Chapter 26

Partial Differential Equations

Section 1

A partial differential equation is an equation which contains one or more partial derivatives. The order of the partial differential equation is that of the derivative of highest order in the equation. Let $z = f(x, y)$ where x and y are independant variables, and z is the dependant variable on x and y. Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives; $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives. For example the equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$ ...(1) is called the first order partial differential equation. The equation $x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + xy \frac{\partial^2 z}{\partial x dy} = 0$ ...(2) is called the second order partial differential equation. For convenience we use the following conventional notations for partial derivatives.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = t, \frac{\partial^2 z}{\partial y^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s$$

Then equations (1) and (2) can be written as

$$x^2 p + y^2 q = z^2 \qquad ...(3)$$

$$x^2 t + xys + y^2 r = 0 \qquad ...(4)$$

In ordinary differential equation we have seen that the differential equations are formed by eliminating arbitary constants. In the same way partial differential equations are also formed by eliminating the arbitary constants from a given relation between the variables. Also partial differential equations may be formed by the elimination of arbitary functions of the variables.

Hence partial differential equations may be formed by

(i) Eliminating arbitary constants.

(ii) Eliminating arbitary functions.

Consider z to be a dependent variable and x and y to be independent variables. Let the relationship between z, x, and y be defined by $f(x, y, z, a, b) = 0$ ...(1) Here there are arbitary constants a and b. To elimnate a and b we need three equations.

Differentiating (1) partially with respect to x and y we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \ (ie) \quad \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \ (ie) \quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \qquad ...(3)$$

In general we can eliminate the arbitrary constants from (1), (2) and (3) and the resulting equation will be of the form $\phi(x, y, z, p, q) = 0$. (4)

Let us now consider for example, forming the partial differential equation by eliminating arbitrary constants.

If may be noted that if the number of constants to be eliminated is equal to the number of independnt variables the eliminant of arbitrary constants will result in a first order partial diffferential equation. If the number of arbitrary constants is more than the number of independent variables a second order partial differential equation will be formed by eliminating arbitary constants.

Example

1. From the partial differential equation by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$

Solution :

$$z = (x^2 + a)(y^2 + b).$$

Differentiating partially with respect to x and y we get

$$p = 2x(y^2 + b)$$

$$q = 2y(x^2 + a) \qquad ..(1)$$

Multiplying (1) and (2) we get ...(2)

$$pq = 4xy(x^2 + a)(y^2 + b) \text{ or } pq = 4xyz$$

This is the required partial differential equation.

Example 2

Find the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$,

Solution :

$$\log(az - 1) = x + ay + b,$$

Differentiating partially with respect to x and y we get

$$\frac{1}{az - 1}ap = 1$$

(1)

$$\frac{1}{az-1}aq = a. \qquad ..(2)$$

Dividing (2) by (1) we get

$$\frac{q}{p} = a \qquad ...(3)$$

From equation (1) $ap = az - 1$.

$$a(z-p) = 1 \text{ or } \frac{q}{p}(z-p) = 1)$$

$$q(z-p) = p.$$

This is the required partial differntial equation.

Example 3

The equation of any sphere of radius 'r' having its centre in the xoy plane is $(x-a)^2 + (y-b)^2 + z^2 = r^2$

where a and b are arbitrary constants. Form a partial differential equation by eliminating the constants a and b.

Solution :

$$(x-a)^2 + (y-b)^2 + z^2 = r^2 \qquad ..(1)$$

Differentiating partially with respect to x and y we get

$$2(x-a) + 2zp = 0 \qquad ...(2)$$

$$2(y-b) + 2zq = 0 \qquad ...(3)$$

$$(x-a) = -zp. \qquad ...(4)$$

$$(y-b) = -zq. \qquad ...(5)$$

Using (4) and (5) in (1) we get

$$z^2\left[(p^2 + q^2 + 1)\right] = r^2$$

This is the required partial differential equation.

Example 4

Form the partial differential equation by eliminating a and b fro

$z = ax + by + a^2 + b^2$

Solution :

$$z = ax + by + a^2 + b^2 \; ...(1)$$

Diffrentiating partially with respect to x and y we get

$$p = a\,;\; q = b.$$

Substituting these in (1) we get

$$z = px + qy + p^2 + q^2$$

This is the requirment partial differential equation.

Example 5

Form the partial differential equation by eleminating a, b, c

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Differentiating partially with respect to x, y

$$\frac{2x}{a^2}+\frac{2z}{c^2}p=0.$$

$$\frac{2y}{b^2}+\frac{2zq}{c^2}=0$$

Differentiating (2) partially with respect to y,

$$0+\frac{2}{c^2}(zs+qp)=0.$$

$$zs+qp=0$$

Note: More than one partial differential equation is possible in this problem.

These partial differential equations are

$xzr+xp^2-zp=0.$

$yzt+yq^2-zq=0.$

Example 6

Form the partial differential equation by eliminating the arbitary constants form

$z=axe^y+a^2e^{2y}+b.$

Solution :

$z=axe^y+a^2e^{2y}+b.$ ...(1)

Differentiate (1) partially with respect to x and y

$p=ae^y$

$q=axe^y+2a^2e^{2y}.$

(i.e) $q=xp+2p^2$

This is the required partial differential equation.

Example 7

Form the partial differential equation of the family of spheres of radius r with the centre at (a, b, c).

Solution :

Let the centre be (a, b, b) and radius be r. The equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-b)^2 = r^2 \qquad ...(1)$$

Differentiating partially with respect to x, y we get

$$2(x-a) + 2(z-b)p = 0.$$
$$2(y-b) + 2(z-b)q = 0.$$
$$x - a = -(z-b)p. \qquad ...(2)$$
$$y - b) = -(z-b)q. \qquad ...(3)$$

From (1), (2) (3) we get

$$(z-b)^2 (p^2 + q^2 + 1) = r^2 \qquad ...(4)$$

Subtracting (3) from (2); $x - y = -(z-b)(p-q)$

$$z - b = -\frac{x-y}{p-q} \qquad ...(5)$$

Substituting (5) in (4) we get

$$\frac{(x-y)^2}{(p-q)^2}\left[p^2 + q^2 + 1\right] = r^2$$

$$(x-y)^2 \left[p^2 + q^2 + 1\right] = r^2 (p-q)^2$$

This is the required partial differential equation.

Example 8

Form the equation of all spheres with centres on the z axis.

Solution :

Let the centre equation of the sphere be.

$$x^2 + y^2 + (z-c)^2 = r^2.$$

Differentiating partially with respect to x and y we get

$$2x + 2p(z-c) = 0$$
$$2y + 2(y-c)q = 0$$
$$(z-c)p = -x.$$
$$(z-c)q = -y.$$

Dividing (3) by (4) we get

$$\frac{p}{q} = \frac{x}{y} \text{ (or) } py = xq.$$

This is the required partial differential equation.

Exercise 1

Form the partial differential equation by eliminating the constants.

1. $z = (x + a)(y + b)$
2. $z = xy + y\sqrt{x^2 + a^2} + b^2$
3. $z = ax + by + ab.$
4. $z = ax + by + a^2 + b^2$
5. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
6. $(x - a)^2 + (y - b)^2 + z^2 = a^2 + b^2$
7. $z = (x - a)^2 + (y - b)^2$
8. $z = axy + b.$
9. $z = xy + y\sqrt{x^2 - y^2} + b^2.$
10. $ax + by + z = 1$
11. $z = (x - a)^2 + (y - b)^2 + 1.$
12. $z = ax^2 + by^2 + ab$
13. $z = ax^3 + by^3$
14. $z = ax + by + \sqrt{a^2 + b^2}.$
15. $z = a(x + \log y) - b^2$
16. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$
17. $a(x^2 + y^2) + bz^2 = 1.$

Section 2

Elimination of Arbitary Functions

Here we consider a few example to show how partial differential equations can be obtained by eliminating the arbitrary functions.

Example 1

Form the partial differential equation by eliminating the arbitrary function from $\phi(u,v) = 0$.

Solution :

The arbitrary function is $\phi(u,v)=0$.

Differentiating partially with respedt to x and y we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0.$$

(i e) $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0.$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0.$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ we get

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)=\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)\left(\frac{\partial v}{\partial y}\right.$$

Grouping the terms containing p,q and other terms we ge

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\right.$$

This equation is in the form.

$$Pp+Qq=R \quad ...(1)$$

where

$$p=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$Q=\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)$$

$$R=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)$$

∴ The required partial differential equation is $Pp+Qq=$

Example 2

Form the partial differential equation by eliminating th function from $z=f(x^2+y^2+z^2)$

Solution :

$$z=f(x^2+y^2+z^2) \quad ...(1)$$

Differentiating (1) partially with respect to x and y,

$$p = f'(x^2 + y^2 + z^2)\ (2x + 2zp)$$

$$q = f'(x^2 + y^2 + z^2)\ (2y + 2zq) \quad ...(2)$$

Dividing (2) by (3), $\frac{p}{q} = \frac{x + 2p}{y + 2q}$

$$py + 2qp = qx + 2pq$$

$$py = qx$$

Example 3

Form the partial differenting equation by eliminating arbitrary f from $xyz = \phi(x^2 + y^2 - z^2)$

Solution :

$$xyz = \phi(x^2 + y^2 - z^2)$$

Differentiating partially with respect to x, y

$$xyp + yz = \phi'(x^2 + y^2 - z^2)\ (2x - 2zp)$$

$$xyq + xz = \phi'(x^2 + y^2 - z^2)\ (2y - 2zq)$$

Dividing (1) by (2)

$$\frac{yz + xyp}{xz + xyq} = \frac{x - zp}{y - zq.}$$

$$(yz + xyp)\ (y - zq) = (x - zp)\ (xz + xyq).$$

$$zy^2 + xy^2p - yz^2q - xyz\,pq = x^2z + x^2yq - xz^2p - xyz\,pq$$

$$\text{or. } px(y^2 + z^2) - qy(z^2 + x^2) = z(x^2 - y^2)$$

This is the required partial differential equation.

Example 4

Form a partial differential equation by eliminating the arbitrary function from $z = \phi(x)\,\phi(y)$

Solution

$$z = \phi(x)\,\phi(y)$$

Differentiating partially with respect to x, y we get

$$p = \phi'(x)\,\phi(y) \quad ...(2)$$

$$q = \phi(x)\,\phi'(y) \quad ...(3)$$

Differentiating (2) with respect to y

$$s = \phi'(x)\,\phi'(y)$$

Multiplying (2) and (3)

$$pq = \left[\phi'(x)\,\phi(y)\right]\ \left[\phi(x)\,\phi'(y)\right]$$

$$pq = zs$$

This is the required partial differential equation.

Example 5

Form the partial differential equation by climinating the arbitrary function from $z = f\left(\frac{y}{x}\right)$.

Solution :

$$z = f\left(\frac{y}{x}\right). \qquad ...(1)$$

Differentiating partially with respect to x and y

$$p = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \qquad ...(2)$$

$$q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \qquad ...(3)$$

Dividing (2) by (3)

$$\frac{p}{q} = \frac{-y}{x} \text{ (or) } px + qy = 0.$$

This is the required partial differential equation.

Example 6

Form the partial differential equation by eliminating the arbitrary function from $z = f(x^2 - y^2)$.

Solution :

$$z = f(x^2 - y^2). \qquad ...(1)$$

Differentiating with respect to x and y we get

$$p = f'(x^2 - y^2)(2x) \qquad ...(2)$$

$$q = f'(x^2 - y^2)(-2y) \qquad ...(3)$$

Dividing (2) by (1), $\frac{p}{q} = \frac{-x}{y}$ *or* $py + qx = 0$.

This is the required partial differential equation.

Example 7

Form the partial differential equation by eliminating the arbitrar function from $lx + my + nz = f(x^2 + y^2 + z^2)$

Solution

$$lx + my + nz = f(x^2 + y^2 + z^2)$$

Differentiating with respect to x and y respectively,

$$l + np = f'(x^2 + y^2 + z^2)(2x + 2zp)$$

$$m + nq = f'(x^2 + y^2 + z^2)(2y + 2zq)$$

Dividing (2) by (3)

$$\frac{l + np}{m + nq} = \frac{x + zp}{y + zq}$$

$$(l + np)(y + zq) = (x + zp)(m + nq).$$

$$ly + npy + lzq = mx + mzp + nxq$$

$$(m - ny)p + (nx - lz)q = ly - mx.$$

This is the required partial differential equation.

Example 8

Form the partial differential Equation by eliminating the ar[bitrary] function in

$z = xf_1(x + t) + f_2(x + t).$

Solution

$$z = xf_1(x + t) + f_2(x + t).$$

Differentiating partially with respect to x and t we get

$$\frac{\partial z}{\partial x} = f_1(x + t) + xf_1'(x + t) + f_2'(x + t) \quad ...($$

$$\frac{\partial z}{\partial t} = xf_1'(x + t) + f_2'(x + t) \quad ...(3$$

$$\frac{\partial^2 z}{\partial x^2} = 2f'(x + t) + xf_1''(x + t) + f_2''(x + t) \quad ...($$

$$\frac{\partial^2 z}{\partial t^2} = xf''_1(x + t) + f_2''(x + t) \quad ...(5)$$

$$\frac{\partial^2 z}{\partial x \, \partial t} = f_1'(x + t) + xf_1''(x + t) + f_2''(x + t). \quad ..(6)$$

From (4), (5), (6) we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = \frac{2\,\partial^2 z}{\partial x \, \partial t}$$

$$t + r = 2s.$$

This is the required partial differential equation.

Example 9

Eliminate the arbitary function in

$z = f_1(y + 2x) + f_2(y - 3x)$

Solution

$$z = f_1(y + 2x) + f_2(y - 3x)$$

Differentiating with respect to x, y we get ...(1)

$$\frac{\partial z}{\partial x} = 2f_1'(y + 2x) - 3f_2(y - 3x) \quad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = f_1'(y + 2x) + f_2'(y - 3x) \quad \text{...(3)}$$

$$\frac{\partial^2 z}{\partial x^2} = 4f_1''(y + 2x) + 9f_2''(y - 3x) \quad \text{...(4)}$$

$$\frac{\partial^2 z}{\partial y^2} = f_1''(y + 2x) + f_2''(y - 3x) \quad \text{...(5)}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = 2f_1''(y + 2x) - 3f_2''(y - 3x) \quad \text{...(6)}$$

Eliminating the arbitrary functions we get.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

Example 10

Form the partial differential equation by eliminating the arbitrary function in $z = y f(x) + x g(y)$

Solution :

$$z = y f(x) + x g(y) \quad \text{..(1)}$$

Differentiating with respect to x and y we get

$$p = yf'(x) + g(y) \quad \text{...(2)}$$

$$q = f(x) + xg'(y) \quad \text{...(3)}$$

$$s = f'(x) + g'(y)$$

$$xp = xyf'(x) + xg(y)$$

$$yq = yf(x) + xyg'(y)$$

$$xp + yq = xyf'(x) + yf(x) + xg(y) + xyg'(y)$$

$$xp + yq = xys + z$$

This is the required partial differential equation.

Example 11

Find the partial differential equation by eliminating the arbitrary function in $f(x + y + z, x^2 + y^2 - z^2) = 0$ ....(1)

Solution

Let $u = x + y + z,$

$v = x^2 + y^2 - z^2$

Then the give equation is $f(u, v) = 0$. Differentiating (1) pa... respect to x and y we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0\text{, Also } \frac{\partial u}{\partial x} = 1 + p \text{ and } \frac{\partial v}{\partial x} = 2x \ldots$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0;\quad \frac{\partial u}{\partial x} = 1 + q \text{ and } \frac{\partial v}{\partial y} = 2y - 2zq$$

(i e) $$\frac{\partial f}{\partial u}(1 + p) + \frac{\partial f}{\partial v}(2x - 2zp) = 0.$$

$$\frac{\partial f}{\partial u}(1 + q) + \frac{\partial f}{\partial v}(2y - 2zq) = 0.$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from the above two equations we get

$$\begin{vmatrix} 1+p & 2x - 2zp \\ 1+q & 2y - 2zq \end{vmatrix} = 0.$$

$$(1 + p)(2y - 2zq) - (1 + q)(2x - 2zp) = 0.$$

$$(1 + p)(y - zq) - (1 + q)(x - zp) = 0.$$

$$y + yp - zq - zpq - x - xq + zp + zpq = 0.$$

$$(y + z)p - (x + z)q = x - y.$$

This is the required partial differential equation

Example 12

Form the partial differential equation by eliminating the arbitra... function in

Solution : $$f(x + y + z, xyz) = 0.$$

Let $u = x + y + z$

$v = xyz$

Thus the given equation becomes $f(u, v) = 0$. Differentiating partially with respect to x and y we get,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0\text{, Also } \frac{\partial u}{\partial x} = 1 + p \text{ and } \frac{\partial v}{\partial x} = yz + xyp$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0.\quad \frac{\partial u}{\partial y} = 1 + q \text{ and } \frac{\partial v}{\partial y} = xz + xyq$$

(i e) $\frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(yz+xyp)=0.$

$$\frac{\partial f}{\partial u}(1+q)+\frac{\partial f}{\partial v}(xz+xyq)=0.$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ we get

$$\begin{vmatrix} 1+p & yz+xyp \\ 1+q & xz+xyq \end{vmatrix}=0.$$

$$(1+p)(xz+xyq)-(1+q)(yz+xyp)=0.$$

$$xz+xzp+xyq+xy\,pq-(yz+yzq+xyp+xy\,pq)=0.$$

$$x(y-z)p+y(z-x)q=z(x-y).$$

This is the required partial differential equation

Example 13

Form the partial differential equation by eliminating the arbitrar function in $f\left(\frac{z}{x},\frac{y}{x}\right)=0.$

Solution

Let $u=\frac{z}{x}$, $v=\frac{y}{x}.$

∴ The given equation becomes $f(u,v)=0.$

Differentiating partially with respect x and y.

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0.$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0.$$

$$\frac{\partial f}{\partial u}\left(\frac{-z}{x^2}+\frac{1}{x}p\right)+\frac{\partial f}{\partial v}\left(\frac{-y}{x^2}\right)=0.$$

$$\frac{\partial f}{\partial u}\left(\frac{1}{x}q\right)+\frac{\partial f}{\partial v}\left(\frac{1}{x}\right)=0$$

Eliminating $\frac{\partial f}{\partial v}$ and $\frac{\partial f}{\partial u}$ we get

$$\begin{vmatrix} \frac{-z}{x^2}+\frac{p}{x} & \frac{-y}{x^2} \\ \frac{q}{x} & \frac{1}{x} \end{vmatrix}=0$$

$$(ie)\quad \begin{vmatrix} -z+px & -y \\ q & 1 \end{vmatrix} = 0.$$

$$-z+px+yq=0 \quad \text{or} \quad px+qy=z.$$

Example 14

Form the partial differential equation by eliminating t function in

$$xy+z^2=f(x+y+z).$$

Solution

$$xy+z^2=f(x+y+z).$$

This equation can be written as $f(x+y+z, xy+z^2)=$ $f(u,v)=0$ where $u=x+y+z$, $v=xy+z^2$. Differentiating, equ partially with respect to x and y,

$$\frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(y+2zp)=0.$$

$$\frac{\partial f}{\partial u}(1+q)+\frac{\partial f}{\partial v}(x+2xq)=0.$$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ we get

$$\begin{vmatrix} 1+p & y+2zp \\ 1+q & x+2zq \end{vmatrix} = 0.$$

$$(1+p)(x+2zq)-(1+q)(y+2zp)=0.$$

$$x+2zq+xp+2zqp-y-2zp-qy-2zpq=0$$

$$p(x-2z)-q(y-2z)=y-x.$$

This is the required partial differential equation.

Exercise 2

I Form the partial differential equation by eliminating arbitrar function from the following:

1. $z=f(xy)$
2. $z=f(x-y)$
3. $z=x+y+f(xy)$
4. $z=e^x f(y-x)$.
5. $z=f(x^2-y^2)$.
6. $z=(x+y)f(x^2-y^2)$.

7. $z = f\left(\frac{xy}{z}\right)$

8. $z = f(my - lx)$

9. $z = f(2x - 3y) + xg(2x - 3y)$.

10. $z = y^2 + 2f\left(\frac{1}{x} \text{lo}, \ y\right)$

11. $z = (x + y) f(x^2 - y^2)$.

12. $x = f(y) + g(z)$.

13. $z = x^2 f(x - y)$.

14. $z = f(x + z) + g(x + y)$.

15. $z = f(x) + e^y g(x)$.

16. $x + y + z = f(xy + z^2)$.

17. $\frac{z}{2} = \phi(x^2 - y^2)$

18. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

19. $ax + by + cz = f(x^2 + y^2 + z^2)$

20. Find the partial differential equation of all spheres of radiu having centre in XOY plane.

21. Find the partial differential equation of all planes having equa and y intercepts.

II. Form the partial differential equation by eliminating the arbit functions from the following.

(1) $z = \phi(x + ct) + \phi(x - ct)$.

(2) $z = \phi(x + 2y) + \phi(x - 2y)$.

(3) $z = \phi(x) + \phi(y)$.

(4) $z = y\phi(x) + x\phi(y)$.

(5) $V = \frac{1}{r}\left[f(r - at) + f(r + at)\right]$

(6) $z = f(x + y)\phi(x - y)$.

(7) $z = x\phi\left(\frac{y}{x}\right) y\phi(x)$.

(8) $z = \phi(2x + 3y) + y\phi(2x + 3y)$.

(9) $z = ax^2 + g(y)$

(10) $z = f(x + iy) + \phi(x - iy)$

III Form the partial differential equation by eliminating function from the following

1. $\phi(x^2 + y^2 + z^2, xyz) = 0$.
2. $\phi(x + y + z, xyz) = 0$.
3. $\phi(x^2 + y^2 + z^2, xy + z) = 0$.
4. $\phi(x + y + z, xy + z^2) = 0$.
5. $\phi\left(xyz, x^2 + y^2 + z^2\right) = 0$.
6. $z = xy\phi(x + 2y + xz^2)$
7. $\phi(x^3 - y^3, x^2 - z^2) = 0$.
8. $\phi\left(\frac{x}{y}, x - z\right) = 0$.
9. $\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$.
10. $\phi(x - z, y - z) = 0$.

Non - Linear Differential Equations of First Order – Standard Typ

A partial differential equation involving the first order parti derivatives $\frac{\partial z}{\partial x}(=p)$ and $\frac{\partial z}{\partial y}(=q)$ is called a Linear partial differenti equation. Partial differential equations which involue the higher power of p, q, pq are called non-linear partial differential equations. Linear partial differential equations are obtained by eliminating arbitrary constants or arbitrary functions.

Complete solution

A solution of a partial differential equation which contains as many arbitary constants as the number of independant variables is called a complete solution. The complete solution is also referred to as complete integral. For example the partial differential equation may be obtained from the equation $f(x, y, z, p, q) = 0$. (1)

$g(x, y, a, b) = 0$. (2)

In this case (2) is called the complete solution of the partial differential equation (1). There are two independant variables. In the solution (2). there are two arbitary constants.

Singular Solution

The complete solution represents two parameter family of surfaces which may or may not have an envelope.

The envelope if it exists, is obtained by eleminating 'a' and 'b' from

$$g(x, y, z, a, b) = 0. \quad (1).$$

$$\frac{\partial}{\partial a} g(x, y, z, a, b) = 0, \quad (2).$$

$$\frac{\partial}{\partial b} g(x, y, z, a, b) = 0. \quad (3).$$

If the eliminent $\phi(x, y, z) = 0$ exists it is called the singular solution of equation (1). Hence if $f(x, y, z, p, q) = 0$ is the partial differential equation whose complete solution is $g(x, y, z, a, b) = 0$ then the eliminent (if it exists) of

$$g(x, y, z, a, b) = 0.$$

$$\frac{\partial}{\partial a} g(x, y, z, a, b) = 0.$$

$$\frac{\partial}{\partial b} g(x, y, z, a, b) = 0.$$

is called the singular solution or singular integral. The singular solution represents the envelope of family of surface given by the general solution $g(x, y z, a, b) = 0$. The envelope in general may or may not exist.

General Solution

In the non-linear partial differential equation $f(x, y z, a, b) = 0$, if $g(x, y z, a, b) = 0$ is the complete solution it may be possible for the existence of a relationship between the arbitrary constants a and b. Suppose this relationship given by. $b = \phi(a)$. Then the complete solution takes the form $g\left[x, y, z, a, \phi(a)\right] = 0$. This is a one parameter family of surfaces of the partial differential equation $f(x, y z, p, q) = 0$. If the family has an envelope it is obtained by eliminating a from

$$g\left(x, y, z, a, \phi(b)\right) = 0 \quad (1)$$

$$\text{and } \frac{\partial}{\partial a} g\left(x, y z. a. \phi(b)\right) = 0. \quad (2)$$

If this eliminent if it exists if called the general soluti... (1). This solution is also referred to as general integral.

Example

Consider the partial differential equation

$z = px + qy - (p^2 + q^2)$ (1).

Solution

It can be easily shown as

$z = ax + by - (a^2 + b^2)$ is the complete solutio...

Here $g\left(x, y z, a, \phi(b)\right) = z - ax - by + a^2 + b^2 = ...$

$\frac{\partial}{\partial a} g\left(x, y z, a, b\right) = 0 = -x + 2a = 0.$

$\frac{\partial g}{\partial b}\left(x, y z, a, b\right) = 0 = -y + 2b = 0.$

Eliminating a and b from the above equations we get $x^2 + ...$ which is the envelope of the surface given by (2). This solution i... the singular solution.

Suppose $b = a$ is complete solution for (2)

$g\left(x, y z, a, b\right) = 0 \Rightarrow,$

$g\left(x, y, z, a, b\right) = z - a(x + y) + 2a^2 = 0.$ (3)

$\frac{\partial g}{\partial a} = 0 = -(x + y) + 4a = 0$ (4)

Eliminating 'a' from (3) and (4) we get

$8z = (x + y)^2$ (3). This is called the general solution of equation...

Let us now consider four different types of non linear pa... differential equation and the procedure for obtaining their comp... solution.

Standard type I $f(p, q) = 0$. Here the partial differential equati... involves only p and q and the variable x, y, z are absent.

Standard Type II $z = px + qy + f(p, q)$

This is called the Clairaut's form

Standard type III $F(z, p, q) = 0$.

Here the partial differential equation will contain p, q and the depend... variable z only. The independent variables x and y are absent.

Standard Type IV: Partial differential equation of the fc $(x, p) = f_2(y, q)$. Here z is absent and also it is in a separable fc .e) the terms containing x and p only on one side and terms contain and q only on the other side.

Charpits method :

This is a general method of solving linear partial differential equat where the given partial differential equation cannot be reduced to one of the four general forms.

Section 4

Standard Type 1 $F(p, q) = 0$. (1)

Consider the equation $z = ax + by + c$ (2) where a and b are connec by the relation $f(a,b) = 0$. Differentiating the above equation parti. with respect to x and y we get $p = a, q = b$.

Substituting these in the given partial differential equation we $f(a, b) = 0$.

From the relationship $f(a, b) = 0$ we can find b in terms of a, say b

Substituting this in (2) we get, $z = ax + \phi(a) y + c$

We see that (3) is the complete solution of equation (1).

Hence we note that for the linear partial differential equation of t (1), the complete solution is of the form

$z = ax + by + c$ where a and b are connected by relat $\phi(a, b) = 0$ or $b = g(a)$.

Here a and b are arbitary constants.

To find the singular soluion we have to determine the eliminent o and c from

$z = ax + g(a) y + c$.

$0 = x + g'(a) y$ (differentiating partially with respect to a)

Differentiating partially with respect to c, $0 = 1$.

As the last equation is inconsistent no singular solutions will exist the partial differential equation of the type $F(p, q) = 0$

Take $c = \phi(a)$.

The complete solution becomes one parameter family

$z = ax + g(a) y + \phi(a)$

Differentiating partially with respect to a,

$0 = a + g'(a) + \phi'(a)$

Eliminating 'a' from the last two equations we get the ge[illegible] which is the envelope of the single parameter family given

Example : 1

Solve $p^2 + q^2 = 1$.

Solution :

This is of the type $f(p,q) = 0$.

Therefore the complete solution of this partial differential [illegible] in the form

$$z = ax + by + c \qquad (1)$$

Differentiatng (1) partially with respect to x and y we get

$$p = a,\ q = b.$$

Substituting in the given partial differential equation we hav[illegible]

$$a^2 + b^2 = 1. \therefore b = \sqrt{1 - a^2}.$$

The complete solution is $z = ax + \left(\sqrt{1 - a^2}\right) y + c$ (1)

where a and c are arbitrary constants

There is no singular integral to the partial differential equation type. To obtain the general solution take $c = \phi(a)$.

$$\therefore z = ax + \left(\sqrt{1 - a^2}\right) y + \phi(a) \qquad \text{..........} (2)$$

Differentiating partially with respect to a we get

$$0 = x - \left(\frac{a}{\sqrt{1 - a^2}}\right) y + \phi'(a) \quad (3)$$

The eliminent of a between (2) and (3) gives the general solutio[illegible]

Example 2 :

Solve $\sqrt{p} + \sqrt{q} = 1$.

Solution :

The equation is of the type $f(p, q) = 0$.

$\therefore$ The complete solution is of the form $z = ax + by + c$ (1)

Differentiating (1) partially with respect to x and y we get

$$p = a \text{ and } q = b.$$

Therefore the given equation becomes $\sqrt{a} + \sqrt{b} = 1$.

the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

where a and c are arbitrary constants.

This type of equation has no singular solution.

Take $c = \phi(a)$.

$$\therefore z = ax + (1 - \sqrt{a})^2 y + \phi(a) \quad (2).$$

Differentiating partially with respect to a we get

$$0 = x - (1 - \sqrt{a})\frac{1}{\sqrt{a}} y + \phi'(a) \qquad (3)$$

The eliminent of 'a' between (2) and (3) gives the general solution.

Example 3

Solve $p + q = pq$.

Solution :

This equation is of the type $f(p, q) = 0$.

$\therefore$ The complete solution is of the form $z = ax + by + c$ (1)

Differentiating (I) with respect to x and y we get

$$p = a, q = b.$$

Therefore the given equation becomes

$$a + b = ab.$$

$$a = b(a - 1)\ ;\ b = \frac{a}{a - 1}.$$

Therefore the complete solution is

$$z = ax + \left(\frac{a}{a - 1}\right) y + c \quad (1).$$

This type of eqution has no singular solution.

Let $c = \phi(a)$.

$$z = ax + \left(\frac{a}{a - 1}\right) y + \phi(a). \quad (2)$$

Differentiating partially with respect to a

$$0 = x + \left[\frac{(a - 1)\,1 - a}{(a - 1)^2}\right] y + \phi'(a).$$

$$0 = -\frac{1}{(a - 1)^2} y + \phi'(a)$$

The eliminent of 'a' between (2) and (3) gives the general

Example 4 :

Solve $x^2p^2 + y^2q^2 = z^2$

Solution :

Dividing by z^2 we get

$$\left(\frac{x}{z}p\right)^2 + \left(\frac{y}{z}q\right)^2 = 1.$$

Put $X = \log x$, $Y = \log y$, $Z = \log z$.

$$\frac{dx}{x} = dX, \frac{dy}{y} = dY, \frac{dz}{z} = dZ$$

$$\frac{x}{z}\frac{dz}{dx} = \frac{dZ}{dX}, \frac{y}{z}\frac{dz}{dy} = \frac{dZ}{dY}$$

Therefore the given equation becomes.

$$\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dZ}{dY}\right)^2 = 1.$$

where

$$\frac{dZ}{dX} = P, \frac{dZ}{dY} = Q$$

This equation is of the form $f(P,Q) = 0$. Therefore the solution i

$$z = aX + bY + c \qquad (1) \text{ where}$$

$$a^2 + b^2 = 1 \text{ (or) } b = \sqrt{1 - a^2}$$

Therefore the complete solution is

$$Z = aX + \sqrt{1 - a^2}\, y + c.$$

$$\log z = a \log x + \sqrt{1 - a^2}\,(\log y) + \log k.$$

$$\therefore z = kx^a y^{\sqrt{1 - a^2}}$$

Example 5 :

Solve $z^2 = xypq$.

Solution :

Dividing by z^2, $\left(\frac{xp}{z}\right)\left(\frac{yq}{z}\right) = 1$

Take $X = \log x$, $Y = \log y$, $Z = \log z$.

$$\frac{x}{z}p = P, \quad \frac{y}{z}q = Q$$

Therefore the given equation becomes $PQ = 1$.

This equation is in the form $F(P, Q) = 0$. Therefore the complete solution is $Z = aX + bY + c$ where

$$ab = 1 \therefore b = \frac{1}{a}.$$

Therefore the complete solution is

$$Z = aX + \frac{1}{a}Y + c.$$

$$(i,e) \quad \log z = a \log x + \frac{1}{a} \log y + \log k$$

$$z = kx^a . y^{1/a}$$

Exercise 3

Obtain the complete solution for the following partial differential Equation.

1. $pq = 1$
2. $p^2 + q^2 = 9$
3. $q^2 = p$
4. $p^2 + p = q^2$
5. $p^2 + q^2 = npq$
6. $pq + p + q = 0.$
7. $3p^2 - 2q^2 = 4pq$
8. $p^3 - q^3 = 0.$
9. $q^2 - 3q + p - 2 = 0.$

Section 5

Standard Type II

Equation of the type $z = px + qy + f(p, q)$. This type of equation may be considered as analogous to clairaut's form $y = px + f(p)$ in the ordinary differential equation. Here $p = \frac{dy}{dx}$

Now consider the equation $z = ax + by + f(a, b)$

where a and b are arbitrary constants.

Differentiating (1) partially with respect to x and y we get

$p = a$ (2) and $q = b$ (3)

Therefore by eliminating the arbitrary constants from (1) (2), and (3) we get,

$z = px + qy + f(p, q)$ (4).

Therefore (1) can be considered as the complete solution of the partial differential equation. This type of partial differential equation is known as extended Clairaut's form. The complete solution consists of two parameter family of planes. The singular solution if it exists is a surface having the complete solution as tangent planes.

Example 1 :

Solve $z = px + qy + ab$

Solution :

This equation is a partial differential equation of Clairauts type.

Therefore the complete solution is got by replacing p by a and q by b where a and b are arbitrary constants.

(i e) the complete solution is $z = ax + by + ab$ (1). Differentiating (1) partially with respect to a and b we get

$0 = x + b$ (2)

$0 = y + a$ (3)

Eliminating a and b from (1), (2) and (3) we get

$z = -xy - xy + xy.$

$(i\,e)\, z + xy = 0.$

This gives the singular solution of the given partial differential equation and to get the general solution,

put $b = \phi(a)$ (1).

$z = ax + \phi(a) y + a\phi(a)$ (4)

Differentiating this partially with respect to 'a' we get

$0 = x + \phi'(a) y + a\phi'(a) + \phi(a)$ (5)

Eliminating 'a' from (4) and (5) we get the general solution.

Example 2 :

Solve $z = px + qy + p^2 q^2$.

Solution :

This equation is a partial differential equation of Clairauts type. Therefore the complete solution is

$z = ax + by + a^2 b^2$ (1) where a and b are arbitrary constants.

Differentiating (1) partially with respec to a and b we get

$0 = x + 2a b^2$ (2)

$0 = y + 2a^2 b$ (3)

$$x = -2a b^2$$

$$= \frac{-2a y^2}{4a^4} = \frac{-y^2}{2a^3}.$$

$$z = x \sqrt[3]{\frac{y^2}{2x}} - y \sqrt[3]{\frac{x^2}{2y}}$$

$$z = -2^{\frac{2}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}}$$

This is the singular solution where $b = f(a)$. The complete solution is $z = ax + f(a) y + a^2 (f(a))^2$ (2) Differentiating partially with respect to 'a'.

$0 = a + f'(a)y + 2a^2 f(a) f'(a) + 2a (f(a))^2$ Eliminant of 'a' from (2) and (3) gives the general solution.

Example 3 :

Obtain the complete solution and singular solution of $z = px + qy + p^2 + pq + q^2$.

Solution :

This equation is of Clairaut's form. Therefore the complete solution is $z = ax + by + a^2 + ab + b^2$ (1) where a and b are arbitary constant.

Differentiating (1) partially with respect to a and b we get

$) = x + 2a + b$ (2)

$) = y + 2b + a$ (3)

$2x - y = 3a$ and $2y - x = 3b$.

$a = \frac{2x - y}{3}, b = \frac{2y - x}{3}$

Substituting this in equaction (1) we get

$\left(\frac{2x - y}{3}\right) x + \left(\frac{2x - x}{3}\right) y + \left(\frac{2x - y}{3}\right)^2 + \frac{(2x - y)(2y - x)}{9}$

plifying we get $3z = xy - x^2 - y^2$

This is the singular solution,

mple 4 :

Obtain the complete and singular solution of

$\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

ation :

$\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

$z = px + qy + (pq)^{3/2}$

This is a Clairaut's form of partial differential equation.

Therefore the complete solution is

$z = ax + by + (ab)^{3/2}$

Differentiating (1) partially with respect to a and b

$0 = x + \frac{3}{2}(a\,b)^{1/2} b$. (2)

$0 = y + \frac{3}{2}(a\,b)^{1/2} a$. (3)

$x = \frac{-3}{2}(a\,b)^{1/2} b$

$y = \frac{-3}{2}(a\,b)^{1/2} a \therefore \frac{x}{y} = \frac{b}{a}$.

Multiplying, $xy = \frac{9}{4}(a\,b)^2 ; xy = \frac{9}{4} a^2 \frac{a^2 x^2}{y^2}$

$$z = \left(\frac{4y^3}{9x}\right)^{1/4} x + \left(\frac{4x^3}{9y}\right)^{1/4} y + \left(\frac{4y^3}{9x}\cdot\frac{4x^3}{9y}\right)^{3/8}$$

$$z = \frac{4^{\frac{1}{4}} y^{\frac{3}{4}} x^{\frac{3}{4}}}{9^{1/4}} + \frac{4^{\frac{1}{4}} x^{\frac{3}{4}} y^{\frac{3}{4}}}{9^{1/4}} + \left(\frac{4}{9}xy\right)^{\frac{3}{4}}$$

$$z = 4\left(\frac{4xy}{9}\right)^{3/4}$$

This is the singular solution.

Example 5 :

Find the complete and singular solution of $z = xp + yq + p^2 - q^2$.

Solution :

$z = xp + yq + p^2 - q^2$.

This is in Clairaut's form.

Therefore the complete solution in $z = ax + by + a^2 - b^2$. (1) where a and b are arbitrary constants.

Differentiating (1) partially with respect to a and b we get

$0 = x + 2a$ (2)

$0 = y - 2b$ (3)

$a = -x/2$

$b = y/2$

Substituting in (1) we get

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$$

$$z = -\left(\frac{x^2 - y^2}{2}\right) + \frac{x^2 - y^2}{4}$$

$4z = y^2 - x^2$ which is the singular solution.

Exercise 4

Find the complete solution and singular solution of the following partial differential equations.

1. $z = px + qy + 2\sqrt{pq}$

2. $z = pq + qy + \sqrt{1 + p^2 + q^2}$

3. $z = px + qy + \log pq.$

4. $z = px + qy + p^2 + q^2$

5. $(1 - x)p + (2 - y)q = 3 - z.$

6. $z = p(x - q) + qy.$

7. $z = px + qy + \sqrt{pq}$

Section 6

Standard type III

Partial differential equation of the form $F(z, p, q) = 0.$

For this partial differential equation we assume $z = F(x + ay)$ solution.

Let $u = x + ay$, then $z = F(u).$

Then $p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$

$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$

Then the given differential equation becomes

$F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of first order.

Procedure for solving partial differential equation of the type $F(z, p, q) = 0.$

Step I: Assume $u = x + ay.$

Step II : Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in the given partial differential equation.

Step III. Solve the resulting ordinary differential equation of first order.

Example 1 :

Find the complete solution of $1 + q^2 = q(z - b)$

Solution :

Assume $z = F(x + ay) = F(u)$

Then $p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$.

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

The given equation becomes

$$1 + a^2\left(\frac{dz}{du}\right)^2 = a\frac{dz}{du}(z-b).$$

$$(i\,e)\ a^2\left(\frac{dz}{du}\right)^2 - a(z-b)\frac{dz}{du} + 1 = 0.$$

$$\frac{dz}{du} = \frac{a(z-b) \pm \sqrt{a^2(z-b)^2 - 4a^2}}{2a^2}$$

$$= \frac{z-b \pm \sqrt{(z-b)^2 - 4}}{2a}$$

$$\frac{dz}{(z-b) \pm \sqrt{(z-b)^2 - 4}} = \frac{du}{2a}$$

$$\frac{z-b) \pm \sqrt{(z-b)^2 - 4}}{4}\,dz = \frac{du}{2a}$$

Integrating we get

$$\frac{1}{4}\left[\frac{(z-b)^2}{2} \pm \frac{z-b}{2}\sqrt{(z-b)^2 - 4} - \frac{4}{2}\cos h^{-1}\frac{z-b}{2}\right] = \frac{u}{2a} + k.$$

$$\frac{1}{4}\left[(z-b)^2 \pm (z-b)\sqrt{(z-b)^2 - 4} - 4\cos h^{-1}\frac{z-b}{2}\right] = \frac{u}{a} + k_1.$$

$$\frac{1}{4}\left[(z-b)^2\right] \pm \frac{1}{4}\left[\{(z-b)\sqrt{(z-b)^2 - 4} - 4\cos h^{-1}\frac{z-b}{z}\right]$$

$$= \frac{x+ay}{a} + k_1.$$

Example 2 :

Solve $z = p^2 + q^2$ for complete and singular solutions.

Solution :

Assume $z = F(x+ay) = F(u)$.

then $p = \frac{dz}{du}$, $q = a\frac{dz}{dy}$

The given equation becomes

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$= \left(\frac{dz}{du}\right)^2 (1 + a^2)$$

$$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$(i\,e)\ \frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}}$$

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + c.$$

$$= \frac{1}{\sqrt{1+a^2}}(u + b)$$

$$\therefore 4(1 + a^2)z = (x + ay + b)^2 \quad (1)$$

This is the complete solution.

Differentiating equation (1) partially with respect a and b we get

$8az = 2(x + ay + b)y \quad (2)$

$0 = 2(x + ay + b)\,1 \quad (3)$

Substuting (3) in (2) we get $z = 0$

This is the singular solution.

Example 3 :

Find the complete solution of $p(1 + q) = qz$.

Solution :

Assume that $z = f(x + ay)$

then $p = \frac{dz}{du}, q = \frac{adz}{du}$

The given equation becomes

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$(i\,e)\ 1 + \frac{adz}{du} = az.$$

$$\frac{dz}{du} = \frac{az - 1}{a}$$

$$\frac{adz}{az-1} = du.$$

$$\log(az-1) = u + c.$$

$$\log(az-1) = x + ay + c.$$

This is the complete solution.

Example 4 :

Solve $z^2(p^2 + q^2 + 1) = b^2$

Solution :

Let $z = F(x + ay) = F(u)$ where

$$p = \frac{dz}{du}, q = \frac{adz}{du}.$$

The given equation becomes

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = b^2$$

$$z^2\left[\left(\frac{dz}{du}\right)^2(1+a^2) + 1\right] = b^2$$

$$z^2(1+a^2)\left(\frac{dz}{du}\right)^2 = b^2 - z^2.$$

$$\left(\frac{dz}{du}\right)^2 = \frac{b^2 - z^2}{z^2(1+a^2)}$$

$$\frac{dz}{du} = \pm\frac{1}{\sqrt{1+a^2}}\frac{\sqrt{b^2-z^2}}{z}$$

$$\frac{zdz}{\sqrt{b^2-z^2}} = \pm\frac{1}{\sqrt{1+a^2}}du.$$

Integrating we get

$$-\sqrt{b^2-z^2} = \pm\frac{u}{\sqrt{1+a^2}} + c.$$

$$-\sqrt{b^2-z^2} = \pm\frac{(x+ay)}{\sqrt{1+a^2}} + c.$$

This gives the complete integral.

Example 5 :

Solve for complete solution for $z^2(p^2z^2 + q^2) = 1$.

Solution :

Assume $z = F(x + ay) = F(u)$ Then

$$p = \frac{dz}{du}, q = a\frac{dz}{du}.$$

The given equation becomes

$$z^2\left[\left(\frac{dz}{du}\right)^2 z^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1.$$

$$z^2\left[\left(\frac{dz}{du}\right)^2 (z^2 + a^2)\right] = 1$$

$$z\sqrt{z^2 + a^2}\, dz = \pm\, du.$$

Integrating we get.

$$\int \sqrt{z^2 + a^2}\, d(z^2) = \pm \int 2\, du.$$

$$\frac{(z^2 + a^2)^{3/2}}{3/2} = \pm\, 2u + c$$

$$\frac{2}{3}(z^2 + a^2)^{3/2} = \pm\, 2(x + ay) + c.$$

$$\frac{1}{3}(z^2 + a^2)^{3/2} = \pm\, (x + ay) + c.$$

Example 6 :

Solve $p^2 z^2 + q^2 = 1$.

Solution :

Assume $z = F(x + ay) = F(u)$. then

$$p = \frac{dz}{du}, q = a\frac{dz}{du}.$$

The given equation becomes.

$$z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = 1.$$

$$(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1.$$

$$\left(\sqrt{z^2 + a^2}\right) dz = \pm\, du.$$

Integrating we get

$$\frac{z}{2}\left(\sqrt{z^2+a^2}+\frac{a^2}{2}\log\left(z+\sqrt{z^2+a^2}\right)=\pm u+c\right.$$

$$z\left(\sqrt{z^2+a^2}\right)+a^2\log\left(z+\sqrt{z^2+a^2}\right)=\pm 2(x+ay)+c$$

Example 7 :

Solve $p^3+q^3=8z$.

Solution :

Assume $z=F(x+ay)=F(u)$. then

$$p=\frac{dz}{du},\ q=a\frac{dz}{du}$$

The given equation becomes

$$\left(\frac{dz}{du}\right)^3\left(1+a^3\right)=8z.$$

$$\therefore \frac{dz}{du}=\frac{2(z)^{1/3}}{(1+a^3)^{1/3}}$$

$$\frac{dz}{z^{1/3}}=\frac{2\,du}{(1+a^3)^{1/3}}.$$

Integrating we get

$$\frac{3z^{2/3}}{2}=\frac{2u}{(1+a^3)^{1/3}}+c.$$

$$\therefore 3(1+a^3)^{1/3}z^{2/3}=4(x+ay)+b \quad (2)$$

This is the complete solution.

Example 8 :

Solve $x^2p^2+y^2q^2=z$.

Solution :

Take $X=\log x,\ Y=\log y$

$$\therefore dX=\frac{dx}{x},\ dY=\frac{dy}{y}$$

The given equation is

$$\left(x\frac{\partial z}{\partial x}\right)^2+\left(y\frac{\partial z}{\partial y}\right)^2=z. \quad (1)$$

This equation becomes $\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = z$

Assume $z = F(X + aY) = F(u)$ where $u = X + aY$

$$\frac{\partial z}{\partial X} = \frac{dz}{du}, \frac{\partial z}{\partial Y} = a\frac{dz}{du}.$$

Therefore from equation (2)

$$\left(\frac{\partial z}{\partial u}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z.$$

$$\frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$\frac{dz}{\sqrt{z}} = \pm\frac{1}{\sqrt{1+a^2}}\,du.$$

$$2\sqrt{z} = \pm\frac{u}{\sqrt{1+a^2}} + c.$$

$$2\sqrt{z} = \pm\frac{X+aY}{\sqrt{1+a^2}} + c.$$

$$2\sqrt{z} = \pm(\log x + a\log y)\sqrt{1+a^2} + c.$$

Example 9 :

Solve $z^2\left(p^2x^2 + q^2\right) = 1.$

Solution :

The equation is

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \quad (1)$$

Take $X = \log x$,

The given equation becomes

$$z^2\left[\left(\frac{dz}{dX}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] = 1. \quad (2)$$

This is of the type $F(z, p, q) = 0$.

Assume that

$z = F(X + ay) = F(u)$.

Then equation (2) becomes

$$z^2\left[\left(\frac{dz}{du}\right)+a^2\left(\frac{dz}{du}\right)^2\right]=1.$$

$$\sqrt{1+a^2}\,z\frac{dz}{du}=\pm 1.$$

$$\therefore \sqrt{1+a^2}\,zdz=\pm du.$$

Integrating $\sqrt{1+a^2}\,\frac{z^2}{2}=\pm u+c.$

Therefore the complete solution is

(i e) $\sqrt{1+a^2}\,\frac{z^2}{2}=\pm(\log x+ay)+c.$

Exercise 5

Solve the following equations

1. $p^2+pq=z^2.$
2. $ap+pq+cz=0.$
3. $zq=p(1+q)$
4. $z^2=p^2-q^2.$
5. $4(1+z^3)=9z^4pq$
6. $1+p^2=qz.$
7. $p^2+q^2=64z^2.$
8. $pq=z.$
9. $p^2+pq^2=4z.$
10. $pz=1+q^2.$
11. $z^2=1+p^2+q^2$
12. $q^2y^2=z(z-p).$

Section 7

Standard Type IV

Partial Differential Equation of the type

$f_1(x,p) = f_2(y,q)$

In this type of equation z is absent. Also the terms containing p and x can be separated from those containing q and y

Let $f_1(x,p) = f_2(y,q) = k$ (1)

(i e) $f_1(x,p) = k$. Solving for p we get, $p = F_1(x)$.

$f_2(y,q) = k$. Solving for q we get $q = F_2(y)$.

Also $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial x} dy$.

$= pdx + qdy$

$= F_1(x)\, dx + F_2(y)\, dy$.

Integrating $\int dz = \int F_1(x)\, dx + \int f_2(y)\, dy$.

$z = \int F_1(x)\, dx + \int F_2(y)\, dy + c$.

This is the complete solution.

Example 1 :

Solve $p^2 + q^2 = x + y$

Solution :

$p^2 + q^2 = x + y$.

$p^2 - x = q^2 - y = k$.

$\therefore p^2 - x = k$; $q^2 - y = k$.

$p = \pm\sqrt{x+k}$. $q = \pm\sqrt{y+k}$.

$dz = pdx + qdy$.

$= \pm(\sqrt{x+k})\, dx \pm (\sqrt{y+k})$

Integrating we get the complete solution is

$z = \pm\frac{2}{3}(x+k)^{3/2} \pm \frac{2}{3}(y+k)^{3/2} + c$.

$= \pm\frac{2}{3}\left[(x+k)^{3/2} + (y+k)^{3/2}\right] + c$.

Example 2 :

$$p^2 + q^2 = z^2 (x + y).$$

Solution :

Dividing by z^2,

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x + y.$$

Take $Z = \log z$

$$dZ = \frac{dz}{z}$$

$$\frac{\partial Z}{\partial x} = \frac{1}{z} \cdot \frac{dz}{dx}$$

The given equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y.$$

$$(i.e) \left(\frac{\partial Z}{\partial x}\right)^2 - x = -y + \left(\frac{\partial Z}{\partial y}\right)^2 = k.$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 - x = k ; y - \left(\frac{\partial Z}{\partial y}\right)^2 = k.$$

$$\frac{\partial Z}{\partial x} = \pm \sqrt{x + k} ; \frac{\partial Z}{\partial x} = \pm \sqrt{y + k}$$

$$\text{Therefore } dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$$

$$= \pm \sqrt{x + k}\, dx \pm \sqrt{y - k}\, dy.$$

$$\text{Integrating } Z = \pm \frac{2}{3} \left[(x + k)^{3/2} + (y - k)^{3/2}\right] + c.$$

$$\log z = \pm \frac{2}{3} \left[(x + k)^{3/2} + (y - k)^{3/2}\right] + c$$

Example 3 :

$$p^2 - q^2 = \frac{x^2 - y^2}{z}$$

Solution :

$$(\sqrt{z}\, p)^2 - (\sqrt{z}\, q)^2 = x^2 - y^2$$

Let $\sqrt{z}\, dz = dZ \therefore Z = \frac{2}{3} z^{3/2}$

The given equation becomes.

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 - y^2.$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 - x^2 = \left(\frac{\partial Z}{\partial y}\right)^2 - y^2 = k.$$

Then $\frac{\partial Z}{\partial x} = \sqrt{k + x^2}$ $\frac{\partial z}{\partial y} = \sqrt{k + y^2}$

$$\partial Z = \frac{\delta Z}{\partial x} dx + \frac{\delta Z}{\partial y} dy$$

$$= \sqrt{k + x^2}\, dx + \sqrt{k + y^2}\, dy.$$

Integrating we get

$$Z = \frac{x}{2}\sqrt{k + x^2} + \frac{k}{2}\sinh^{-1}\left(\frac{x}{\sqrt{k}}\right) + \frac{y}{2}\sqrt{k + y^2} + \frac{k}{z}\sinh^{-1}\left(\frac{y}{\sqrt{k}}\right) +$$

$$\frac{2}{3}z^{3/2} = \frac{x}{2}\sqrt{k + x^2} + \frac{y}{2}\sqrt{k + y^2} + \frac{k}{z}\left[\sinh^{-1}\left(\frac{x}{\sqrt{k}}\right) + \sinh^{-1}\left(\frac{y}{\sqrt{k}}\right)\right] +$$

This is the required complete solution.

Example 4 :

$p + q = \sin x + \sin y.$

Solution :

$p - \sin x = \sin y - q = k.$

$\therefore p = k + \sin x \;;\quad q = \sin y - k.$

$dz = pdx + qdy$

$= (k + \sin x)dx + (\sin y - k)dy.$

Integrating we get

$z = (kx - \cos x) - (ky + \cos y) + c.$

$z = k(x - y) - (\cos x + \cos y) + c.$

This is the complete solution.

Example 5 : $z^2(p^2+q^2)=x^2+y^2$

Solution :

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial q}\right)^2=x^2+y^2$$

Take $Z=z^2$

$$\frac{\partial Z}{\partial x}=2z\frac{\partial z}{\partial x} \qquad \frac{\partial Z}{\partial y}=2z\frac{\partial z}{\partial y}$$

$$\frac{1}{2}\left(\frac{\partial Z}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial Z}{\partial y}\right)^2=x^2+y^2$$

$$\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial y}\right)^2=4(x^2+y^2)$$

$$\left(\frac{\partial Z}{\partial x}\right)^2-4x^2=4y^2-\left(\frac{\partial Z}{\partial y}\right)^2=k$$

$$\frac{\partial Z}{\partial x}=\sqrt{4x^2+k} \qquad \frac{\partial Z}{\partial y}=\sqrt{4y^2-k}$$

$$\partial Z=\frac{\partial Z}{\partial x}\partial x+\frac{\partial Z}{\partial y}\partial y$$

$$=\sqrt{4x^2+k}\;dx+\sqrt{4y^2-k}\;dy$$

$$=2\sqrt{x^2+\frac{k}{4}}\;dx+2\sqrt{y^2-\frac{k}{4}}\;dy$$

$$Z=2\left[\frac{x}{2}\sqrt{x^2+\frac{k}{4}}+\frac{k}{8}\log\left(x+\sqrt{x^2+\frac{k}{4}}\right)\right]$$

$$+2\left[\frac{y}{2}\sqrt{y^2-\frac{k}{4}}+\frac{k}{8}\log\left(y+\sqrt{y^2-\frac{k}{4}}\right)\right]$$

$$=x\sqrt{x^2+\frac{k}{4}}+\frac{k}{4}\log\left(x+\sqrt{x^2+\frac{k}{4}}\right)$$

$$z^2=+y\sqrt{y^2-\frac{k}{4}}+\frac{k}{4}\log\left(y+\sqrt{y^2-\frac{k}{4}}\right)+c$$

Example 6 :

Solve $q = -px + p^2$

Solution :

$$p^2 - px = q = k.$$

$$p^2 - px = k \therefore p^2 - px - k = 0.$$

$$p = \frac{x \pm \sqrt{x^2 + 4k}}{2}$$

$$dz = pdx + qdy = \frac{x \pm \sqrt{x^2 + 4k}}{2} dx + kdy.$$

Integrating, $z = \frac{x^2}{4} \pm \frac{x}{4} \sqrt{x^2 + 4k} + 2k \sinh^{-1}\left(\frac{x}{2\sqrt{k}}\right) + ky$

Exercise 6

1. $p - q = x^2 + y^2$
2. $q = -px + p^2$
3. $yp - x^2 q^2 = x^2 y$
4. $p^2 + q^2 = x^2 + y^2$
5. $p^2 y(1 + x^2) = qx^2.$
6. $pq = xy$
7. $\sqrt{p} - \sqrt{q} + x = 0.$
8. $p^2 - x = q^2 - y.$
9. $(1 - y^2) xq^2 + y^2 p = 0.$
10. $\frac{x}{p} + \frac{y}{q} = 1$
11. $px + qy = y.$
12. $\left(\frac{p}{2} + x\right)^2 + \left(\frac{q}{2} + y\right)^2 = 1$
13. $q = xyp^2$
14. $xp - y^2 q^2 = 1.$
15. $p^2 y (1 + x^2) = qx^2.$
16. $p + q = px + qy.$

10. $z = pq$.

Lagrange's Linear Partial Differential Equations

The partial differential equation $Pp + Qq = R$ where P, Q, R are [f]unctions of x, y, z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ is called Langrange's linear [d]ifferential equation.

We shall now consider the method of solving such an equation.

It has already been shown that a partial differential equation of this type is abtained by eliminating the arbitrary function from the equation $F(x, y, z) = 0$.

In other words $F(u, v) = 0$ is the solution of the partial differental equation $Pp + Qq = R$. We are now required to find the values of u and v and have the solution $F(u, v) = 0$.

Let us assume that $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, be two solutions of the given linear partial differential equation. Taking the differentials of the above two solutions,

$$du = \frac{\partial u}{\partial x} du + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0. \qquad (1).$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0. \qquad (2).$$

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\delta v}{\delta z} - \frac{\partial u}{\partial z}\frac{\delta v}{\delta y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}$$

$$= \frac{dz}{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}}$$

$$\text{(i.e)} \ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (3).$$

It is now easily seen that $u = c_1$ and

$v = c_2$ are solutions of the equation (3). Then,

$\phi(u, v) = 0$. (i,e) $u = \phi(v)$ is the solution of the linear partial differential equation $Pp + Qq = R$. Hence

Procedore For Solving $Pp + Qq = R$.

Step 1 : Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step 2 : Find two independent solutions of the auxiliary equations say $u = c_1$ & $v = c_2$

Step 3 : Then the solution of the given partial differential equation is $\phi(u, v) = 0$, *or* $u = \phi(v)$.

Example 1 :

Solve $x^2 p + y^2 q = z^2$

Solution:

This is a Lagrange's linear equation

The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Consider $$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating $$\frac{1}{x} - \frac{1}{y} = c.$$

Consider $$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating $$\frac{1}{y} - \frac{1}{z} = c.$$

∴ **The solution is** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0.$

Example 2 :

Solve $(y + z)p + (z + x)q = x + y.$

Solution :

This is a Lagrange's linear equation.

The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

(ie) $$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} = \frac{dx + dy + dz}{2(x + y + z)}.$$

Considering first two members and integrating we get

$$\frac{x - y}{y - z} = c$$

Considering first and last members and integrating we get

$(x - y)^2 (x + y + z) = c$

∴ The solution is

$$\phi\left[\frac{x - y}{y - z}, (x - y)^2 (x + y + z)\right] = 0$$

Example 3

Solve $y^2 p + x^2 q = x^2 y^2 z^2$

Solution :

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$$

Considering 1st and 2nd members,

$$x^2\,dx = y^2\,dy.$$

Integrating we get $\frac{x^3}{3} = \frac{y^3}{3} + c$

$$x^3 - y^3 = c$$

Considering 2nd and 3rd members,

$$y^2\,dy = \frac{dz}{z^2}$$

Integrating $\frac{y^3}{3} = \frac{-1}{z} + c$

$$\Rightarrow \frac{y^3}{3} + \frac{1}{z} = c$$

$\therefore$ The solution is $\phi\left(x^3 - y^3, \frac{y^3}{3} + \frac{1}{z}\right) = 0.$

Example 4

Solve $y^2 zp + x^2 zq = y^2 x.$

Solution :

The auxiliary equations are

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}.$$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} \Rightarrow x^2 dx = y^2 dy.$$

Integrating $x^2\,dx - y^2\,dy = 0$ we get

$$x^3 - y^3 = c$$

Consider 1st and 3rd members,

$$\frac{dx}{y^2z} = \frac{dz}{y^2x} \Rightarrow xdx = zdz$$

Integrating $xdx - zdz = 0$ we get

$$x^2 - z^2 = c$$

$\therefore$ **The solution is** $\phi\left(x^3 - y^3, x^2 - y^2\right) = 0.$

Example 5

Solve $(mz - ny)p - (nx - lz)q = ly - mx.$

Solution :

The auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using multipliers x, y, z we get each ratio

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{xdx + ydy + zdx}{0}$$

$\therefore x^2 + y^2 + z^2 = c$ is one solution.

Also using multipliers $l, m, n,$ we get.

$$\text{each ratio} = \frac{ldx + mdy + ndy}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{ldx + mdy + ndz}{0}$$

$\therefore lx + my + nz = c$ is the 2nd solution.

$\therefore$ The general solution is $\phi\,(x^2 + y^2 + z^2, lx + my + nz) = 0.$

Example 6

Solve $(y - z)p + (z - x)q = x - y.$

Solution :

The auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx + dy + dz}{0.}$$

$$\therefore dx + dy + dz = 0$$

Integrating we get $x + y + z = c$

Also each ratio $= \dfrac{xdx + ydy + zdz}{x(y-z) + y(z-x) + z(x-y)}$

$$= \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0.$$

Integrating we get $x^2 + y^2 + z^2 = c.$

$\therefore$ The general solution is $\phi\,(x + y + z,\, x^2 + y^2 + z^2) = 0.$

Example 7

Solve $(3z - 4y)p + (4x - 2z)\,q = 2y - 3x.$

Solution :

The auxiliary equations are

$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

Multiplying by x, y, z, respectively and adding each ratio is equal to

$$\frac{xdx + ydy + zdz}{x(3z - 4y) + y\,(4x - 2z) + z\,(2y - 3x)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0.$$

Integrating we get $x^2 + y^2 + z^2 = c$

using mulitipliers 2, 3 and 4 and adding, each ratio

$$= \frac{2dx + 3dy + 4dz}{2\,(3z - 4y) + 3\,(4x - 2z) + 4\,(2y - 3x)}$$

$$= \frac{2dx + 3dy + 4dz}{0}$$

$$\Rightarrow 2dx + 3dy + 4dz = 0.$$

Integrating $\quad = 2dx + 3dy + 4dz = c\,.$

$\therefore$ **The general solution is** $\phi\,(x^2 + y^2 + z^2,\, 2x + 3y + 4z) = 0.$

Example 8

Solve $dx + dy = (1 + 2xy + 3x^2y^2)\,(x + y)\,z.$

Solution :

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{(1 + 2xy + 3x^2y^2)\,(x + y)\,z}$$

$$\frac{dx}{1} = \frac{dy}{1} \Rightarrow x - y = c$$

Also $\dfrac{ydx + xdy}{(x+y)} = \dfrac{dz}{(1 + 2xy + 3x^2y^2)(x+y)z.}$

$$\therefore \frac{d(xy)}{x+y} = \frac{dz}{(1 + 2xy + 3x^2y^2)(x+y)z.}$$

$$(ie)\ (1 + 2xy + 3x^2y^2)(dxy) = \frac{dz}{z}$$

Integrating we get $xy + x^2y^2 + x^3y^3 = \log z + c$

$\therefore$ The general solution is $\phi\,(x - y, xy + x^2y^2 + x^3y^2 - \log z) = 0.$

Example 9

Solve $x\,(2y^4 - z^{4)}p + y\,(z^4 - 2x^4)\,q = z\,(x^4 - y^4).$

Solution :

The auxiliary equations are

$$\frac{dx}{x\,(2y^4 - z^4)} = \frac{dy}{(z^4 - 2x^4)y} = \frac{dz}{z\,(x^4 - y^4)}$$

Each ratio $= \dfrac{x^3\,dx + y^3\,dy + z^3\,dz}{x^4\,(2y^4 - z^4) + y^4\,(z^4 - 2x^4) + z^4\,(x^4 - y^4)}$

$$\Rightarrow \frac{x^3\,dx + y^3\,dy + z^3\,dz}{0}$$

Integrating we get $x^4 + y^4 + z^4 = c$ (1)

Also each ratio $= \dfrac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{2dz}{z}}{2y^4 - z^4 + z^4 - 2x^4 + 2x^4 - 2y^4}$

$$= \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + 2\dfrac{d}{z}}{0}$$

$$= \frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z} = 0$$

Integrating we get $xyz^2 = c$

$\therefore$ The general solution is $\phi(x^4 + y^4 + z^4, xyz^2) = 0$

Example 10

Solve $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$.

Solution :

The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Thus each ratio is equal to

$$\frac{dx - dy}{(x + y + z)(x - y)} = \frac{dy - dz}{(x + y + z)(y - z)} = \frac{dz - dx}{(x + y + z)(z - x)}$$

$$\frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z} = \frac{d(z - x)}{z - x}$$

Integrating we get $\dfrac{x - y}{y - z} = c$ ... (1)

Also each ratio $= \dfrac{xdx + ydy + zdz}{(x^3 + y^3 + z^3) - 3xyz}$

Also each ratio $= \dfrac{dx + dy + dz}{x^2 + y^2 + y^2 - xy - yz - zx}$

$$\therefore \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$

(i e) $\dfrac{xdx + ydy + zdz}{x + y + z} = d(x + y + z)$.

$$xdx + ydy + zdz = (x + y + z)\,d(x + y + z).$$

Integrating we get $x^2 + y^2 + z^2 = (x + y + z)^2 + c$

$\therefore$ The general Solution is $\phi\left(\dfrac{x - y}{y - z}, xy + yz + zx\right) = 0$.

Example 11

Solve $(x^2 - y^2 - z^2)\,p + 2xy\,q = 2xz$.

Solution :

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}.$$

Consider $\frac{dy}{2xy} = \frac{dz}{2xz}$ $\therefore \frac{dy}{y} = \frac{dz}{z}$

Integrating we get

$$\log y - \log z = c$$

$\therefore$ one solution is $\frac{y}{z} = c$

Also $$\frac{xdx + ydy + zdx}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2}$$

$$= \frac{xdx + ydy + zdx}{x(x^2 + y^2 + z^2)}$$

$$\therefore \frac{xdx + ydy + zdx}{x^2 + y^2 + z^2} = \frac{dy}{2y}$$

$$\frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2} = \frac{dy}{y}$$

Integrating we get

$$\log(x^2 + y^2 + z^2) = \log y + \log c$$

Solution is $x^2 + y^2 + z^2 = cy$ *or* $\frac{x^2 + y^2 + z^2}{y} = c$

$\therefore$ The general solution is

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{y}\right) = 0.$$

Example 12

Solve $bc(b - c)yzp + ca(c - a)xzq = ab(a - b)xy.$

Solution :

The subsidiary equations are

$$\frac{dx}{bc(b - c)yz} = \frac{dy}{ca(c - a)xz} = \frac{dz}{ab(a - b)xy}$$

Using multipliers ax, by, cz each ratio is equal to

$$\frac{ax\,dx + by\,dy + cz\,dz}{ax\,bc(b - c)yz + by\,ac(c - a)xz + cz(a - b)ab\,xy}$$

$$= \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

$\Rightarrow ax\,dx + by\,dy + cz\,dz = 0.$

Integrating we get

$$ax^2 + by^2 + cz^2 = c. \quad \dots (1)$$

Using multipliers a^2x, b^2y, c^2z each ratio is equal to

$$\frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{a^2x\,bc\,(b-c)\,yz + b^2y\,ac\,(c-a)\,xz + ab\,c^2z\,(a-b)\,xy}$$

$$= \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0}.$$

(ie) $a^2x\,dx + b^2y\,dy + c^2z\,dz = 0.$

Integrating, we get $a^2x^2 + b^2y^2 + c^2z^2 = c$

Thus, the general solution is

$$\phi(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0.$$

Example 13

Solve $(y^3x - 2x^4)\,dx + (2y^4 - x^3y) = 9z\,(x^3 - y^3)$

Solution :

The subsidiary equations are.

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z\,(x^3 - y^3)}$$

(i e)
$$\frac{\frac{dx}{x}}{y^3 - 2x^3} = \frac{\frac{dy}{y}}{2y^3 - x^3} = \frac{\frac{dz}{3z}}{3\,(x^3 - y^3)}$$

$$\Rightarrow \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z}}{y^3 - 2x^3 + 2y^3 - x^3 + 3x^3 - 3y^3} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z}}{0}$$

$$= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z} = 0. \quad \dots (1$$

Integrating we get $x^3y^3z = c$

Also each ratio $= \dfrac{x^2\,dx + y^2\,dy}{x^3\,(y^3 - 2x^3) + y^3\,(2y^3 - x^3)}$

$$= \frac{x^2\,dx + y^2\,dy}{2\,(y^6 - x^6)}$$

$$\therefore \frac{dz}{9z(x^3 - y^3)} = \frac{x^2 dx + y^2 dy}{2(y^6 - x^6)}$$

$$(i.e) \frac{dz}{3z} = \frac{-d(x^3 + y^3)}{2(x^3 + y^3)}$$

Integrating we get

$$\log(x^3 + y^3) = \frac{-2}{3}\log z + \log c$$

$$(ie) \quad (x^3 + y^3)^3 z^2 = c$$

$\therefore$ The general solution is $\phi(x^3 y^3 z, (x^2 + y^3)^3 z^2) = 0$

Example 14

Sove $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$.

Solution :

The auxiliary equations are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$$

$$(ie) \frac{\frac{dx}{x}}{y^2 + z} = \frac{\frac{dy}{y}}{-(x^2 + z)} = \frac{\frac{dz}{z}}{x^2 - y^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Integrating we get $xyz = c$.

Also multiplying $x, y, -1$, each ratio is equal to

$$\frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = 0.$$

$$\Rightarrow \frac{xdx + ydy - dz}{0} = 0$$

$$\therefore xdx + ydy - dz = 0.$$

Integrating we get $x^2 + y^2 - 2z = c$.

$\therefore$ The general solution is $\phi\,(xyz, x^2 + y^2 - 2z) = 0$.

Example 15

Solve $(x + y)zp + (x - y)\,zq = x^2 + y^2$.

Solution :

The subsidiary equations are.

$$\frac{dx}{(x+y)z} = \frac{dy}{(x-y)z} = \frac{dz}{x^2 + y^2}$$

Using multipliers x, $-y$, $-z$ each ratio is equal to

$$\frac{xdx - ydy - zdz}{0} = 0.$$

$$\therefore\ xdx - ydy - zdz = 0.$$

Integrating we get $x^2 + y^2 - z^2 = c$

Also each ratio is equal to

$$\frac{ydx + xdy}{yz\,(x+y) + xz\,(x-y)} = \frac{ydx + xdy}{z\,(x^2 + y^2)}.$$

Taking this along with the 3rd member we have

$$\frac{ydx + xdy}{z\,(x^2 + y^2)} = \frac{dz}{x^2 + y^2} \Rightarrow d\,(xy). = zdz.$$

Integrating we get $2xy - z^2 = c$.

$\therefore$ The general solution is $\phi\,(x^2 - y^2 - z^2, 2xy - z^2) = 0$.

Example 16

Solve $(x^2 + y^2)p + 2xyq = (x + y)^2\, z^2$

Solution :

The auxiliary equations are

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)^2 z^2}$$

From first two members each ratio is equal to

$$\frac{dx + dy}{(x+y)^2} = \frac{dx - dy}{(x-y)^2}.$$

(i.e) $\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2}$.

Integrating we get

$$\frac{-1}{x+y} = \frac{-1}{x-y} + c$$

(i.e) $\frac{1}{x-y} - \frac{1}{x+y} = c$

$$\Rightarrow \frac{2y}{x^2-y^2} = c$$

Also $\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)^2 z^2}$

(ie) $d(x+y) = \frac{dz}{z^2}$

Integrating we get

$$x+y = -\frac{1}{z} + c.$$

$$x+y+\frac{1}{z} = c.$$

$\therefore$ The general solution is $\phi\left(\frac{2y}{x^2-y^2}, x+y+\frac{1}{z}\right) = 0.$

Example 17

$(1+y)p + (1+x)q = z.$

Solution :

The auxiliary equations are.

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z.}$$

(ie) $(1+x)\,dx = (1+y)\,dy.$

Integrating we get

$$(x+1)^2 - (y+1)^2 = c$$

$$(x-y)(x+y+2) = c \quad (1)$$

Also $\frac{dx-dy}{-(x-y)} = \frac{dz}{z.}$

Integrating we get $z(x-y) = c$

$\therefore$ The general solution is $\therefore \left[(x-y)(x+y+2), z(x-y)\right] = 0.$