous on $[0, N]$ if $f(t)$ is piecewise continuous on $[0, N]$ for $N > 0$. For [ex]ample the function

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 2 & 1 < t < 2 \\ (t-2)^2 & 2 \le t \le 3 \end{cases}$$

is piecewise continuous as $[0, 3]$

A function that is piecewise continuous in a finite interval is necessarily [int]egrable on that interval. However piecewise continuity on $[0, \infty]$ is not [en]ough to guarantee the existence (as a finite number) of the improper [int]egral over $[0, \infty\}$

We also need to consider the growth of the integrand for large ... We shall show that the LT of a piecewise continuous function will [ex]ist provided that the function does not grow "faster than an exponential".

[Ex]ponential Order

A function $f(t)$ is said to be of exponential order α [i]f there exists positive constants T and M such that

$$|f(t)| \le Me^{\alpha t} \text{ for all } t \ge T.$$

For example $f(t) = e^{3t} \sin 2t$ is a function of exponential order $\alpha = 3$ since

$$|e^{3t} \sin 2t| \le e^{3t}.$$

Here $M = 1$ and T is any positive constant.

We use the phrase 'exponential order' to mean that for some value of α, the function f(t) satisfies the definition of exponential order. (i.e) $f(t)$ does not grow faster than a function of the form $Me^{\alpha t}$. For [ex]ample the function e^{t^2} is not of exponential order since

$$\lim_{t \to 0} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t \to 0} e^{t(t-\alpha)} = \infty \text{ for any } \alpha,$$

e^{t^2} grows faster than $e^{\alpha t}$ for every choice of α.

The usual functions encountered in solving linear differential [eq]uations with constant coefficients (polynomial, exponential, sine and cosine functions) are of exponential order. The LT of such functions exists for large value of t. It can be shown that if $f(t)$ is piecewise continuous in $[0, \alpha]$ and of exponential order α then $L\{f(t)\}$ exists for $s > \alpha$.

Definition and properties

Definition

Let $f(t)$ be a real valued function of the variable 't' defined for $t > 0$. Then the Laplace transform of $f(t)$ is defined by

$$\int_0^\infty e^{-st} f(t)\, dt$$

where the parameter s is real and $f(t)$ is denoted by the symbol $\bar{f}(s)$ or $L\{f(t)\}$

$$\text{i.e. } \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note: The Laplace transform of f(t) is said to exist only if the integral (1) converges for some values of s. Otherwise it does not exist.

Laplace transform of elementary functions

Laplace transform of 1

$$L\{1\} = \int_0^\infty e^{-st}.1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right] = \frac{1}{s} \text{ if } s > 0$$

Laplace transform of t

$$L(t) = \int_0^\infty e^{-st}\, t\, dt$$

$$= \left[\frac{e^{-st}}{-s} t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt$$

$$= 0 + \left[\frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2} \text{ if } s > 0.$$

Laplace transform of t^2

$$\therefore L\{t^2\} = \int_0^\infty e^{-st} t^2\, dt$$

$$= \left[\frac{e^{-st}}{-s} t^2\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 2t\, dt$$

$$= 0 + \frac{2}{s}\int_0^\infty e^{-st} \cdot t\, dt \text{ if } s > 0$$

$$= \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$$

Similarly $L\{t^3\} = \int_0^\infty e^{-st} t^3\, dt$

$$= \left[\frac{e^{-st}}{-s} t^3\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 3t^2 dt$$

$$= 0 + \frac{3}{s}\int_0^\infty e^{-st} t^2 dt \text{ if } s > 0$$

$$= \frac{3}{s} \cdot \frac{2}{s^3} = \frac{\underline{|3}}{s^4}$$

Similarly $L\{t^n\} = \dfrac{\underline{|n}}{s^{n+1}}$ when n is a postive integer

Laplace transform of e^{-at}

$$L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{s+a} \text{ if } s > -a$$

Note:

$$L\{e^{at}\} = \frac{1}{s-a} \text{ if } s > a$$

Laplace transform of sin at

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

We know that

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + k$$

$$\therefore L\{\sin at\} = \left[\frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= \frac{a}{s^2+a^2} \text{ if } s > 0$$

Laplace transform of cos at

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (- s \cos at + a \sin at) \right]_0^\infty$$

we know that

$$\int_0^\infty e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a\cos bx + b\sin bx) + k$$

$$\therefore L\{\cos at\} = \frac{s}{s^2 + a^2} \quad \text{if } s > 0$$

Laplace transform of sinh *at*

$$L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{e^{at} - e^{-at}}{2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t} dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t} dt$$

$$\frac{1}{2(s-a)} - \frac{1}{2(s+a)} \quad \text{if } s > |a|$$

$$= \frac{s + a - s + a}{2(s^2 - a^2)} = \frac{a}{s^2 - a^2}$$

Laplace transform of cosh *at*

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t} dt + \frac{1}{2}\int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2(s+a)}$$

$$= \frac{s}{s^2 - a^2} \quad \text{if } s > |a|$$

I. Linearity Property

If $f_1(t)$ and $f_2(t)$ are two functions of t defined for positive values of t and if c is a constant then

$$L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\} \text{ and}$$
$$L\{c(t)\} = cL\{f(t)\}$$

$$L\{f_1(t) + f_2(t)\} = \int_0^\infty e^{-st} [f_1(t) + f_2(t) dt]$$

$$= \int_0^\infty e^{-st} f_1(t) \, dt + \int_0^\infty e^{-st} f_2(t) \, dt$$

$$= L\{f_1(t)\} + L\{f_2(t)\}$$

$$L\{cf_1(t)\} = \int_0^\infty e^{-st} cf_1(t) \, dt$$

$$= c\int_0^\infty e^{-st} f_1(t) \, dt$$

$$= cL\{f_1(t)\}$$

...Shifting property

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{-at} f(t)\} = \bar{f}(s + a)$

...oof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = \bar{f}(s)$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{-at} f(t) \, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t) at$$

$$= \bar{f}(s + a)$$

...ote: $L\{e^{at} f(t)\} = \bar{f}(s - a)$

...Change of scale property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

...roof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at) \, dt$$

Put at $= t'$

$$a\,dt = dt' \text{ or } dt = \frac{1}{a}dt'$$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

$$= \int_0^\infty e^{-\frac{s}{a}t'} f(t')\frac{1}{a}dt'$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t'} f(t')\,dt'$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t} f(t)\,dt$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

IV Laplace Transform of Derivatives

If $L\{f(t)\} = \bar{f}(s)$ then

(i) $L\{f'(t)\} = s\bar{f}(s) - f(0)]$

(ii) $L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s)f(t)\,dt \text{ (By parts rule)}$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\bar{f}(s) \quad \text{.... (1)}$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st}f'(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s)f'(t)\,dt$$

$$= 0 - f'(0) + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\left[s\bar{f}(s) - f(0)\right] \text{ by (1)}$$

$$= s^2\bar{f}(s) - sf(0) - f'(0)$$

In general

$$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - sf^{n-2}(0) - f^{n-1}(0)$$

Laplace transform of Integrals

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{\bar{f}(s)}{s}$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,du$$

$$\text{Let } g(t) = \int_0^t f(u)\,du$$

then $g(0) = 0$

Also $g'(t) = f(t)$

Taking laplace transform on both sides

$$L\{g'(t)\} = L\{f(t)\}$$

$$\text{i.e. } s\bar{g}(s) - g(0) = \bar{f}(s)$$

$$s\bar{g}(s) = \bar{f}(s)$$

$$\therefore \bar{g}(s) = \frac{\bar{f}(s)}{s}$$

$$\text{i.e. } L\left\{\int_0^t f(u)\,du\right\} = \frac{\bar{f}(s)}{s}$$

Multiplication by t

If $L\{f(t)\} = \bar{f}(s)$ then $L\{t\,f(t)\} = -\frac{d}{ds}\bar{f}(s)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\text{i.e. } \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} (-t) f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, tf(t)\, dt$$

$$= -L\left\{tf(t)\right\}$$

$$\therefore L\left\{tf(t)\right\} = \frac{-d}{ds}\bar{f}(s)$$

$$L\left\{t^2 f(t)\right\} = (-1)^2 \frac{d^2}{ds^2}\bar{f}(s)$$

Differentiating (1) again w.r.t s,

$$\frac{d^2}{ds^2}\bar{f}(s) = \frac{-d}{ds}\int_0^\infty e^{-st}(t)f(t)\, dt$$

$$= -\int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}(-t)\,[tf(t)]\, dt$$

$$= \int_0^\infty e^{-st}\, t^2 f(t)\, dt$$

$$= L\left\{t^2 f(t)\right\}$$

$$\therefore L\left\{t^2 f(t)\right\} = (-1)^2 \frac{d^2}{ds^2}\bar{f}(s)$$

Similarly $L\left\{t^3 f(t)\right\} = (-1)^3 \frac{d^3}{ds^3}\bar{f}(s)$

In general, $L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

$f(t)$	$\bar{f}(s) = L\{f(t)\}$	
1	$\frac{1}{s}$	$s > 0$
e^{-at}	$\frac{1}{s+a}$	$s > -a$
t^n	$\frac{\lfloor n}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$

Properties of Laplace Transforms

(1)	$L\{f(t)+g(t)\} = L\{f(t)\} + L\{g(t)\}$
(2)	$L\{cf(t)\} = c\,L\{f(t)\}$
(3)	$L\{e^{-at}f(t)\} = \bar{f}(s+a)$
(4)	$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$
(5)	$L\{f'(t)\} = s\bar{f}(s) - f(0)$
(6)	$L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$

(7)	$L\left\{\int_0^t f(u)\,du\right\} = \frac{\bar{f}(s)}{s}$
(8)	$L\left\{tf(t)\right\} = -\frac{d}{ds}\bar{f}(s)$
(9)	$L\left\{t^2 f(t)\right\} = (-1)^2 \frac{d^2}{ds^2}\bar{f}(s)$ . . . $L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

Example 1:

Find the laplace transform of the following functions

(i) t^3 (ii) e^{-4t} (iii) $\sin 4t$ (iv) $\cos\frac{t}{2}$

Solution:

(i) $L\left\{t^3\right\} = \frac{3!}{s^4}$

(ii) $L\left\{e^{-4t}\right\} = \frac{1}{s+4}$

(iii) $L\left\{\sin 4t\right\} = \frac{4}{s^2+16}$

(iv) $L\left\{\cos\frac{t}{2}\right\} = \frac{s}{s^2+\frac{1}{4}} = \frac{4s}{4s^2+1}$

Example 2:

Find the laplace transform of the following functions (i) $\sin^2 3t$ (ii) $\cos^3 t$ (iii) $\sin^3 2t$ (iv) $\cos^4 t$

Solution

(i) $\sin^2 3t = \frac{1-\cos 6t}{2}$

$L\left\{\sin^2 3t\right\} = L\left\{\frac{1}{2} - \frac{1}{2}\cos 6t\right\}$

$= \frac{1}{2}L\left\{1\right\} - \frac{1}{2}L\left\{\cos 6t\right\}$

$= \frac{1}{2s} - \frac{s}{2(s^2+36)}$

(ii) $\cos 3t = 4\cos^3 t - 3\cos t$

$4\cos^3 t = \cos 3t + 3\cos t$

$\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$

$L\left\{\cos^3 t\right\} = \frac{1}{4}L\left\{\cos 3t\right\} + \frac{3}{4}L\left\{\cos t\right\}$

$= \frac{1}{4}\cdot\frac{s}{s^2+9} + \frac{3s}{4(s^2+1)}$

$= \frac{s\left[(s^2+1)+3(s^2+9)\right]}{4(s^2+9)(s^2+1)}$

$= \frac{s(4s^2+28)}{4(s^2+1)(s^2+9)}$

$= \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

(iii) $\sin 3A = 3\sin A - 4\sin^3 A$

$4\sin^3 A = 3\sin A - \sin 3A$

$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$\therefore L\left\{\sin^3 2t\right\} = \frac{3}{4}L\left\{\sin 2t\right\} - \frac{1}{4}L\left\{\sin 6t\right\}$

$= \frac{3}{4}\cdot\frac{2}{s^2+4} - \frac{1}{4}\cdot\frac{6}{s^2+36}$

$= \frac{6\left[s^2+36-(s^2+4)\right]}{4(s^2+4)(s^2+36)}$

$= \frac{48}{(s^2+4)(s^2+36)}$

(iv) $\cos^4 t = (\cos^2 t)^2$

$= \left(\frac{1+\cos 2t}{2}\right)^2$

$$= \frac{1}{4}\left[1 + 2\cos 2t + \cos^2 2t\right]$$

$$= \frac{1}{4}\left[1 + 2\cos 2t + \frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{8}(2 + 4\cos 2t + 1 + \cos 4t)$$

$$= \frac{1}{8}(2 + 4\cos 2t + .1 + \cos 4t)$$

$$= \frac{1}{8}(3 + 4\cos 2t + \cos 4t)$$

$$L\{\cos 4t\} = \frac{1}{8}\left[3L\{1\} + 4L\{\cos 2t\} + L\{\cos 4t\}\right]$$

$$= \frac{1}{8}\left[\frac{3}{s} + 4\cdot\frac{s}{s^2+4} + \frac{s}{s^2+16}\right]$$

Example 3:

Find the laplace transform of the following functions (1) $\cos 4t \sin 3t$ (2) $\sin^2 t \cos^3 t$ (3) $t^4 - t^2 - t + \sin\sqrt{2}t$

Solution

(1) $\cos 4t \sin 3t = \frac{1}{2}[\sin 7t - \sin t]$

$$\therefore L\{\cos 4t \sin 3t\} = \frac{1}{2}L\{\sin 7t - \sin t\}$$

$$= \frac{1}{2}L\{\sin 7t\} - \frac{1}{2}L\{\sin t\}$$

$$= \frac{1}{2}\cdot\frac{7}{s^2+49} - \frac{1}{2}\cdot\frac{1}{s^2+1}$$

(2) $\sin^2 t\cos^3 t = \frac{1-\cos 2t}{2}\cdot\frac{\cos 3t + 3\cos t}{4}$

$$= \frac{1}{8}[\cos 3t - \cos 3t\cos 2t + 3\cos t - 3\cos 2t\cos t]$$

$$= \frac{1}{8}[\cos 3t - \frac{1}{2}\cos 5t - \frac{1}{2}\cos t + 3\cos t - \frac{3}{2}\cos 3t - \frac{3}{2}\cos t]$$

$$= \frac{1}{8}[-\frac{1}{2}\cos 5t - \frac{1}{2}\cos 3t + \cos t]$$

$$= \frac{1}{16}[2\cos t - \cos 3t - \cos 5t]$$

$$= \frac{1}{16}\left[\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right]$$

$$L\{t^4 - t^2 - t + \sin\sqrt{2}t\}$$

$$= L\{t^4\} - L\{t^2\} - L\{t\} + L\{\sin\sqrt{2}t\}$$

$$\frac{\lfloor 4}{s^5} - \frac{\lfloor 2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}$$

Example 4:

Find the laplace transform of the following functions

(i) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

(ii) $f(t) = \begin{cases} e^{2t} & 0 < t < 3 \\ 1 & t > 3 \end{cases}$

Solution:

(i) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\pi e^{-st}\sin t\,dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t + \cos t)\right]_0^\pi$$

$$= \frac{e^{-s\pi}}{s^2+1}(-1) - \frac{1}{s^2+1}$$

$$= \frac{-(e^{-s\pi}+1)}{s^2+1}$$

(ii) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^3 e^{-st}e^{2t}\,dt + \int_3^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^3 + \left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \frac{e^{-3(s-2)}}{-(s-2)} + \frac{1}{s-2} + \frac{1}{s}e^{-3s}$$

Example 5:

Find the laplace transform of the following functions

(1) $e^{7t}\sin^2 t$ (2) $e^{-3t}\cos 2t$ (3) $2t^2e^{-t} - t + \cos 4t$

(4) $e^{-2t}\sin 2t + e^{3t}$ (5) $te^{2t}\cos 5t$ (6) $(1+e^{-2t})^2$

(7) $t\sin^2 t$ (8) $t^2\cos 4t$

Solution:

(1) $L\{e^{7t}\sin^2 t\} = L\left\{e^{7t}\frac{(1-\cos 2t)}{2}\right\}$

$= \frac{1}{2}L\{e^{7t}\} - \frac{1}{2}L\{e^{7t}\cos 2t\}$

$= \frac{1}{2}\cdot\frac{1}{s-7} - \frac{1}{2}\cdot\frac{s-7}{(s-7)^2+4}$

(2) $L\{\cos 2t\} = \frac{s}{s^2+4}$

$L\{e^{-3t}\cos 2t\} = \frac{s+3}{(s+3)^2+4}$

(3) $L\{\cos 4t\} = \frac{s}{s^2+16}$

$L\{t^2\} = \frac{\lfloor 2}{s^3}$

$\therefore L\{e^{-t}t^2\} = \frac{2}{(s+1)^3}$

$\therefore L\{2t^2e^{-t} - t + \cos 4t\} = 2L\{t^2e^{-t}\} - L\{t\} + L\{\cos 4t\}$

$= 2\cdot\frac{2}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+4}$

$= \frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+4}$

(4) $L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{e^{-2t}\sin 2t\} = \frac{2}{(s+2)^2+4}$

$= \frac{2}{s^2+4s+8}$

(5) $L\{\cos 5t\} = \frac{s}{s^2+25}$

$L\{t\cos 5t\} = \frac{-d}{ds}\left(\frac{s}{s^2+25}\right)$

$= -\frac{[(s^2+25)\cdot 1 - s.2s]}{(s^2+25)^2}$

$= \frac{s^2-25}{(s^2+25)^2}$

$L\{e^{2t}t\cos 5t\} = \frac{(s-2)^2-25}{((s-2)^2+25)^2}$

$= \frac{s^2-4s-21}{(s^2-4s+29)^2}$

(6) $(1+e^{-2t})^2 = 1 + 2e^{-2t} + e^{-4t}$

$L\{(1+e^{-2t})^2\} = L\{1+2e^{-2t}+e^{-4t}\}$

$= L\{1\} + 2L\{e^{-2t}\} + L\{e^{-4t}\}$

$= \frac{2}{s} + \frac{2}{s+2} + \frac{s}{s+4}$

(7) $L\{t\sin^2 t\} = L\left\{\frac{t(1-\cos 2t)}{2}\right\}$

$= \frac{1}{2}L\{t - t\cos 2t\}$

$= \frac{1}{2}L\{t\} - \frac{1}{2}L\{t\cos 2t\}$

$= \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1}{2}\frac{d}{ds}L\{\cos 2t\}$

$= \frac{1}{2s^2} + \frac{1}{2}\frac{d}{ds}\frac{s}{s^2+4}$

$= \frac{1}{2s^2} + \frac{1}{2}\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}$

$= \frac{1}{2s^2} + \frac{1}{2}\frac{4-s^2}{(s^2+4)^2}$

(8) $L\{\cos 4t\} = \frac{s}{s^2+4}$

$L\{t^2\cos 4t\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$

$$= \frac{d}{ds}\left[\frac{(s^2+4)\,1 - s\cdot 2s}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)}\right]$$

$$= \frac{(s^2+4)^2(-2s)-(4-s^2).2(s^2+4).2s}{(s^2+4)}$$

$$= \frac{-2s(s^2+4)-4s(4-s^2)}{(s^2+4)^3}$$

$$= \frac{2s^3-24s}{(s^2+4)^3}$$

Example 6:

Find the Laplace transform of the following functions (i) $t^2\cosh at$ (ii) $\frac{\sin^2 t}{t}$ (iii) $\frac{\cos 3t - \cos 2t}{t}$ (iv) $\frac{e^{3t}-e^{-2t}}{t}$

Solution:

(1) $L\{t^2\cosh at\} = L\left\{\frac{t^2\left(e^{at}+e^{-at}\right)}{2}\right\}$

$$= \frac{1}{2}L\left\{\frac{t^2e^{at}}{2}\right\} + \frac{1}{2}\left\{t^2e^{-at}\right\}$$

$$= \frac{1}{2}\cdot\frac{2}{(s-a)^3} + \frac{1}{2}\cdot\frac{2}{(s+a)^3}$$

$$= \frac{(s+a)^3+(s-a)^3}{(s^2-a^2)^3}$$

$$= \frac{2s^3+6a^2s}{(s^2-a^2)^3}$$

(ii) $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2}L\{1-\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$\therefore L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \left[-\frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}\right]_s^\infty$$

$$= \left[-\frac{1}{2}\log\sqrt{1+\frac{4}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\log\sqrt{1+\frac{4}{s^2}}$$

$$= \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$$

(iii) $L\{\cos 3t - \cos 2t\} = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$$\therefore L\left\{\frac{\cos 3t - \cos 2t}{t}\right\} = \int_s^\infty\left(\frac{s}{s^2+9}-\frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+9)\right] - \frac{1}{2}\log(s^2+4)\Big]_s^\infty$$

$$= \frac{1}{2}\left[\log(s^2+9) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+9}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{9}{s^2}}{1+\frac{4}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0-\log\left(\frac{1+\frac{9}{s^2}}{1+\frac{4}{s^2}}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

(iv) $L\{e^{3t}-e^{-2t}\} = \frac{1}{s-3} - \frac{1}{s+2}$

$$L\left\{\frac{e^{3t}-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s-3} - \frac{1}{s+2}\right) ds$$

$$= \left[\log\left(\frac{s-3}{s+2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1-\frac{3}{s}}{1+\frac{2}{s}}\right)\right]_s^\infty$$

$$= 0 - \log\frac{1-\frac{3}{s}}{1+\frac{2}{s}} = \log\left(\frac{s+2}{s-3}\right)$$

Exercise 1

I Find the Laplace transform of the following functions

(1) $t^5 - 4t^3 + 3$ (2) $\cosh 4t$ (3) $\sin^4 t$
(4) $\sin 3t \cos 2t$ (5) $\sin 7t \cos 3t$ (6) $\cos 4t \sin 2t$
(7) $\sinh 5t$ (8) $f(t) = \begin{cases} 0 & 0 < t < 2 \\ t & t > 2 \end{cases}$
(9) $5 - e^{2t} + 6t^2$ (10) $6e^{-3t} - t^2 + 2t - 8$

II Find the Laplace transform of the following functions

(1) te^{-3t} (2) $t^2 e^{5t}$ (3) $(te^t + e^{-t})^2$ (4) $t \sin 3t$
(5) $t^2 \cos 3t$ (6) $t^2 \sin 4t$ (7) $t\cosh 2t$ (8) $t^2 \sinh 3t$
(9) $\cos at - at \sin at$ (10) $e^{3t}\cos 6t - t^3 + e^t$
(11) $t^4 e^{5t} - e^t \cos\sqrt{7}t$ (12) $t^3 - te^t + e^{4t}\cos t$
(13) $\frac{1-\cos t}{t}$ (14) $\frac{\cos t - \cos 2t}{t}$
(15) $\frac{\cos at - \cos bt}{t}$ (16) $\frac{\sinh t}{t}$
(17) $e^{2t} t^n$ (18) $(at+b)^2 e^{2t}$
(19) $te^{3t}\cos 4t$ (20) $te^{-2t}\sin 3t$

III Find the Laplace transform of the following functions

(1) $t^2 + e^t \cos t$ (2) $e^{-t}\sin 3t + e^{6t} - 1$ (3) $(1+e^{-t})^2$
(4) $e^{7t}\sin 2t$ (5) $t \sin 2t \sin 5t$ (6) $te^{2t}\cos 5t$
(7) $(t-1)^4$ (8) $2t^2 e^{-t} + \cos 4t - t$ (9) $\sin^2 t \cos t$

(10) $\sin^3 t \cos^2 t$ (11) $(\sin t - \cos t)^2$
(12) $e^{-t}(3\sinh 2t - 5\cosh 2t)$ (13) $t^3 \cos t$
(14) $t(3\sin 2t - 2\cos 2t)$

Evaluate the following integrals using laplace transform

(1) $\int_0^\infty \frac{e^{-3t} - e^{-4t}}{t} dt$ (2) $\int_0^\infty \frac{e^{-t}\sin t}{t} dt$

(3) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$ (4) $\int_0^\infty te^{-3t}\sin t\, dt$

(5) $\int_0^\infty te^{-2t}\cos t\, dt$ (6) $\int_0^\infty t^3 e^{-t}\sin t\, dt$

Section - 3

Periodic functions

Definition

A function $f(t)$ is said to be periodic of period T if $f(t+T) = f(t)$ for all t in the domain of f.

Theorem:

If $f(t)$ has period T and is piecewise continuous on [0, T], then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + \ldots \quad (1)$$

Put $t = u + T$ in the 2nd integral, $dt = du$

$$\int_T^{2T} e^{-st} f(t)\, dt = \int_0^T e^{-s(u+t)} f(u+T)\, du$$

$$= e^{-sT}\int_0^T e^{-su} f(u)\, du \quad [\text{since } f(u+T) = f(u)]$$

$$= e^{-sT}\int_0^T e^{-st} f(t)\, dt$$