CHAPTER IV

Couples

§ 1. Couples: *Definition*

In the last chapter we have seen that the general method of finding the resultant of two equal and unlike parallel froces fails i.e. the effect of two equal and unlike parallel forces cannot be replaced by a single force. A pair of such forces is called a couple.

8M [*Two equal and unlike parallel forces not acting at the same point are suid to constitute a couple.*]

Examples of a couple are the forces used in winding a clock or turning a tap. Such forces acting upon a rigid body can have only a rotatory effect on the body and they cannot produce a motion of translation.

Let P, P be the magnitudes of the forces forming a couple and O any point in their plane.

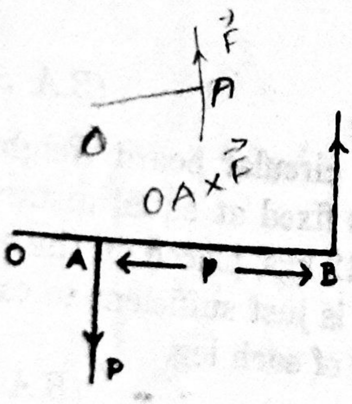


Fig.1

Draw OAB perpendicular to the forces to meet their lines of action in A and B.

The algebraic sum of the moments of the forces about O is

$$= P.OB - P.OA$$

$$= P(OB - OA) = P.AB$$ couple

and this value is independent of the position of O.

Thus the algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and is equal to the product of either of the forces and the perpendicular distance between them. This algebraic sum measures the total turning effect of the forces of the couple upon the body and is called the *moment of the couple.*

moment of couples

[Thus, *the moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.*] 2m

[The perpendicular distance AB ($=p$) between the two equal forces P of a couple is called the *arm of the couple.* A couple each of whose forces is P and whose arm is p, as in fig. 1 is usually denoted by (P, p).] 2m distance

A couple is positive when its moment is positive i.e. if the forces of the couple tend to produce rotation in the anticlockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

§ 2. Equilibrium of two couples: important theorem.

Theorem: *If two couples, whose moments are equal and opposite, act in the same plane upon a rigid body, they balance one another.*

Let (P, p) and (Q, q) be two given couples such that Pp=Qq in magnitude but opposite in sign.

Case 1: Let the forces P and Q be parallel.

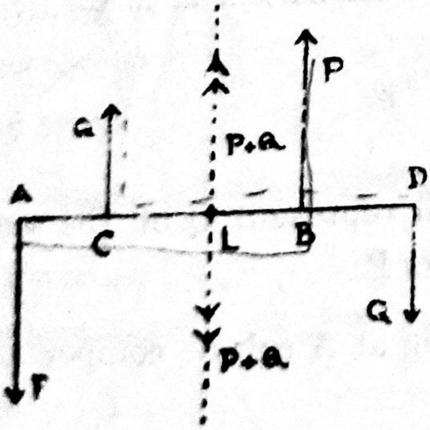


Fig. 2

Draw a straight line perpendicular to the lines of action of the forces, meeting them at A,B,C,D as in fig. 2.

Since the moments of the couples are equal, we have

$$P.AB = Q.CD \quad \dots \dots (1)$$

The downward like parallel forces P at A and Q at D can be compounded into a single force P + Q acting at L such that

$$P.AL = Q.DL \quad \dots \dots \dots (2)$$

(1) – (2) gives

$P(AB - AL) = Q(CD - DL)$

i.e. $P.BL = Q.CL$ ... ... (3)

Result (3) shows that the resultant of the upward like parallel forces P at B and Q at C will also pass through L. The magnitude of this resultant is also $(P + Q)$ but it is opposite in direction to the previous resultant. Thus the two resultants balance each other. Hence the four forces forming the couples are in equilibrium.

Case 2: Let the forces P and Q intersect

Let the two forces P of the couple (P, p) meet the two forces Q of the couple (Q, q) at the points A,B,C,D. Clearly ABCD is a parallelogram.

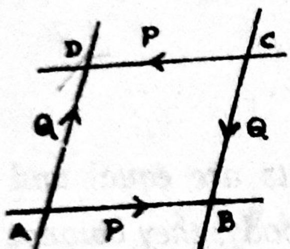


Fig. 3

Let AB represent P on some scale.

As the moments of the two couples are equal, we have

$P.p = Q.q$ ... ... (1)

Also $AB.p = AD.q$ (each being equal to the area of the || gm. ABCD) ... ... (2)

(1) ÷ (2) gives

$$\frac{P}{AB} = \frac{Q}{AD} \quad \text{... ... (3)}$$

(3) shows that the side AD will represent Q on the same scale in which the side AB represents P.

The two forces P and Q meeting at A can be compounded by || gm. law so that

$(P+Q)$ at A $= \overline{AB} + \overline{AD} = \overline{AC}$

Similarly $(P+Q)$ at C $= \overline{CD} + \overline{CB} = \overline{CA}$.

The two resultants $\overline{AC}$ and $\overline{CA}$ being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

§ 3. Equivalence of two couples:

Theorem: *Two couples in the same plane whose moments are equal and of the same sign are equivalent to one another.*

Let (P, p) and (Q, q) be two couples in one plane having the same equal moments in magnitude and direction. Let (R, r) be a third couple, in the same plane, whose moment is equal to the moment of either (P, p) or (Q, q) only in magnitude but opposite in direction. By the previous theorem, the couple (R, r) will balance the couple (P,p). It will also balance the couple (Q,q). Hence the effects of the couples (P, p) and (Q, q) must be the same. In other words, they are equivalent.

This is a fundamental theorem on coplanar couples. From this, it follows that a couple in a plane can be replaced by any other couple in the same plane, provided that the moment of the latter replacing couple is equal in magnitude and direction to the moment of the first couple. The replacing couple may act in any manner in that plane i.e. it does not matter in what direction its forces act; the magnitude of its forces and its arm length may be anything. The only important criterion is that the moment of the new couple must be equal to that of the first couple in magnitude and sense.

Thus a couple (P,p) may be replaced by a couple $\left(F, \frac{Pp}{F}\right)$ in the same plane with its constituent forces each equal to F and the arm length being equal to $\frac{Pp}{F}$. The moment of this couple is $= F\frac{Pp}{F} = Pp$ moment of the first couple. Also one force F may be taken to be acting in any line and direction, the other at the distance $\frac{Pp}{F}$ being on that side so as to make the sign of the moment same as that of (P,p).

Similarly, the couple (P,p) may be replaced by a couple $\left(\frac{Pp}{x}, x\right)$ with a given arm x anywhere in the plane.

§ 4. Couples in Parallel Planes: Theorem.

The effect of a couple upon a rigid body is not altered if it is transferred to a parallel plane provided its moment remains unchanged in magnitude and direction.

Consider a couple of forces P at the ends of arm AB in a given plane. Let AL and BM be the lines of action of the forces.

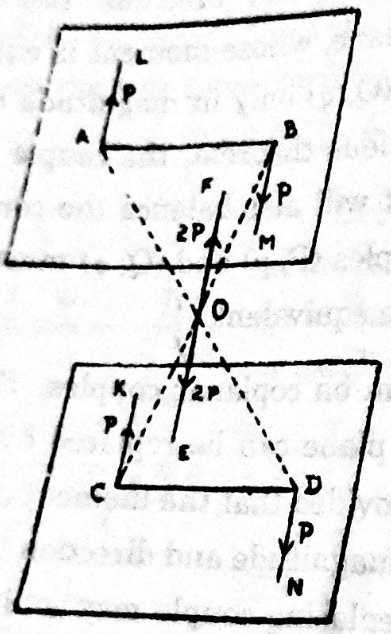


Fig. 4

In any parallel plane, take a straight line CD equal and parallel to AB.

Then ABDC will be a parallelogram. The diagonals AD and BC will bisect each other, say at O.

At O, introduce two equal and opposite forces of magnitude 2P along EF, parallel to the forces P at A and B. By this, the effect of the given couple is not altered.

Now the unlike parallel forces P along AL and 2P along OE can be compounded into a single force P acting at D, since $\frac{AD}{OD} = \frac{2}{1} = \frac{2P}{P}$ This resultant force P acts along DN in the second plane. Similarly, the unlike parallel forces P along BM and 2P along OF can be compounded into a single force P acting at C along CK. We are therefore left with a couple of forces P at the ends of the arm CD in a plane parallel to that of the original couple.

Thus the given couple with the arm AB is equivalant to another couple of the same moment in a parallel plane, having its arm CD equal and parallel to AB. Now this couple with arm CD can be replaced in its own plane by another couple, provided the moment is unchanged in magnitude and direction as in § 3. Hence we conclude that a couple in any plane can be replaced by another couple acting in a parallel plane, provided that the moments of the two couples are the same in magnitude and sign.

§ 5. Representation of a couple by a vector:

From articles 3 and 4, it is clear that a couple is not localised in any particular plane, for it may be replaced by another couple of the same moment in the same plane or in any parallel plane. Thus the effect of a couple remains unaltered so long as its moment remains the same in magnitude and sense, whatever be the magnitude of its constituent forces, the length of its arm and its position in any one of a set of parallel planes in which it may be supposed to act.

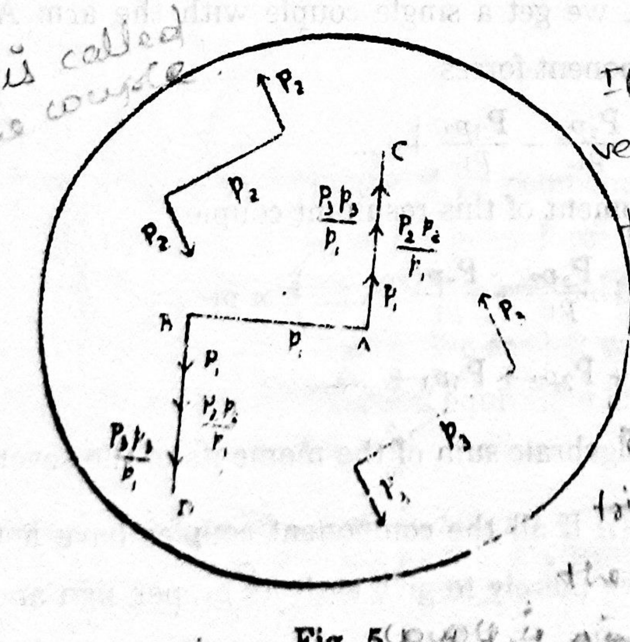
A couple is therefore completely specified if we know (i) the direction of the set of parallel planes (ii) the magnitude of its moment (iii) the sense in which it acts. These three aspects of a couple can be conveniently represented by a straight line drawn (i) perpendicular to the set of parallel planes to indicate the direction (ii) of a measured length, to indicate the moment of the couple and (iii) in a definite direction, to indicate the sense of the moment.

Such a vector which is drawn to represent a couple is called the *axis of the couple*. Definition

§ 6. Resultant of coplanar couples: Theorem:

The resultant of any number of couples in the same plane on a rigid body is a single couple whose moment is equal to the algebraic sum of the moments of the several couples.

where $\hat{n}$ is called axis of the couple



If $\hat{n}$ is the unit vector perpendicular to the plane of a couple, such that $\vec{AB}, \vec{F}, \hat{n}$ form a right handed triad triangle Then the moment of the couple is $AB \cdot F \sin\theta\, \hat{n}$

Fig. 5

$= (AB \sin\theta) F \hat{n}$

$= PF \hat{n}$

Let (P_1, p_1), (P_2, p_2), (P_3, p_3) etc. be a number of couples acting in the same plane upon a body. Let AB represent the arm p_1 of the first couple (P_1, p_1) whose component forces P_1 act along AC and BD.

The moment of the second couple $(P_2, p_2) = P_2 . p_2$. This couple can be replaced by an equivalent couple, having its arm along AB and having its forces along AC and BD.

If F is the force of such a replacing couple,

we have $F . p_1 = P_2 p_2$

$$\therefore F = \frac{P_2 p_2}{p_1}$$

Thus the couple (P_2, p_2) is replaced by another couple whose arm coincides with AB and whose component forces along AC and BD are of magnitude $\frac{P_2 p_2}{p_1}$.

Similarly the couple (P_3, p_3) is replaced by a couple $\left(\frac{P_3 p_3}{p_1}, p_1\right)$ with the forces $\frac{P_3 p_3}{p_1}$ along AC and BD. This process is repeated for the other couples.

Finally, we get a single couple with the arm AB, each of whose component forces

$$= P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \ldots\ldots$$

The moment of this resultant couple

$$= \left(P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \ldots\ldots\right) \times p_1$$

$$= P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots\ldots$$

= the algebraic sum of the moments of the several couples.

Note: (i) If all the component couples have not the same sign, we have merely to give each its proper sign and the same proof will apply.

(ii) If all the couples do not lie in the same plane but in different parallel planes, they can all be transferred into equivalent couples in one plane parallel to the given planes and then their resultant can be found.

§ 7. Resultant of a couple and a force: Theorem.

A couple and a single force acting on a body cannot be in equilibrium but they are equivalent to the single force acting at some other point parallel to its original direction.

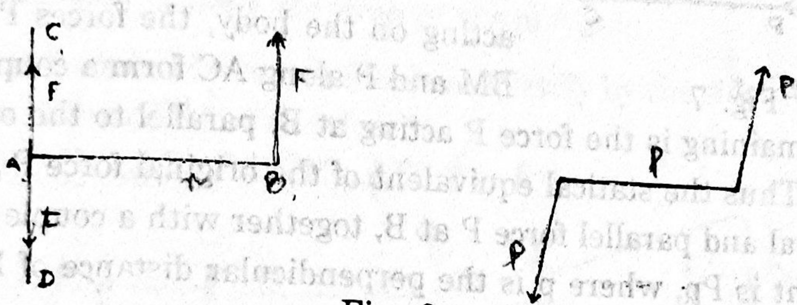


Fig. 6

Let the given couple be (P, p) and the given force be F lying in the same plane. Let F act along AC.

Replace the couple (P, p) by another couple whose each force is equal to F. If x be the length of the arm of this new couple, its moment = F.x = Pp.

$$\therefore\ x = \frac{Pp}{F}$$

Place this couple such that one of its component forces F acts at A along the line of action of the given force F but in the opposite direction i.e. it acts along AD. The original force F along AC and the force F along AD balance. We are left with a force F acting at B parallel to AC, as the statical equivalent of the system.

Also $AB = x = \dfrac{Pp}{F}$

Hence the couple (P, p) and the force F are equivalent to an equal force F, parallel to its original direction, at a distance $\frac{Pp}{F}$ from its original line of action.

5m

§ 8. **Theorem:** *A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body, together with a couple.*

Let P be a force acting at A along AC and B any arbitrary point. Let p be the distance of B from AC.

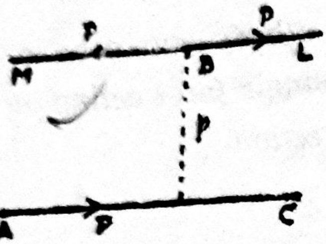


Fig. 7

At B, apply two equal and opposite forces each equal and parallel to P along BL and BM. These two new forces being equal and opposite, will have no effect on the body. Of the three forces now acting on the body, the forces P along BM and P along AC form a couple and the remaining is the force P acting at B, parallel to the original force. Thus the statical equivalent of the original force P at A is an equal and parallel force P at B, together with a couple whose moment is Pp, where p is the perpendicular distance of B from AC.

Note: The moment of the couple is equal to the moment of the original force at A about B.

memory Theorey ?

§ 9. **Theorem:** *If there forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is twice the area of the triangle.*

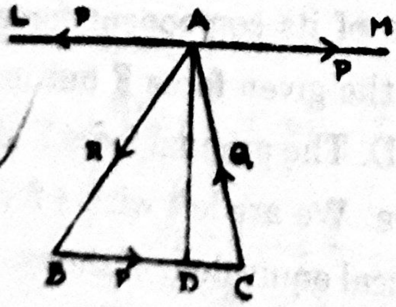


Fig.8

Let, P, Q, R be three forces acting on a rigid body and represented in magnitude, direction and line of action by the sides BC, CA, AB of the triangle ABC taken in order. Through A draw LAM parallel to BC and AD perpendicular to BC. At A, along AL and AM introduce two equal and opposite forces, each equal to P. These two new forces, being equal and opposite, have no effect on the body.

Now the three forces P along AM, Q along CA, and R along AB act at the point A and they are completely represented by the sides of the ΔABC taken in order. Hence, by the triangle of forces, they are in equilibrium. We are left with a force P along AL and a force P along BC. These being two equal and opposite forces form a couple whose moment ½bh

$$= P.AD = BC \,.\, AD = 2\,\Delta ABC.$$

§ 10 Theorem: *If any number of forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a polygon taken in order, they are equivalent to a couple whose moment is twice the area of the polygon.*

Let the forces be represented completely by the sides AB, BC, CD, DE, EF, and FA of the closed polygon ABCDEF. Join AC, AD and AE.

Introduce along AC, AD and AE, pairs of equal and opposite forces represented completely by these lines. These new forces do not affect the resultant of the system.

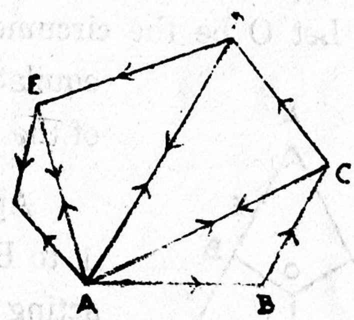


Fig. 9

Applying the theorem of § 9, we have

$\overline{AB} + \overline{BC} + \overline{CA}$ = a couple whose moment is equai to $2\Delta ABC$.

$\overline{AC} + \overline{CD} + \overline{DA}$ = a couple whose moment is equal to $2\Delta ACD$.

$\overline{AD} + \overline{DE} + \overline{EA}$ = a couple whose moment is equal to $2\Delta ADE$.

$\overline{AE} + \overline{EF} + \overline{FA}$ = a couple whose moment is equal to $2\Delta AEF$.

Adding vectorially,

$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FA}$

= resultant of the four couples

= a single couple whose moment is equal to

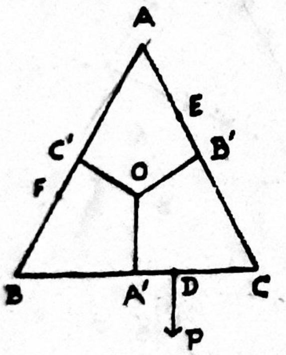
$2(\Delta ABC + \Delta ACD + \Delta ADE + \Delta AEF)$

i.e. The resultant is a couple whose moment is equal to twice the area of the polygon ABCDEF.

WORKED EXAMPLES

Ex. 1. *ABC is an equilateral triangle of side a: D.E.F divide the sides BC, CA, AB respectively in the ratio 2:1. Three forces ecch equal to P act at D, E, F perpendicularly to the sides and outward from the triangle. Prove that they are equivalent to a couple of moment 1/2 Pa.* (B.Sc. 81 M.U.)

Let O be the circumcentre (also the orthocentre) of the equilateral Δ and A', B', C' the middle points of the sides. OA' is ⊥ to BC.



Applying § 8, the force P acting at D ⊥ to BC is equivalent to a parallel force P acting at O along OA' together with a couple whose moment

$$= P.A'D = P.(A'C - DC)$$

$$= P.\left(\frac{a}{2} - \frac{a}{3}\right) = \frac{Pa}{6}$$

Fig. 10

Similarly, the force P acting at E ⊥ to CA is replaced by a parallel force P acting at O along OB' together with a couple whose moment $= \frac{Pa}{6}$

The force P acting at F ⊥ to AB is replaced by a parallel force P acting at O along OC' together with a couple whose moment $= \frac{Pa}{6}$

The three equal forces P acting O ⊥ to the sides of the triangle are in equilibrium by the perpendicular triangle of forces.

The three couples having the same moment $\frac{Pa}{6}$ each in the same direction are equivalent to a single couple whose moment $= 3 \times \frac{Pa}{6} = \frac{Pa}{2}$

Ex. 2. *Five equal forces ac alongthe sides AB, BC, CD, DE, EF of a regular hexagon. Find the sum of the moments of these forces about a point Q on AF at a distance x from A. Interpret the result and explain why it is so.* (3)

very important

|| CD

(B.Sc. 44)

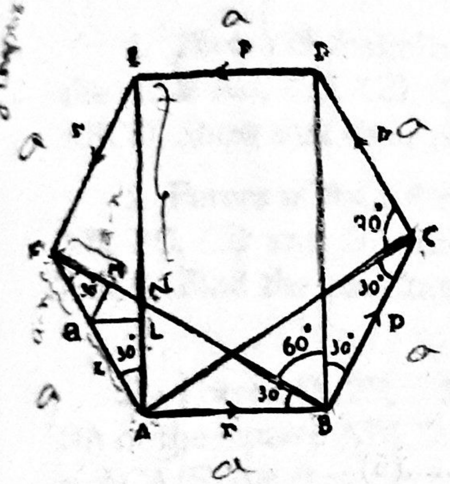


Fig. 11

Let a be the length of each side of the regular hexagon. Each interior angle of the regular hexagon $=120°$

We know that AB || DE, BC || EF and DC || AF, FB ⊥ BC, AE and DB are ⊥ to AB.

Let equal force P act along the sides AB, BC, CD, DF and EF. Q is a point on AF such that AQ = x.

From Q, draw QL ⊥ to EA and QM ⊥ to BF.

Let AN be ⊥ to BF. $\cos 30° = \frac{FN}{FA}$

$FB = FN + NB = a\cos 30° + a\cos 30° = 2a\cos 30°$ $FA\cos 30° = FN$

$AC = AE = BF = 2a\cos 30°$ $a\cos 30° = FN$

Moment of P along AB about Q

AB $= P.AL = P.\,x\cos 30°$ (from rt. $\angle d\Delta AQL$)

$$= P.x\frac{\sqrt{3}}{2}\text{.........}(1)$$

(cont)

Moment of P along BC about Q

$= P.MB = P\,(FB - FM)$

$= P\,[2a\cos 30° - (a-x)\cos 30°]$

$= P(2a - a + x)\cos 30^\circ$

$= P(a + x)\frac{\sqrt{3}}{2}$ ... ... (2)

Moment of P along CD about Q

$= P.AC$ ($\because AF \parallel CD$ and AC is $\perp$ to CD)

$= P.2a\cos 30^\circ = P.2a\frac{\sqrt{3}}{2} = Pa\sqrt{3}$ ... (3)

Moment of P along DE about Q

$= P.EL = P(AE - AL)$

$= P(2a\cos 30^\circ - x\cos 30^\circ)$

$= P(2a - x)\frac{\sqrt{3}}{2}$ ... ... (4)

Moment of P along EF about Q

$= P.MF = P(a - x)\cos 30^\circ$

$$= P(a - x)\frac{\sqrt{3}}{2}\text{.........}(5)$$

AF || CD

AC ⊥ to CD

Adding up, the sum of the moments of the five forces about Q

$$= Px\frac{\sqrt{3}}{2} + P(a+x)\frac{\sqrt{3}}{2} + Pa\sqrt{3} + P(2a - x)\frac{\sqrt{3}}{2} + P(a-x)\frac{\sqrt{3}}{2}$$

$$= P\frac{\sqrt{3}}{2}(x + a + x + 2a - x + a - x)$$

$$= P\frac{\sqrt{3}}{2}6a = 3Pa\sqrt{3} = \text{a constant, independent of } x.$$

The sum of the moments of the five forces about any point on the sixth side AF is constant.

Introduce two equal and opposite forces, each equal to P along the sixth side. These new forces do not affect the resultant of the system. We have now seven forces. The moment of the new force P introduced along AF about Q is = 0

The other six forces act along the sides of the hexagon and are represented in magnitude, direction and line of action by the sides of the hexagon.

Hence by § 10, they are equivalent to a couple whose moment is

$= 2 \times$ area of the hexagon

$= 2 \times 6 \times a^2 \frac{\sqrt{3}}{4}$

$= 3\, a^2\sqrt{3} = 3a\sqrt{3}P$ (as P is represented in magnitude by a)