Chapter 24

Linear Homogeneous Equation and Variation of Parameters

So far we have studied the methods of solving linear second order differential equations with constant co-efficients. Let us now consider methods of solving linear differential equation with variable co-efficients. Let us consider a linear differential equation of the type

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{(n-1)} \frac{dy}{dx} + a_n y = F(x).$$

where $a_1, a_2, a_3, \ldots\ldots a_n$ are constants. This equcation is called homogeneous linear equation. This is also called Cauchy - Euler equation. The substitution $x = e^z \left(or\, z = \log_e x\right)$ converts this differential equation with variable co.efficients into a differential equation with constant co-efficients.

For the substitution $x = e^z \left(or\, z = \log_e x\right)$ it follows,

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = x \frac{dy}{dx} \quad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

Hence $x \frac{dy}{dx} = \frac{dy}{dz}$ (ie) $x \frac{dy}{dx} = Dy$ ...(1) where $D = \frac{d}{dz}$.

$$\frac{d^2 y}{dz^2} = \frac{d}{dz}\left(x \frac{dy}{dx}\right)$$

$$= \frac{dx}{dz} \frac{dy}{dx} + x \frac{d}{dz}\left(\frac{dy}{dx}\right)$$

$$= \frac{dx}{dz} \cdot \frac{dy}{dx} + x \frac{d^2 y}{dx^2} \cdot \frac{dx}{dz}$$

$$= x \frac{dy}{dx} + x^2 \frac{d^2 y}{dx^2}$$

$$= \frac{dy}{dz} + x^2 \frac{d^2 y}{dx^2}$$

[using (1)]

$$\therefore x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$$=(D^2-D)y \quad ...(2) \text{ or } D(D-1)y \text{ where } D=\frac{d}{dz}$$

Similarly we can show that

$$x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y \quad ...(3)$$

Substituting (1), (2), (3). etc in the given differential equation (I), we get a differential equcation with constant co-efficients. Earlier we have seen the methods of solving a linear differential equation with constant co-efficients. Therefore we can solve the converted linear equation with constant co-efficients and hence the solution for (I) can be obtained in terms of x.

Lagrange's Linear Equation

An equation of the type

$$a_0(ax+b)^n\frac{d^ny}{dx^n}+a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+ \ldots\ldots a_ny=F(x).$$

where $a_0, a_1, a_2 \ldots a_n$ are constants is called Lagrange's linear equation.

put $ax+b=X$

Then $\frac{dX}{dx}=a$

Now $\frac{d}{dx}=\frac{d}{dX}\left(\frac{dX}{dx}\right)=a\frac{d}{dX}$

Also $\frac{d^2}{dx^2}=a^2\frac{d^2}{dX^2}$ etc.

$\therefore$ the Lagrange's equation becomes

$$a_0a^nX^n\frac{d^ny}{dX^n}+a_1a^{n-1}X^{n-1}\frac{d^{n-1}y}{dX^{n-1}}+\ldots+a_ny=F\left(\frac{x-b}{a}\right)$$

This being the homogeneous differential equation of Cauchy Euler type we can solve this equation.

Example 1

Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

Solution :

This is a homogeous linear equation of Cauchy - Euler type.

Let $x = e^z$

$\therefore z = \log_e x.$

Then, we know that

$$x\frac{dy}{dx} = Dy \text{ where } D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

The given differential equation becomes

$$D(D-1)y - 2Dy - 4y = e^{4z}.$$

$$(D^2 - D - 2D - 4)y = e^{4z}.$$

$$(D^2 - 3D - 4)y = e^{4z}$$

The auxiliary equation is

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$\therefore m = 4, -1.$$

The complemantary function is $y = Ae^{4z} + Be^{-z}$. The particular Integral is

$$P.I = \frac{1}{D^2 - 3D - 4}e^{4z}$$

$$= \frac{1}{(D-4)(D+1)}e^{4z}$$

$$= \frac{1}{5}\cdot\frac{1}{D-4}e^{4z}$$

$$= \frac{1}{5}ze^z$$

$\therefore$ The complete solution is

$$y = Ae^{4z} + Be^{-z} + \frac{1}{5}ze^{4z}$$

$$y = Ae^{4z} + \frac{B}{e^z} + \frac{1}{5}ze^{4z}$$

$$y = Ax^4 + \frac{B}{x} + \frac{1}{5}x^4\log x$$

Example 2

Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(\log x)$.

Solution :

This is a homogenous linear equation

$$\text{Let } x = e^z \text{ (i e) } z = \log_e x.$$

$$\text{Then } x\frac{dy}{dx} = Dy \text{ where } D = \frac{d}{dz}.$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y.$$

The given differential equation becomes

$$\left[D(D-1)y + 4Dy + 2y\right] = \sin z$$

$$\left[D^2y - Dy + 4Dy + 2y\right] = \sin z$$

$$\left[D^2 + 3D + 2\right]y = \sin z$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0 \therefore m = -1, -2.$$

The complementary function is

$$y = Ae^{-z} + Be^{-2z}$$

The Particular Integral is

$$PI = \frac{1}{D^2 + 3D + 2}\sin z$$

$$= \frac{1}{-1 + 3D + 2}\sin z \quad \text{put } D^2 = -1.$$

$$= \frac{1}{3D + 1}\sin z$$

$$= \frac{(3D - 1)}{9D^2 - 1}\sin z$$

$$= -\frac{(3D - 1)}{10}\sin z = -\frac{(3\cos z - \sin z)}{10}$$

The complete solution is $y = Ae^{-z} + Be^{-2z} - \dfrac{3\cos z - \sin z}{10}$

$$\therefore y = \frac{A}{x} + \frac{B}{x^2} - \frac{3\cos(\log x) - \sin(\log x)}{10}$$

Example 3

$$\text{Solve } x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} = x + 1$$

Solution :

$$\text{Let } x = e^z \therefore z = \log_e x$$

$$\text{Then } x\frac{dy}{dx} = Dy \text{ where } D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

The given differential equation becomes

$$[D(D-1) - 3D]\, y = e^z + 1$$

$$(D^2 - 4D)\, y = e^z + 1$$

The auxiliary equation is $m^2 - 4m = 0$

$m = 0, 4.$

The complementary function is

$y = A + Be^{4z}$

The Particular Integral is

$$P.I = \frac{1}{D^2 - 4D}e^z + \frac{1}{D^2 - 4D}1$$

$$= \frac{1}{-3}e^z + \frac{1}{D(D-4)}e^{0z}$$

$$= \frac{-1}{3}e^z - \frac{1}{4}\cdot\frac{1}{D}e^{0z}$$

$$= \frac{-1}{3}e^z - \frac{1}{4}z$$

$\therefore$ The complete solution is $y = A + Be^{4z} - \frac{1}{3}e^z - \frac{1}{4}z.$

$$y = A + Bx^4 - \frac{x}{3} - \frac{1}{4}\log x.$$

Example 4

$$\text{Solve } x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2\log x. \qquad \ldots(1)$$

Solution :

$$\text{Let } \frac{dy}{dx} = V. \text{ Then } \frac{d^2y}{dx^2} = \frac{dV}{dx} \text{ and } \frac{d^3y}{dx^3} = \frac{d^2V}{dx^2}.$$

$\therefore$ the given DE becomes

$$x^2\frac{d^2V}{dx^2}+3x\frac{dV}{dx}+V=x^2\log x. \quad ...(2)$$

Let $x=e^z \therefore z=\log_e x$

Then equation (2) becomes

$$\left[D(D-1)+3D+1\right]y=ze^{2z}$$

(ie) $\left(D^2+2D+1\right)y=z\,e^{2z}$

The auxiliary equation is $m^2+2m+1=0$

$\therefore m=-1,-1.$

The complementary function is

$$V=e^{-z}(Az+B)$$

The particular Integral is

$$PI=\frac{1}{D^2+2D+1}ze^{2z}$$

$$=\frac{1}{(D+1)^2}ze^{2z}$$

$$=e^{2z}\frac{1}{(D+3)^2}z.$$

$$=e^{2z}\frac{1}{9}\left(1+\frac{D}{3}\right)^{-2}z$$

$$=e^{2z}\frac{1}{9}\left[\left(1-\frac{2D}{3}\right)\right]z$$

$$=\frac{e^{2z}}{9}\left[z-\frac{2}{3}\right]$$

$\therefore$ the solution is

$$V=e^{z}(Az+B)+\frac{1}{9}e^{2z}\left(z-\frac{2}{3}\right)$$

$$(i\,e)\ \frac{dy}{dx}=e^{z}(Az+B)+\frac{1}{9}e^{2z}\left(z-\frac{2}{3}\right)$$

$$=x(A\log x+B)+\frac{x^2}{9}\left(\log x-\frac{2}{3}\right)$$

Integrating, $y=A\int x\log x\,dx+B\int x\,dx+\int\frac{x^2}{9}\log x dx-\int\frac{2x^2}{27}dx$

$$y = A\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + \frac{Bx^2}{2} + \frac{x^3}{27}\log x - \frac{x^3}{81} - \frac{2x^3}{81} + c$$

$$\Rightarrow y = A\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + \frac{Bx^2}{2} + \frac{x^3}{27}\log x - \frac{x^3}{27} + c$$

Example 5

Sove $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Solution :

Let $x = e^z \therefore z = \log x$

The given equation becomes

$$\left[D(D-1) - 2D - 4\right]y = e^{2z} + 2z$$

$$\left[D^2 - 3D - 4\right]y = e^{2z} + 2z$$

The Auxiliary equation is

$$(m^2 - 3m - 4) = 0, (m-4)(m+1) = 0$$

$$\therefore m = 4, -1.$$

The complementary function is

$$y = Ae^{4z} + Be^{-z}$$

$$PI_1 = \frac{1}{D^2 - 3D - 4}e^{2z}$$

$$= \frac{1}{4-6-4}e^{2z} = \frac{-1}{6}e^{2z}$$

$$PI_2 = \frac{1}{D^2 - 3D - 4}2z$$

$$= \frac{-1}{4\left(1 - \frac{D^2 - 3D}{4}\right)}2z$$

$$= -\frac{1}{4}\left(1 - \frac{D^2 - 3D}{4}\right)^{-1}z$$

$$= -\frac{1}{2}\left[1 + \frac{D^2 - 3D}{4}\right]z$$

$$= -\frac{1}{2}\left[z - \frac{3}{4}\right]$$

$\therefore$ The complete solution is

$$y = Ae^{4z} + Be^{-z} - \frac{1}{6}e^{2z} - \frac{1}{2}\left(z - \frac{3}{4}\right)$$

$$y = Ax^4 + \frac{B}{x} - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$$

Example 6

$$\text{Solve } \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$$

Solution :

The given differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

$$\text{Let } x = e^z \text{ (ie) } z = \log_e x$$

The given equation becomes

$$[D(D-1) + D]\,y = 12z$$

$$D^2 y = 12z$$

The auxiliary equation is

$$m^2 = 0 \therefore m = 0, 0$$

$\therefore$ The complementary function is $y = Az + B$

$$PI = \frac{1}{D^2}12z$$

$$= \frac{1}{D}6z^2$$

$$= 2z^3$$

$\therefore$ The complete solution is

$$y = Az + B + 2z^3$$

$$y = A\log x + B + 2(\log x)^3$$

Example 7

Find the equation of the curve which satisfies the differential equatio

$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$ and touches x axis at an angle of $60°$ at $x = 1$

Solution :

$$\text{Let } x = e^z$$

The given differential equation becomes

$$\left[4D(D-1)-4D+1\right]y=0$$

$$(4D^2-8D+1)y=0.$$

The auxiliary equation is $4m^2-8m+1=0.$

$$\therefore m=\left(1\pm\frac{\sqrt{3}}{2}\right)$$

$\therefore$ the complementary function is

$$y=Ae^{\left(1+\frac{\sqrt{3}}{2}\right)z}+Be^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

$$y=Ax^{1+\frac{\sqrt{3}}{2}}+Bx^{1-\frac{\sqrt{3}}{2}}$$

Given $y=0$ when $x=1$. $\therefore A+B=0.$

$$\frac{dy}{dx}=A\left(1+\frac{\sqrt{3}}{2}\right)x^{\sqrt{3}/2}+B\left(1-\frac{\sqrt{3}}{2}\right)x^{-\sqrt{3}/2}$$

Given $\frac{dy}{dx}=\tan 60^\circ=\sqrt{3}$ at $x=1$

$$\therefore A\left(1+\frac{\sqrt{3}}{2}\right)+B\left(1-\frac{\sqrt{3}}{2}\right)=\sqrt{3}$$

$$\text{(ie) } A\frac{\sqrt{3}}{2}-B\frac{\sqrt{3}}{2}=\sqrt{3}$$

$$\therefore A-B=2$$

$$A+B=0$$

$$2A=2 \quad A=1,\ B=-1.$$

$$\therefore y=x^{\left(1+\frac{\sqrt{3}}{2}\right)}-x^{\left(1-\frac{\sqrt{3}}{2}\right)}$$

Example 8

The radial displacement u in a rotating disc at a distance r from the axis is given by

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+kr=0$$

find its displacement.

Solution :

The given equation is

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}-u=-kr^3$$

Let $r=e^z \quad \therefore z=\log r$

The given equation becomes

$$\left[D(D-1)+D-1\right]u=-ke^{3z}$$

$$\left(D^2-1\right)u=-ke^{3z}$$

The auxiliary equation is $m^2-1=0$ $\therefore m=\pm 1.$

The complementary function is $u=Ae^z+Be^{-z}$

$$PI=-\frac{1}{D^2-1}ke^{3z}$$

$$=-\frac{k}{8}e^{3z}$$

$\therefore$ the complete solution is

$$u=Ae^z+Be^{-z}-\frac{k}{8}e^{3z}$$

$$u=Ar+\frac{B}{r}-\frac{k}{8}r^3$$

Example 9

Solve $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+5y=x^2\sin(\log x)$

Solution :

$$x=e^z \quad \therefore z=\log x.$$

The given differential eqution becomes

$$\left[D(D-1)-3D+5\right]y=e^{2z}\sin z$$

$$(D^2-4D+5)y=e^{2z}\sin z$$

The auxiliary equation is $m^2-4m+5=0$

$$m=2\pm i$$

The complementary function is

$$y=e^{2z}\left(A\cos z+B\sin z\right)$$

$$P.I=\frac{1}{D^2-4D+5}e^{2z}\sin z$$

$$=e^{2z}\left[\frac{1}{(D+2)^2-4(D+2)+5}\right]\sin z$$

$$=e^{2z}\left[\frac{1}{D^2+1}\right]\sin z$$

$$= e^{2z}\, IP \text{ of } \frac{1}{(D+i)(D-i)} e^{iz}$$

$$= e^{2z}\, IP \text{ of } \frac{1}{2i} z\, e^{iz}$$

$$= e^{2z}\, IP \left[\frac{-i}{2} z\,(\cos z + i \sin z)\right]$$

$$= -\frac{e^{2z} z \cos z}{2}$$

The solution is $y = e^{2z}\left[A \cos z + B \sin z\right] - \dfrac{e^{2z} z \cos z}{2}$

$$y = x^2\left[A \cos(\log x) + B \sin(\log x)\right] - \frac{x^2}{2} \log x \cos(\log x)$$

Example 10

Solve $\left(x^2 D^2 + 3x D + 1\right) y = \dfrac{1}{(1-x)^2}$

Solution

Let $x = e^z \therefore z = \log_e x$

The given equation becomes

$$\left[D(D-1) + 3D + 1\right] y = \left(1 - e^z\right)^{-2}$$

$$\left(D^2 + 2D + 1\right) y = \left(1 - e^z\right)^{-2}$$

The auxiliary equation is $m^2 + 2m + 1 = 0$

$\therefore m = -1, -1.$

The complementary function is $y = e^{-z}(Az + B)$

$$PI = \frac{1}{(D+1)^2} \frac{1}{(1-e^z)^2}$$

$$= \frac{1}{D+1}\left[e^{-z} \int \frac{e^z}{(1-e^z)^2} dz\right]$$

$$= \frac{1}{D+1}\left[-e^{-z} \int (1-e^z)^{-2}(-e^z) dz\right]$$

$$= \frac{1}{D+1}\left[-e^{-z} \int (1-e^z)^{-2} d(-e^z)\right]$$

$$= \frac{1}{D+1}\left[-e^{-z}\frac{(1-e^{z})^{-1}}{-1}\right]$$

$$= \frac{1}{D+1}\frac{1}{e^{z}(1-e^{z})}$$

$$= e^{-z}\int e^{z}\frac{1}{e^{z}(1-e^{z})}\,dz$$

$$= e^{-z}\int\left(1+\frac{e^{z}}{1-e^{z}}\right)dz$$

$$= e^{-z}\left(z-\log(1-e^{z})\right)$$

$\therefore$ The solution is $y = e^{-z}(Az+B) + e^{-z}\left[z-\log(1-e^{z})\right]$

$$(ie)\; y = \frac{1}{x}\left(A\log x + B\right) + \frac{1}{x}\left[\log x - \log(1-x)\right]$$

$$y = \frac{A\log x + B}{x} + \frac{1}{x}\log\left(\frac{x}{1-x}\right)$$

Example 11

Solve $(1+x)^{2}\frac{d^{2}y}{dx^{2}} + (1+x)\frac{dy}{dx} + y = 2\sin\left[\log(1+x)\right]$

Solution

put $1+x = X$

The given equation becomes

$$X^{2}\frac{d^{2}y}{dX^{2}} + X\frac{dy}{dX} + y = 2\sin(\log X)$$

Let $X = e^{z}$ $\therefore z = \log_e X$

$\therefore$ The equation (1) becomes

$$\left[D(D-1)+D+1\right]y = 2\sin(z)$$

$$\left(D^{2}+1\right)y = 2\sin z$$

The auxiliary equation is $m^{2}+1 = 0$ $\therefore m = \pm i$

The complementary function is $y = A\cos z + B\sin z$

$$PI = \frac{1}{D^{2}+1}2\sin z$$

$$= IP\,\frac{1}{D^{2}+1}2e^{iz}$$

$$= I P \frac{1}{(D+i)(D-i)} 2e^{iz}$$

$$= I P \frac{1}{2i} z 2e^{iz}$$

$$= -z \cos z$$

$\therefore$ The complete solution is

$$y = A \cos z + B \sin z - z \cos z$$

$$y = A \cos(\log X) + B \sin(\log X) - \log X . \cos(\log X)$$

$$y = A \cos \log(1+x) + B \sin \log(1+x) - \log(1+x) . \cos(\log(1+$$

Example 12

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y$$

Solution :

$$= \log(1+x)^4 + \cos\left[\log(1+x\right.$$

$$1 + x = X$$

$$X^2 \frac{d^2y}{dX^2} + X\frac{dy}{dX} + y = 4 \log X + \cos \log X$$

$$z = \log X, \quad D = \frac{d}{dz}$$

$$[D(D-1) + D + 1]\, y = 4z + \cos z$$

$$\left(D^2 + 1\right) y = 4z + \cos z$$

$$\text{CF is } y = c_1 \cos z + c_2 \sin z$$

$$\text{PI} = 4 \frac{1}{1+D^2} z + \frac{1}{D^2+1} \cos z$$

$$= 4z + \frac{z \sin z}{2}$$

$\therefore$ The complete solution is

$$y = c_1 \cos \log(1+x) + c2 \sin \log(1+x) +$$

$$4 \log(1+x) + \frac{1}{2} \log(1+x)\left[\sin \log(1+x)\right]$$

Exercise 1

Solve the following Differential Equations :

1. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x.$

2. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = ax.$

3. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = x^2.$

4. $3x^2 \frac{d^2y}{dx^2} + 11x \frac{dy}{dx} - 3y = 8 - 3 \log x$

5. $x^2 y_2 + 6xy_1 + 4y = 0$

6. $x y_2 + 3y_1 - \frac{3}{x} y = x^2$

7. $x^2 y_2 - y = (\log x)^2 - 1$

8. $x^2 y_2 + 3xy_1 + 5y = x^2$

9. $x^2 y_2 + 3xy_1 + 4y = \cos(4 \log x)$

10. $x^2 y_2 + 7xy + 5y = 0$

11. $x^2 y_2 + 3xy_1 + y = x \log x$

12. $x^2 y_2 + 9xy_1 + y = x$

13. $x^2 y_2 - 3xy_1 + 5y = x^2 \log x$

14. $x^2 y_2 + 7xy_1 + 5y = x^5$

15. $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy + \sin(\log x)$

16. $r \frac{d^2y}{dr^2} + \frac{dy}{dr} - \frac{y}{r} = - ar^2$

17. $u = r \left[\frac{d}{dr}\left(r \frac{du}{dr}\right)\right] + ar^3$

18. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 4y = x^4$

19. $x^2 y_2 + y = 3x^2$

20. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 5y = x \cos(\log x) + 3$

21. $(x^2 D^2 + 4xD + 2) y = x^2 + \sin(\log x)$

22. $x^2 y_2 - xy_1 + y = 2 \log x$

23. $4x^2 y_2 + 4x y_1 - y = 4x^2$

24. $x^2 \frac{d^2y}{dx^2} - 7x \frac{dy}{dx} + 11y = x^2$

25. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} = (x+3)(2x+4)$

26. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 2 sin \left(\log(1+x) \right)$

27. $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

28. $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1) \frac{dy}{dx} - 12y = 6x$

29. $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}$

30. $x^2 y_2 - 4xy_1 + 6y = \frac{42}{x^4}$

31. $x^2 y_2 + 4xy_1 + 2y = x \log x$

32. $x^2 y_5 - 4x y_4 + 6y_3 = 4$

33. $(1+2x)^2 \frac{d^2y}{dx^2} + (1+2x) \frac{dy}{dx} + y = 8(1+2x)^2$

Variation of Parameter

Earlier we have seen the method of solving a linear differential equation with constant co-efficients when the non-homogeneous term is of a special type. Here we present a more general method called variation of parameters which can be used to determine particular integral. This method applies even when the co-efficients of the differential equation are functions of x provided we know a fundamental solution set for the corresponding homogeneous equation. Consider the non-homogereous linear second order equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = h(x) \qquad ...(1)$$

where the co-efficient of $\frac{d^2y}{dx^2}$ is taken as unity. Let $y_1(x)$ and $y_2(x)$ be a fundamental solution set for the corresponding homogereous equation.

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0 \qquad ...(2)$$

Then the general solution of the homogeneous equation (2) is

$$y = c_1 y_1(x) + c_2 y_2(x) \qquad ...(3)$$

where c_1 and c_2 are arbitary constants. To find the particular solution to the non-homogerous equation, the idea behind variation of parameters is to replace the constants in (3) by the functions of x.

That is , we seek a solution of (1) of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \qquad ...(4)$$

Since we have introduced two unknown functions $v_1(x)$ and $v_2(x)$ it is reasonable to expect that we will need two equations involving these functions in order to determine them. Naturally one of these two equations should come from (1). Let us therefore substitute $y_p(x)$ in (4) into (1) - for this we must compute

$$y_p'(x) \text{ and } y_p''(x)$$

from (4) we obtain

$$y'_p = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2') \qquad ...(5)$$

To simplity the computation and to avoid second order derivatives for the unknowns v_1, v_2 in the expression for y_p'' we take

$$v_1' y_1 + v_2' y_2 = 0 \qquad ...(6)$$

Then (5) becomes

$$y_p' = v_1 y_1' + v_2 y_2' \qquad ...(7)$$

Now $y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$ ...(8)

Substituting y_p, y_p', y_p'' as given in (4), (5), (6) into (1) we find

$(v_1{}'y_1{}' + v_1 y_1{}'' v_2{}'y_2{}' + v_2 y_2{}'') + p(v_1 y_1{}' + v_2 y_2{}') +$
$q(v_1 y_1 + v_2 y_2) = h$

$\Rightarrow (v_1{}'y_1{}' + v_2{}'y_2{}') + v_1(y_1{}'' + py_1{}' + qy_1) +$
$v_2(y_2{}'' + py_2{}' + qy_2) = h$

$v_1{}'y_1{}' + v_2{}'y_2{}' = h$ (10) since y_1 and y_2 are the solutions of homogeneous equation. We can solve for $v_1{}'$ and $v_2{}'$ from (6) and

$$y_1 v_1{}' + y_2 v_2{}' = 0.$$
$$y_1{}'v_1{}' + y_2{}'v_2{}' = h.$$

From $v_1{}'$ and $v_2{}'$ integrating we get v_1 and v_2 as functions of x.

∴ The particular integral is given by

$$y_p(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x).$$

The complete solution is

$$y = c_1 y_1(x) + c_2 y_2(x) + v_1(x)\, y_1(x) + v_2(x)\, y_2(x)$$

Let us illustrate this method in the following examples.

Example 1

Solve $\quad \dfrac{d^2y}{dx^2} + y = \tan x.$

Solution :

$$(D^2 + 1)\, y = \tan x.$$

The auxiliary equation is $m^2 + 1 = 0 \therefore m = \pm i$

∴ The complementary function is $y = A\cos x + b\sin x$ (1)

Let the particular integral be

$y = A\cos x + B\sin x$ (2) where A and B are functions of x

Differentiating (2) with respect to x,

$$y = -A\sin x + A'\cos x + B\cos x + B'\sin x$$
$$= (-A\sin x + B\cos x) + A'\cos x + B'\sin x$$

Let us Assume that

$A'\cos x + B'\sin x = 0$

The from (3) we have $y' = -A\sin x + B\cos x$

Further differentiating with respect to x, we get

$y'' = -A\cos x - A'\sin x + B'\cos$ ...

Substituting (6) and (1) in the given differential equation we get

$$-(A\cos x + B\sin x) - A'\sin x + B'\cos x + A\cos x + B\sin x = \tan x$$

$$(ie)\ -A'\sin x + B'\cos x = \tan x \qquad (7)$$

Let us now solve for A' and B'

$$A'\cos x + B'\sin x = 0 \qquad ...(4)$$

$$-A'\sin x + B'\cos x = \tan x \qquad ...(7)$$

(4) $\cos x$ − (7) $\sin x$ gives $A' = -\tan x\sin x$

$$\therefore B' = \frac{\tan x\sin x\cos x}{\sin x} = \sin x.$$

$$A = -\int \tan x\sin x\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx$$

$$= \int \frac{\cos^2 x - 1}{\cos x}\, dx$$

$$= \sin x - \log(\sec x + \tan x)$$

$$B = \int \sin x\, dx = -\cos x$$

$$\therefore PI = \left[\sin x - \log(\sec x + \tan x)\right]\cos x - \sin x\cos x$$

$$= -\cos x\log(\sec x + \tan x)$$

∴ The complete solution is

$$y = c_1\cos x + c_2\sin x - \cos x\log(\sec x + \tan x)$$

where c_1, c_2 are arbitrary constants.

Example 2

Solve $\quad \dfrac{d^2y}{dx^2} + a^2y = \sec ax$

Solution :

$$(D^2 + a^2)y = \sec ax$$

The auxiliary equation is $m^2 + a^2 = 0$

$$m = \pm ai$$

The complementary function is $y = A\cos ax + B\sin ax$ ...(1)

Let the particular integral be

$y = A\cos ax + B\sin ax$ where A and B are arbitrary functions of x.

$$y' = A'\cos ax + B'\sin ax - aA\sin ax + aB\cos ax$$

Now take $A' \cos ax + B' \sin ax = 0$

Then $y' = -aA \sin ax + aB \cos ax$

$$y'' = -aA' \sin ax + aB' \cos ax - a^2 A \cos ax - a^2 B$$

$$= -aA' \sin ax + aB' \cos ax - a^2 y.$$

$$\therefore y'' + a^2 y = a(-A' \sin ax + B' \cos ax).$$

$$(ie).\ -A' \sin ax + B' \cos ax = \frac{1}{a} sec\, ax$$

$$A' \cos ax + B' \sin ax = 0$$

(4) sin ax – (5) cos ax gives $-A' = \frac{1}{a} sec\, ax \sin ax$

$$= \frac{1}{a} \tan ax$$

$$\therefore A' = -\frac{1}{a} \tan x \text{ and } B' = \frac{1}{a} \frac{\tan ax \cos ax}{\sin ax} = \frac{1}{a}$$

$$A \therefore = -\int \frac{1}{a} \tan ax\, dx \text{ and } B = \int \frac{1}{a} dx$$

$$= \frac{1}{a^2} \log \cos ax;\ B = \frac{x}{a}$$

∴ The Particular Integral is

$$PI = (\frac{1}{a^2} \log \cos ax) \cos ax + \frac{x}{a} \sin ax$$

∴ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \cos ax \log \cos ax + \frac{1}{a} x \sin ax$$

where c_1 and c_2 are arbitrary constants.

Example 3

solve $(D^2 + 1)y = x \sin x$

Solution :

The auxiliary equation is $m^2 + 1 = 0$

$$\therefore m = \pm i$$

The complementary function is

$y = A \cos x + B \sin x$ (1) where A, B are arbitrary constant.

Let the particular integral be

$y = A \cos x + B \sin x$ where A, B, are arbitrary functions of x.

$$y' = A' \cos x + B' \sin x - A \sin x + B \cos x$$

Choose $A' \cos x + B' \sin x = 0$ ...(

Then $y' = -A \sin x + B \cos x$

$y'' = -A' \sin x + B' \cos x - A \cos x - B \sin x$

$= -A' \sin x + B' \cos x - y$

$\therefore y'' + y = -A' \sin x + B' \cos x$

$\therefore \quad A' \cos x + B' \sin x = 0$ ...(3

$-A' \sin x + B' \cos x = x \sin x$ ....(4)

(4) $\sin x$ – (3) $\cos x$ gives $-A' = x \sin^2 x$

From (3) $B' = \dfrac{x \sin^2 x \cos x}{\sin x} = x \sin x \cos x$

$$\therefore A = -\int x \sin^2 x \, dx$$

$$= -\frac{x}{2}\left[x - \frac{\sin 2x}{2}\right] + \frac{1}{2}\int \left(x - \frac{\sin 2x}{2} dx\right)$$

$$= \frac{-x^2}{2} + \frac{x \sin 2x}{2} + \frac{x^2}{4} + \frac{\cos 2x}{8}$$

$$= \frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$$

$$B = \int \frac{x \sin 2x}{2} dx$$

$$= \frac{-x \cos 2x}{4} + \int \frac{\cos 2x}{4} dx$$

$$= \frac{-x \cos 2x}{4} + \frac{\sin 2x}{8}$$

The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \left(\frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}\right) \cos x + \left(\frac{\sin 2x}{8} - x\frac{\cos 2x}{4}\right) \sin x$$

Example 4

Solve $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Solution :

$$(D^2 + 1) y = \operatorname{cosec} x$$

The auxiliary equation is given by $m^2 + 1 = \operatorname{cosec} x$

$\therefore m = \pm i.$

The complementary function is $y = A\cos x + B\sin x$ where A and B are arbitrary constants.

Let the particular integral be

$$y = A\cos x + B\sin x$$

where A and B are arbitrary functions.

$$\frac{dy}{dx} = -A\sin x + B\cos x + A'\cos x + B'\sin x$$

Choose $A'\cos x + B'\sin x = 0$

$$\therefore \frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = -A'\sin x + B'\cos x - A\cos x - B\sin x$$

$$= -A'\sin x + B'\cos x - y.$$

$$\therefore \frac{d^2y}{dx^2} + y = -A'\sin x + B'\cos x.$$

$$-A'\sin x + B'\cos x = cosec\, x$$

But $\quad A'\cos x + B'\sin x = 0$

(5) $\sin x$ −(4) $\cos x$ gives $-A' = 1$

$$A' = -1$$

$$\therefore B' = \cot x$$

$$A' = -\int dx = -x$$

$$B' = \int \cot x\, dx = \log \sin x$$

∴ The complete solution is

$y = c_1\cos x + c_2\sin x - x\cos x + (\log\sin x)\sin x$

Example 5

Solve $\quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x \sin x$

Solution :

$$(D^2 - D)\,y = e^x \sin x$$

The auxiliary equation is $m^2 - m = 0$

$$m(m-1)=0$$

$$\therefore m=0,1$$

The complementary function is

$$y = A + Be^x$$

Let the particular integral be ...(1)

$y = A + Be^x$ .... (2) where A and B are arbitrary functions.

Differentiating with respect to x

$$y' = A' + B'e^x + Be^x$$

Choose $A' + B'e^x = 0$ ...(3)

Then

$$y' = Be^x$$

$$y'' = Be^x + B'e^x.$$

The given equation is $y'' - y' = e^x \sin x$.

$$Be^x + B'e^x - Be^x = e^x \sin x$$

$$B'e^x = e^x \sin x$$

$$B' = \sin x$$

$$B = \int \sin x\, dx = -\cos x$$

From (3) $A' = -B'e^x = e^x \cos x$

$$\therefore A = \int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x)$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 x + \frac{e^x}{2}(\cos x + \sin x) - e^x \cos x.$$

$$= c_1 + c_2 x + \frac{e^x}{2}(\sin x - \cos x)$$

Exercise 2

1. $\frac{d^2y}{dx^2} + 4y = \tan 2x$
2. $y'' - 2y' + 2y = e^x \tan x$
3. $(D^2 + 4)y = \operatorname{cosec} 2x$
4. $\frac{d^2y}{dx^2} + 9y = \sec 3x$

5. $\frac{d^2y}{dx^2} - 4y = e^{2x}$

6. $\frac{d^2y}{dx^2} - y = \frac{2}{1 + e^x}$

7. $y'' + 2y' + y = e^{-x} \cos x$

8. $\frac{d^2y}{dx^2} - y = e^x \sin (e^{-x}) + \cos (e^{-x})$

9. $\frac{d^2y}{dx^2} + 16y = sec\ 4x$

10. $y'' + 16y = 4 \tan 4x$

11. $(D^2 + 1)\, y = x \cos 2x$

12. $(D^2 + 1)\, y = \tan x$

13. $(D^2 + 1)\, y = \frac{1}{1 + \sin x}$