# CHAPTER VII

## Friction உராய்வு

§ 1. **Introduction:** Till now, we have been studying the problems involving equilibrium of smooth bodies. When two smooth bodies are in contact with each other, the mutual action between them is entirely along the common normal at the point of contact. There is no force in the tangential direction. Hence there is no force tending to prevent one smooth body from sliding over the other.

But practically, there are no bodies which are perfectly smooth. All bodies are *rough* to some extent. Thus, if we attempt to drag a heavy body along the ground by means of a horizontal force, a resistance is felt to the motion of the body. This resistance is due to the *roughness* of the ground and is called *friction*. Thus, in the case of rough bodies in contact, besides the normal reaction, a tangential reaction i.e. a force acting in a direction perpendicular to the normal reaction is called into play. This tangential force between two bodies in contact prevents the one from sliding over the other. Such a force is called the *force of friction*

Friction

2m (**Definition:** *If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction; the force exerted is called the force of friction* )

§ 2. **Experimental Results:** Suppose a heavy body is placed on a table and is pulled in a horizontal direction by a force P. It is found that, upto a certain value of P, the body does not move. The normal reaction R of the table and the weight W of the body are acting in the vertical direction and so, they have no effect in the horizontal direction. They are not responsible for stopping the motion of the body. Since the body is at rest, there must be some force in the horizontal direction to oppose the force P. This force F is the force of friction between the body and the table.

As P is gradually increased, the force F also increases so as to balance P at each instant. This state will continue till P attains a certain value when the body is *just on the point of motion.* At this stage, the *force of friction has attained its maximum value* and equilibrium is about to be broken. When P is further increased, F cannot increase further, since it has already reached its maximum. The equilibrium is actually broken and the body begins to move.

Thus we find that so long as the body remains at rest, the force of friction depends on P and is just sufficient to resist P. In this case, friction is called *statical friction. Thus statical friction is a self - adjusting force and is just sufficient to maintain equilibrium.* If P ceases to exist, F also vanishes, as otherwise, the body will move in the opposite direction. Also the amount of statical friction varies from zero upto a maximum value.

Types of Friction:

**§ 3. Statical, Dynamical and Limiting Friction:**

*When one body in contact with another is in equilibrium, the friciton exerted is just sufficient to maintain equilibrium and is called statical friction.* 2m

*When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called limiting friction; the equilibrium in this case is said to be limiting.* 2m

*When motion ensues by one body sliding over another, the fricilon exerted is called dynamical friction.* 2m

5m

**§ 4. Laws of Friction:** Friction is not a mathematical concept; it is a physical reality. The results of physical observation and experiment are formulated as the Laws of Friction.

**Law 1.** *When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.*

**Law 2.** *When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.*

**Law 3.** *The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.*

**Law 4.** *The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.*

**Law 5. (Law of dynamical friction)**

*When motion ensues by one body sliding over the other, the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.*

**Note:** These laws are experimental, and cannot be accepted as rigorously accurate but they express fairly accurately the results of a large number of experiments.

**§ 5. Friction - a passive force:** It should be noted that friction is only a resisting force and appears only when necessary to prevent or oppose the motion of the point of contact. It cannot by itself produce motion of a body but it maintains relative equilibrium. It is a self-adjusting force. It assumes such magnitude and direction as to balance other forces acting on the body. Such a type of force is called a *passive force.* Friction is thus a purely passive force.

The force of friction, though considered to be dissipative is really beneficient, for, without it, most forms of motion would be impossible. If there were no friction of the ground, walking would have been impossible. Screws or nails would not stick to wood. Wheels and carriages would not roll. Thus friction is indirectly the agent for producing motion, though often it is recognised as a waste of energy and a source of loss.

**§ 6. Coefficient of Friction:** In § 4, by law 3, we know that limiting friction between two bodies bears a constant ratio to the normal reaction between them. *The ratio of the limiting friction to the normal reaction is called the coefficient of friction.* It is usually denoted by the letter $\mu$.

Let F be the friction and R the normal reaction between two bodies when equilibrium is limiting.

Then $\frac{F}{R} = \mu$, i.e. $\boxed{F = \mu R.}$

The constant $\mu$ depends on the nature of the materials in contact. It is different for different pairs of substances and is ordinarily less than unity.

Since friction is maximum when it is limiting, $\mu R$ is the maximum value of friction. When equilibrium is nonlimiting, F is less than $\mu$ R and $\frac{F}{R} < \mu$

**Note:** 1. In limiting equilibrium, F and R may both vary with the masses of the bodies even for the same pair of substances but the ratio $\frac{F}{R}$ is constant being equal to $\mu$.

2. We must not assume that friction is always equal to $\mu R$. It has this value only when motion is about to take place. Otherwise, it may have any value from zero upto $\mu R$.

**§ 7. Angle of Friction:**

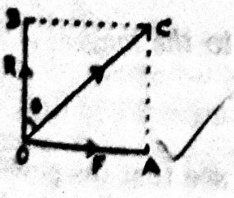


Fig. 1

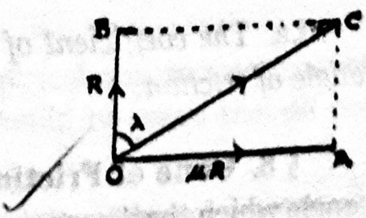


Fig. 2

Resultant Reaction :-

Suppose one body is kept in equilibrium by friction on another. At the point of contact Q, two forces act on the first body, namely the normal reaction and the force of friction. These two act in perpendicular directions and they can be compounded into a single force. This single force is called the *resultant reaction* or the *total reaction.*

In fig. 1, let $\overline{OA} = F$, the force of friction and $\overline{OB} = R$ the normal reaction. Let $\overline{OC}$ be the resultant of F and R.

It $\angle BOC = \theta$,

$$\tan\theta = \frac{BC}{OB} = \frac{OA}{OB} = \frac{F}{R} \quad \ldots \ldots \quad (1)$$

As F increases from 0, the value of $\theta$ increases until the friction F reaches its maximum value. In that case, the equilibrium is limiting and the angle made by the resultant reaction with the normal is called the *angle of friction* and it is denoted by $\lambda$.

Hence the greatest value of $\theta$ is $\lambda$.

~~Relation between~~

*When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called the angle of friction and it is denoted by* $\lambda$.

Relation between co-efficient of friction and angle of friction

In fig. 2, $\overline{OA}$ represents the limiting friction which $= \mu R$, $\mu$ being the coefficient of friction.

$\overline{OA} = \mu R$

$\overline{OC}$ is the resultant of $\overline{OA}$ and $\overline{OB}$.

$\angle BOC = \lambda =$ the angle of friction.

$$\tan\lambda = \frac{BC}{OB} = \frac{OA}{OB} = \frac{\mu R}{R} = \mu.$$

*i.e. The coefficient of friction is equal to the tangent of the angle of friction.*

**§ 8. Cone of Friction:** From § 7, we see that the greatest angle which the direction of the resultant reaction can make with the normal is $\lambda$ i.e. $\tan^{-1}(\mu)$.

Now the motion of one body at O, its point of contact with another, can take place in any direction perpendicular to the normal. Hence when two bodies are in contact, we can consider a cone drawn with the point of contact as the vertex, the common normal as the axis and its semi-vertical angle being equal to $\lambda$, the angle of friction. It is clear that the resultant reaction will have a direction which entirely lies within the surface or on the surface of that cone. It cannot fall outside the cone. Such a cone is called the *cone of friction.* 2M

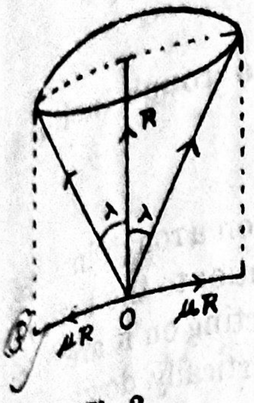


Fig.3

§ 9. **Numerical Values:** The values of the coefficients and angles of friction for various substances have been found by expriment and tabulated. The following table, taken from Prof. Rankine's *Machinery and Millwork*, gives the values of $\mu$ and $\lambda$ for a few substances.

| Substances | | $\mu$ | $\lambda$ |
|---|---|---|---|
| Wood on wood | ...Dry | .25 to .5 | $14$ to $26\frac{1}{2}^\circ$ |
| " " | ...Soaped | .04 to .2 | $2^\circ$ to $11\frac{1}{2}^\circ$ |
| Metals on metals | ...Dry | .15 to .2 | $8\frac{1}{2}^\circ$ to $11\frac{1}{2}^\circ$ |
| " " | ...Wet | .3 | $16\frac{1}{2}^\circ$ |
| Leather on metals | ...Dry | .56 | $29\frac{1}{2}^\circ$ |
| " " | ...Wet | .36 | $20^\circ$ |
| " " | ...Oily | .5 | $8\frac{1}{2}^\circ$ |

**The above values are purely empirical and so they may be affected by experimental errors. It should be noted that all the values of $\mu$ are less than 1.**

**The only substances stated by Rankine to have a coefficient of friction as great as unity are:**

earth on earth, damp clay ... ... 1.0

and earth on earth, shingle and gravel 0.81 to 1.11

Important

## § 10. Equilibrium of a particle on a rough inclined plane:

Let a particle of weight W be placed at A on a rough inclined plane, whose inclination to the horizontal is $\theta$. The forces acting on it are: (i) its weight W acting vertically downwards (ii) the frictional force F acting along the inclined plane upwards. (If there had been no friction, the body would have a tendency to move downwards. Hence friction will act upwards). (iii) the normal reaction R, perpendicular to the plane.

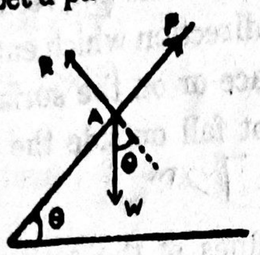


Fig. 4.

Resolving along and perpendicular to the plane, we get

$$F = W\sin\theta \text{......(1)}$$

$$\text{and } R = W\cos\theta \text{......(2)}$$

$$\therefore \frac{F}{R} = \tan\theta \text{ ... ... (3)}$$

We know that $\frac{F}{R}$ is always $< \mu$

Hence for equilibrium, $\tan\theta < \mu$.

i.e. $\tan\theta < \tan\lambda$, $\lambda$ being the angle of friction or $\theta < \lambda$.

Suppose $\theta$, the inclination of the plane, is gradully increased. When $\theta = \lambda$, then $\frac{F}{R} = \tan\lambda = \mu$.

In this case, the equilibrium becomes limiting and the particle is just on the point of sliding down.

Hence we have the following theorem:

Statement

*If a body be placed on a rough inclined plane and be on the point of sliding down the plane under the action of its weight* *and the reaction of the plane only, the angle of inclination of the plane to the horizon is equal to the angle of friction.*

Definition:

The inclination ($\lambda = \tan^{-1}\mu$) of the inclined plane when the body just begins to slip is called the *angle of repose.* Hence the above theorem is stated as:

*The angle of repose of a rough inclined plane is equal to the angle of friction.*

**Important Note:** It should be noted that the angle of repose of a rough inclined plane is equal to the angle of friction, only when there are *no external force* acting on the body.

Important

## § 11. Equilibrium of a body on a rough inclined plane under a force parallel to the plane.

**Theorem:** *A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted upon by a force, parallel to the plane and along the line of greatest slope; to find the limits between which the force must lie.*

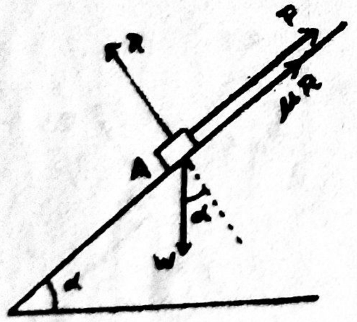


Fig. 5

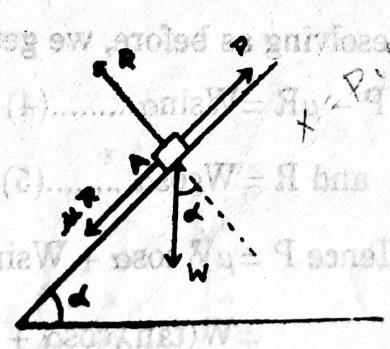


Fig. 6

Let $\alpha$ be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction.

**Case 1.** Refer to fig. 5. Let the body be on the point of moving *down* the plane. Then limiting friction acts *up* the plane and $= \mu R$. Let P be the force required to keep the body at rest.

$0 = P\cos 0 + F\cos 0 + R\cos 90 - W\cos(90-\alpha)$

$= P + \mu R - W\sin\alpha$

$0 = P\sin 0 + F\sin 0 + R\sin 90 - W\sin(90-\alpha)$

$= R - W\cos\alpha$



Resolving along and perpendicular to the plane, we have

$P + \mu R = W\sin\alpha$......(1)

and $R = W\cos\alpha$.....(2)

Substituting for R from (2) in (1), we get

$P = W\sin\alpha - \mu W\cos\alpha.$

If $\lambda$ is the angle of friction, we know that $\mu = \tan\lambda$.

$\therefore P = W(\sin\alpha - \tan\lambda\cos\alpha)$

$$= W\frac{(\sin\alpha\cos\lambda - \sin\lambda\cos\alpha)}{\cos\lambda} = \frac{W\sin(\alpha - \lambda)}{\cos\lambda}$$

Let this value of P be $P_1$.

Then $P_1 = \dfrac{W\sin(\alpha-\lambda)}{\cos\lambda}$ ... ... (3)

Since $\alpha > \lambda$, $P_1$ is positive.

**Case 2.** As in fig. 6, let the body be on the point of moving *up* the plane. Then limiting friction $\mu R$ acts downwards. Let P be the force required to keep the body at rest.

Resolving as before, we get

$P - \mu R = W\sin\alpha$.........(4)

and $R = W\cos\alpha$.........(5)

Hence $P = \mu W\cos\alpha + W\sin\alpha$

$$= W(\tan\lambda\cos\alpha + \sin\alpha)$$

$$= \frac{W(\sin\lambda\cos\alpha + \cos\lambda\sin\alpha)}{\cos\lambda}$$

$$= \frac{W\sin(\alpha + \lambda)}{\cos\lambda} = P_2(\text{say})$$

$\therefore P_2 = \dfrac{W\sin(\alpha + \lambda)}{\cos\lambda}$ ......(6)

Now if P is > $P_2$, the body will move up the plane.

$\therefore$ $P_2$ is the limiting value of P, which is necessary to keep the body in equilibrium, without moving upwards.

If P is < $P_1$, the body will move down the plane.

$\therefore$ $P_1$ is the limiting value of P, which is necessary to keep the body in equilibrium without moving downwards.

Hence, If P lies between $P_1$ and $P_2$, the body will remain in equilibrium and is not in the point of motion in either direction.

Hence, for equilibrium, the force P must lie between the values $\frac{W\sin(\alpha-\lambda)}{\cos\lambda}$ and $\frac{W\sin(\alpha+\lambda)}{\cos\lambda}$

**Note:** The value of $P_2$ may be obtained from that of $P_1$, by changing the sign of $\mu$.

## § 12. Equilibrium of a body on a rough inclined plane under any force

Important

**Theorem:** *A body is at rest on a rough inclined plane of inclination $\alpha$ to the horizon, being acted on by a force making an angle $\theta$ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.*

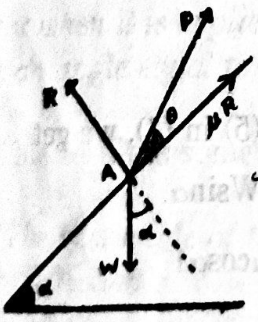


Fig. 7

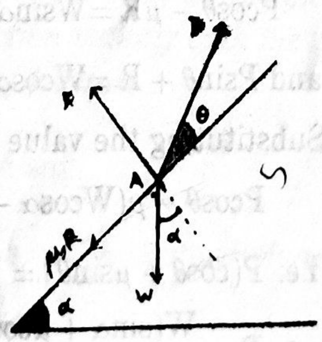


Fig. 8

**Let W be the weight of the body, P the force acting at an angle $\theta$ with the plane and R the normal reaction.**

**Case I: In fig. 7, the body in just on the point of moving *down* the plane. Then limiting friction $\mu R$ acts upwards. Resolving the**

$0 = F\cos 0 + P\cos\theta + R\cos 90^\circ - W\cos(90^\circ - \alpha)$

$= \mu R + P\cos\theta - W\sin\alpha$



forces along and perpendicular to the plane, we get,

$P\cos\theta + \mu R = W\sin\alpha$......(1) $\Rightarrow Y = F\sin 0 + P\sin\theta + R\sin 90^\circ - W\sin(90^\circ - \alpha)$

and $P\sin\theta + R = W\cos\alpha$......(2) $\Rightarrow R = W\cos\alpha - P\sin\theta$

Substituting the value of R from (2) in (1), we get

$$P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$$

i.e. $P(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha)$

$$\therefore P = \frac{W(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

If $\lambda$ is the angle of friction, we know that $\mu = \tan\lambda$.

$$\therefore P = \frac{W(\sin\alpha - \tan\lambda\cos\alpha)}{\cos\theta - \tan\lambda\sin\theta}$$

$$= \frac{W(\sin\alpha\cos\lambda - \sin\lambda\cos\alpha)}{\cos\theta\cos\lambda - \sin\lambda\sin\theta} = \frac{W\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$$

Let this value of be $P_1$

$$\therefore P_1 = \frac{W\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}\text{.......(3)}$$

**Case 2.** In fig. 8, the body is just on the point of moving up the plane. Then limiting friction $\mu R$ acts downwards, Resolving the forces as before,

$P\cos\theta - \mu R = W\sin\alpha$......(4)

and $P\sin\theta + R = W\cos\alpha$......(5)

Substituting the value of R from (5) in (4), we get

$$P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha.$$

i.e. $P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$

$$\therefore P = \frac{W(\sin\alpha + \mu\cos\alpha)}{\cos\theta + \mu\sin\theta}$$

$$= \frac{W(\sin\alpha + \tan\lambda\cos\alpha)}{\cos\theta + \tan\lambda\sin\theta}$$

$$= \frac{W(\sin\alpha\cos\lambda + \sin\lambda\cos\alpha)}{\cos\theta\cos\lambda + \sin\lambda\sin\theta} = \frac{W\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Let this value of P be $P_2$.

$$\therefore P_2 = \frac{W\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}\text{......(6)}$$

$P_1$ and $P_2$ are the limiting values of the force P, necessary to keep the body in equilibrium.

Hence if P lies between $P_1$ and $P_2$, the body will remain in equilibrium.

Note: (1) The value of $P_2$ may be obtained from that of $P_1$ by changing the sign of $\mu$.

(2) By putting $\theta = 0$, the discussion of § 12 gives the results of discussion of § 11.

**Corollary:** We can find the direction and magnitude of the least force required to drag the body up the inclined plane.

From case II, $P = \dfrac{W\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$

Since $\alpha$, W and $\lambda$ are constants,

P is least if $\cos(\theta - \lambda)$ is greatest.

i.e. if $\cos(\theta - \lambda) = 1$.

This happens when $\theta - \lambda = 0$. i.e. when $\theta = \lambda$

In that case, value of $P = W\sin(\alpha + \lambda)$

*Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.*

This result is sometimes stated as:

*"The best angle of traction up a rough inclined plane is the angle of friction".*

## WORKED EXAMPLES

**Ex. 1.** *A particle of weight 30 kgs. resting on a rough horizontal plane is just on the point of motion when acted on by horizontal forces of 6kg wt. and 8kg. wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.*

*(B.Sc. 85 M.K.U.)*

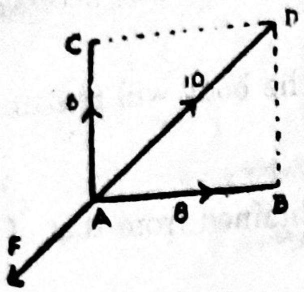


Fig. 9.

Let AB (=8) and AC (=6) represent the directions of the forces, A being the particle.

The resultant force $= \sqrt{8^2 + 6^2}$ = 10kg. wt. and this acts along AD, making an angle $\cos^{-1}(\frac{4}{5})$ with the 8kg force.

The particle tends to move in the direction AD of the resultant force and hence friction acts in the opposite direction DA.

Let F be the frictional force. As motion just begins, magnitude of F is equal to that of the resultant force.

$\therefore$ F = ~~10~~ 30 ... ... (1)    F = 10 —(1)

If R is the normal reaction on the particle,

R = ~~30~~ 10 ... ... (2)    R = 30 —(2)

If $\mu$ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$

i.e. $10 = \mu . 30$    $10 = \mu \cdot 30$

or $\mu = \frac{10}{30} = \frac{1}{3}$. 3    $\mu = 10/30 = 1/3$

**Ex. 2.** *A weight can be supported on a rough inclined plane by a force P acting along the plane or by a force Q acting horizontally. Show that the weight is* $\frac{PQ}{\sqrt{(Q^2 \sec^2\lambda - P^2)}}$ *where $\lambda$ is the angle of friction.*

Let W be the weight and $\alpha$ the angle of inclination of the plane. R is the normal reaction.

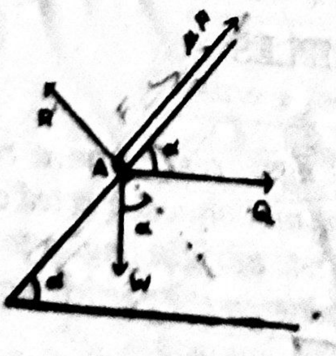


Fig. 10.

When the weight is just on the point of moving down, limiting friction $\mu R$ acts upwards. A horizontal force Q keeps the weight in equilibrium.

Resolving along and perpendicular to the plane,

$X = P_1\cos\theta_1 + P_2\cos\theta_2 + P_3\cos\theta_3 + P_4\cos\theta_4$
$= F\cos 0 + -Q\cos(180-\alpha) - W\cos(90-\alpha) + R\cos 90$
$0 = F - Q\cos\alpha - W\sin\alpha$





$\Rightarrow \mu R = Q\cos\alpha + W\sin\alpha$

$\mu R + Q\cos\alpha = W\sin\alpha$ ... ... (1)

and $R = W\cos\alpha + Q\sin\alpha$

Putting the value of R from (2) in (1), we have

$Y = P_1\sin\theta_1 + P_2\sin\theta_2 + P_3\sin\theta_3 + P_4\sin\theta_4$
$= F\sin 0 - Q\sin(180-\alpha) - W\sin(90-\alpha) + R\sin 90$

$$\mu(W\cos\alpha + Q\sin\alpha) + Q\cos\alpha = W\sin\alpha$$

$0 = -Q\sin\alpha - W\cos\alpha + R$

i.e. $Q(\mu\sin\alpha + \cos\alpha) = W(\sin\alpha - \mu\cos\alpha)$ ... ... (3)

The same weight W is supported by a force P acting along the plane.

From § 11, page 213, we have

$$P = \frac{W\sin(\alpha-\lambda)}{\cos\lambda} = \frac{W}{\cos\lambda}(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) \text{ ... ... (4)}$$

From (3), $\cos\alpha(Q+\mu W) = \sin\alpha(W-\mu Q)$

$$\therefore \frac{\cos\alpha}{W-\mu Q} = \frac{\sin\alpha}{Q+\mu W}$$

and each $= \dfrac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\sqrt{(W-\mu Q)^2 + (Q+\mu W)^2}}$ $\quad W^2 + \mu^2 Q^2 - 2W\mu Q + Q^2 + \mu^2 W^2 + \cancel{2Q\mu W}$

$$= \frac{1}{\sqrt{W^2(1+\mu^2) + Q^2(1+\mu^2)}}$$

$$= \frac{1}{\sqrt{1+\mu^2}\sqrt{W^2+Q^2}}$$

$$= \frac{1}{\sqrt{1+\tan^2\lambda}\sqrt{W^2+Q^2}}$$

$$= \frac{1}{\sec\lambda\sqrt{W^2+Q^2}}$$

$$\therefore \cos\alpha = \frac{W-\mu Q}{\sec\lambda\sqrt{W^2+Q^2}} \text{ and } \sin\alpha = \frac{Q+\mu W}{\sec\lambda\sqrt{W^2+Q^2}}$$

Substituting these in (4), we have

$$P = \frac{W}{\cos\lambda}\left[\frac{(Q+\mu W)\cos\lambda - (W-\mu Q)\sin\lambda}{\sec\lambda\sqrt{W^2+Q^2}}\right]$$

$$= \frac{W}{\sqrt{W^2+Q^2}}\left[Q(\cos\lambda + \mu\sin\lambda) + W(\mu\cos\lambda - \sin\lambda)\right]$$

$$= \frac{W}{\sqrt{W^2+Q^2}}\left[Q\left(\cos\lambda + \frac{\sin\lambda}{\cos\lambda}\sin\lambda\right) + W\left(\frac{\sin\lambda}{\cos\lambda}\cos\lambda - \sin\lambda\right)\right]$$

$$= \frac{W}{\sqrt{W^2+Q^2}} \frac{Q(\cos^2\lambda+\sin^2\lambda)}{\cos\lambda} = \frac{WQ\sec\lambda}{\sqrt{W^2+Q^2}}$$

$$\therefore P^2 = \frac{W^2Q^2\sec^2\lambda}{W^2+Q^2}$$

$$P^2(W^2+Q^2) = W^2Q^2\sec^2\lambda$$

$$\text{or } W^2(Q^2\sec^2\lambda - P^2) = P^2Q^2.$$

$$\therefore W^2 = \frac{P^2Q^2}{Q^2\sec^2\lambda - P^2} \text{ or } W = \frac{PQ}{\sqrt{Q^2\sec^2\lambda - P^2}}$$

**Ex. 3.** *Show that the horizontal force, which can support a heavy mass on a smooth inclined plane of inclination $\alpha$ to the horizon can just sustain it in limiting equilibrium on a rough plane inclined to the horizon at an angle of $(\alpha + \lambda)$ where $\lambda$ is the angle of friction.*

Referring to equation (3) of the previous worked example, we have

$$Q = \frac{W(\sin\alpha - \mu\cos\alpha)}{\mu\sin\alpha + \sec\alpha} \quad \ldots (1)$$

$$= W\frac{\left(\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right)}{\frac{\sin\lambda}{\cos\lambda}\sin\alpha + \cos\alpha} = \frac{W\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} \quad \ldots (2)$$

This gives the horizontal force Q which just keeps the weight W in limiting equilibrium on a rough plane of inclination $\alpha$.

Let $Q_1$ be the horizontal force which keeps W in limiting equilibrium on a rough plane of inclination $\alpha + \lambda$.

$Q_1$ can be got from Q by changing $\alpha$ in to $\alpha + \lambda$.

$$\therefore Q_1 = \frac{W\sin(\alpha + \lambda - \lambda)}{\cos(\alpha + \lambda - \lambda)} = W\tan\alpha \text{......(3)}$$

Let $Q_2$ be the horizontal force which keeps W in equilibrium on a smooth plane of inclination $\alpha$. $Q_2$ can be got from the value of Q give in (1) by putting $\mu = 0$.

$$\therefore Q_2 = \frac{W\sin\alpha}{\cos\alpha} = W\tan\alpha \text{......(4)}$$

From (3) and (4), we find that $Q_1 = Q_2$.

*Ex. 4. A particle is placed on a rough plane hose inclination to the horizon is $\alpha$ and is acted upon by force P acting parallel to the plane and in a direction making an angle $\beta$ with the line of greatest slope in the plane. If the coefficient of friciton be $\mu$ and the equilibrium be ilimiting, find the direction in which the body will begin to move.*

*(B.Sc. 52)*

Let BCDE be the rough plane meeting the horizontal plane in the line BC, the angle between the two planes being $\alpha$.

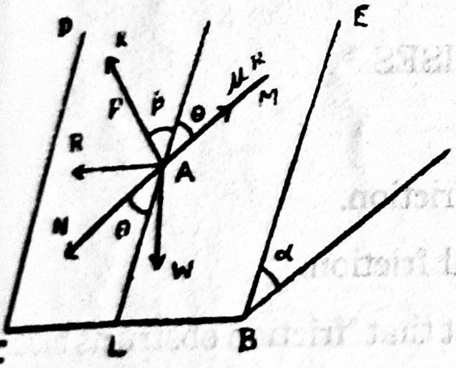


Fig. 11.

A is the particle and AL is the line of greatest slope through A. The forces acting on the particle are:

(i) its weight W acting vertically downwards

(ii) the normal reaction R perpendicular to the inclined plane.

(iii) the force P acting in the plane in a direction making an angle $\beta$ with LA.

(iV) the (limiting) frictional force $\mu$ R acting in the plane.

The weight W can be resolved into two components namely W sin $\alpha$ acting in the plane along AL and W cos $\alpha$ perpendicular to the inclined plane.

Resolving the forces perpendicular to the inclined plane,

$R = W\cos\alpha$ ... ... (1)

The particle will tend to move in the direction given by the resultant of the two forces P along AK and W sin $\alpha$ along AL. Let it move along AN at an angle $\theta$ to AL. Then friction will act in the opposite direction i.e. it will act along AM and its magnitude $= \mu R$, as the equilibrium is limiting.

The forces P, $\mu R$ and W sin $\alpha$ act at A in the inclined plane and keep the particle at rest.

∴ By Lami's theorem.

$$\frac{P}{\sin\theta} = \frac{\mu R}{\sin\beta} = \frac{W\sin\alpha}{\sin(\theta+\beta)}$$

$$\therefore \sin(\theta+\beta) = \frac{W\sin\alpha\sin\beta}{\mu R} = \frac{W\sin\alpha\sin\beta}{\mu W\cos\alpha}$$

$$\text{i.e. } \sin(\theta+\beta) = \frac{\tan\alpha\sin\beta}{\mu} \quad \ldots (2)$$

Hence the particle will begin to move in a direction making an angle $\theta$, given by equation (2), with the line of greatest slope.