CHAPTER 6

INTEGRAL TRANSFORM AND GREEN FUNCTION METHODS

6.1 Introduction

The integral transforms are essentially a mathematical tool, which can be used to solve several problems in science and engineering. Linear integral transforms of functions, $f(t)$, defined on a finite or infinite interval $a < t < b$ are particularly useful in dealing with problems in linear differential equations. A general linear integral transformation of a function $f(t)$ is represented by the equation

$$\bar{f}(s) = T\{f(t)\} = \int_a^b k(s,t)f(t)\,dt \tag{6.1.1}$$

where $k(s, t)$ is called its kernel. It represents a function $\bar{f}(s)$, the image or transform of the function $f(t)$.

The kernels and limits of integration of some of the integral transforms are given in table 6.1

Table 6.1: Integral transforms and their kernels

Name of Transform	k(s, t)	a	b
Laplace transform	e^{-st}	0	∞
Fourier transform	$\frac{1}{\sqrt{2\pi}}e^{ist}$	$-\infty$	∞
Fourier sine transform	$\sqrt{\frac{2}{\pi}}\sin st$	0	∞
Fourier cosine transform	$\sqrt{\frac{2}{\pi}}\cos st$	0	∞
Hankel transform	$t\,J_n(st)$	0	∞
Mellin transform	t^{s-1}	0	∞

The integral transforms defined above are applicable, either for semi-infinite or infinite domains. Similarly, finite integral transforms can be defined on finite domains.

Theorem 6.1 : Prove that the application of an integral transform to a partial differential equation reduces the independent variables by one.

Proof : Suppose we want to solve the partial differential equation of the type

$$A(x)\frac{\partial^2 f}{\partial x^2}+B(x)\frac{\partial f}{\partial x}+C(x)f+Lf=\phi \qquad (6.1.1)$$

where f and ϕ are functions of the independent variables $x, y, z, \ldots$ and L is a linear differential operator in the variables $y, z, \ldots$ but not in x. Let $k(x, s)$ be the kernel of the integral transform.

Now

$$kA\frac{\partial^2 f}{\partial x^2}-f\frac{\partial^2 (Ak)}{\partial x^2}=\frac{\partial}{\partial x}\left(Ak\frac{\partial f}{\partial x}\right)-\frac{\partial}{\partial x}\left[f\left(\frac{\partial (Ak)}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[Ak\frac{\partial f}{\partial x}-f\frac{\partial (Ak)}{\partial x}\right]$$

$$kB\frac{\partial f}{\partial x}+f\frac{\partial (kB)}{\partial x}=\frac{\partial}{\partial x}(fkB)$$

$$kCf-f\,kC=0$$

Therefore

$$\int_a^b\left(A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial f}{\partial x}+Cf\right)k\,dx-\int_a^b f\left(\frac{\partial^2 (Ak)}{\partial x^2}-\frac{\partial (kB)}{\partial x}+kC\right)dx$$

$$=\left[Ak\frac{\partial^2 f}{\partial x^2}+B\frac{\partial (Ak)}{\partial x}+f\,kB\right]_a^b=g(s,y,z,\ldots)\text{ (say)} \qquad (6.1.2)$$

We choose the function k in such a way that

$$\frac{\partial^2 (Ak)}{\partial x^2}-\frac{\partial}{\partial x}(kB)+kC=\lambda k \qquad (6.1.3)$$

Then (6.1.2), becomes

$$\int_a^b\left(A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial f}{\partial x}+Cf\right)k\,dx=\int_a^b \lambda f\,k\,dx+g(s,y,z,\ldots)$$

$$=\lambda\bar{f}(s)+g(s,y,z\ldots) \qquad (6.1.4)$$

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$$=\lambda\bar{f}(s)+g(s,y,z\ldots) \qquad (6.1.4)$$

Multiplying (6.1.1) throughtout by k and integrating between the limits a to b. We choose k as given in (6.1.3) and then from (6.1.2) and (6.1.4), we have

$$L(\bar{f}) + \lambda\bar{f} + g(s, y, z, ...) = \bar{\phi}$$

Therefore, $$(L + \lambda)\bar{f} = \bar{\phi} - g(s, y, z, ...) = \bar{\Phi} \text{ (say)} \qquad (6.1.5)$$

Now (6.1.5) is a partial differential equation for $\bar{f}$ in the independent variables y, z,... but not in x. Thus we conclude that the application of an integral transform to a partial differential equation reduces the number of independent variables by one.

Hence the result.

6.2 Laplace Transforms

This transform was first introduced by Laplace (1749-1827), a French mathematician, in the year 1790 in his work on probability theorems. This technique became popular when Heaviside applied it to the solutions of the ordinary different equations representing problems in electrical engineering.

Definition 6.2.1: Let $f(t)$ be a piecewise continuous function. Then $f(t)$ is said to be of exponential order α if there exists a real and finite positive number M such that

$$\lim_{t\to\infty} |f(t)| e^{-\alpha t} \leq M$$

and we write

$$|f(t)| = 0(e^{\alpha t})$$

Definition 6.2.2: Let $f(t)$ be a continuous and single-valued function of the real variable t defined for all t, $0 < t < \infty$, and is of exponential order. Then the Laplace transform of $f(t)$ is defined as

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (6.2.1)$$

provided the integral on right hand exists. Here s is a parameter, real or complex and clearly $\bar{f}(s)$ is a function of s.

Inverse Laplace Transform Integral

We have $L\{f(t)\} = \bar{f}(s)$, which implies that $f(t) = L^{-1}\{\bar{f}(s)\}$, where L and L^{-1}

Respectively, represent Laplace and inverse Laplace transform operations. The transformed solutions of many problems yield results not contained in tables of Laplace transforms. What is needed, therefore is a so called 'first-principle' method of inverting a transformed solution that can be applied regardless of the availability of the tabulated values. To develop the needed inversion process, it is essential to extend our theory of the Laplace transform by letting s represent a complex variable and then an extension of the Cauchy integral formula provides the desired result.

From complex variables Cauchy's integral formula states that if $f(z)$ is an analytic function inside a closed curve C and if z_0 is a point within C, then

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} \tag{6.2.2}$$

An extension of this formula is stated below.

Theorem 6.2: Let $f(z)$ be analytic for $Re(z) \geq \gamma$, where γ is a real constant greater than zero. Then, for $Re(z_o)>\gamma$,

$$f(z_o)=+\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta}\frac{f(z)\,dz}{z-z_o} \tag{6.2.3}$$

Proof : This theorem can be proved by choosing a closed rectangular contour C $(C_1 + C_2 + C_3 + C_4)$ enclosing the point z_0 and having sides parallel to the real and imaginary axes as shown in Figure 6.2.1.

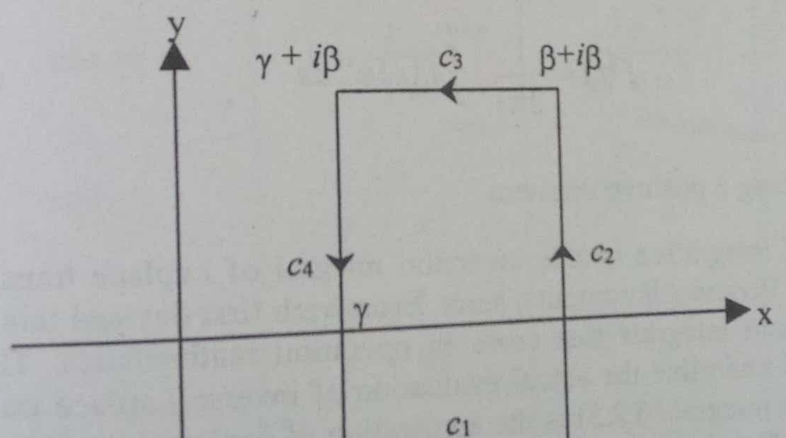

Figure 6.2.1

Then we have

$$f(z_0)=\frac{1}{2\pi i}\left[\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right]\frac{f(z)}{z-z_0}dz \tag{6.2.4}$$

By some limiting arguments and inequalities, it can be show-n that the contribution from the paths C_1, C_2 and C_3 vanishes on letting $\beta \to \infty$ and the expression (6.2.4) becomes

$$\bar{f}(s) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \frac{\bar{f}(z)}{s - z} dz \tag{6.2.5}$$

where z_0 has been replaced by $s = x + iy$ and $\bar{f}(s)$ is assumed to be analytic in the half – plane $Re(s) > \gamma$. Applying the inverse Laplace transform on both sides of this equation, we get

$$f(t) = L^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i} L^{-1}\left\{ \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \frac{\bar{f}(z)}{s - z} dz \right\}$$

$$= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \bar{f}(z) L^{-1}\left(\frac{1}{s - z} \right) dz$$

$$= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \bar{f}(s) ds \qquad \left[\because L^{-1}\left(\frac{L}{s - z} \right) = e^{zt} \right]$$

where the dummy variable z has been represented by s.

Thus, in general, Laplace transform inverse is given by

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s) e^{st} ds \tag{6.2.5}$$

for $t > 0$ and γ being a positive constant.

The path of integration in this inversion integral of Laplace transform, is often called the 'Bromwich contour', since Bromwich first devised this method of handling certain integrals that arose in operation mathematics. The basic concept that will underline the actual evaluation of inverse Laplace transforms by use of inverse integral (6.2.5) is the application of contour integration in the complex plane. The details of the calculations will depend on the nature of transformed function $\bar{f}(s)$. The numerical technique to evaluate the integral (6.2.5) is given in Appendix B.

Laplace transformation of some important elementary functions and their inverse transforms are given in Table 6.2.

Table 6.2: Table of some Laplace transforms

No.	$f(t)$	$\bar{f}(s)=L[f(t)]$	$\bar{f}(s)$	$f(t)=L^{-1}\{\bar{f}(s)\}$
1	0	0	0	0
2	1	$\frac{1}{s}$	$\frac{1}{s}$	1
3	t	$\frac{1}{s^2}$	$\frac{1}{s^2}$	t
4	t^n	$\frac{\Gamma(n)}{s^{n+1}}$	$\frac{\Gamma(n)}{s^{n+1}}$	t^n
5	e^{at}	$\frac{1}{s-a}$	$\frac{1}{s-a}$	e^{at}
6	$\sin at$	$\frac{a}{s^2+a^2}$	$\frac{a}{s^2+a^2}$	$\sin at$
7.	$\cos at$	$\frac{s}{s^2+a^2}$	$\frac{s}{s^2+a^2}$	$\cos at$
8.	$\sinh at$	$\frac{a}{s^2-a^2}$	$\frac{a}{s^2-a^2}$	$\sinh at$
9.	$\cosh at$	$\frac{s}{s^2-a^2}$	$\frac{s}{s^2-a^2}$	$\cosh at$
10.	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$	$\frac{2as}{(s^2+a^2)^2}$	$t\sin at$
11.	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
12.	$H(t-a)$	$\frac{e^{-as}}{s}, s>0$	$\frac{e^{-as}}{s}, s>0$	$H(t-a)$
13.	$f(t-a)H(t-a)$	$e^{-as}f(s)$	$e^{-as}f(s)$	$f(t-a)H(t-a)$

Continued ...

	$f(t)$	$\bar{f}(s) = L[f(t)]$	$\bar{f}(s)$	$f(t) = L^{-1}\{\bar{f}(s)\}$
14.	$\delta(t-a)$	$e^{-as}, a > 0$	$e^{-as}, a > 0$	$\delta(t-a)$
15.	$J_0(t)$	$\frac{1}{\sqrt{1+s^2}}$	$\frac{1}{\sqrt{1+s^2}}$	$J_0(t)$
16.	$TJ_0(t)$	$\frac{s}{(1+s^2)^{3/2}}$	$\frac{s}{(1+s^2)^{3/2}}$	$tJ_0(t)$
17.	$e^{-at}f(t)$	$f(s+a)$	$f(s+a)$	$e^{-at}f(t)$
18.	$T^n f(t)$	$(-1)^n \frac{d^n}{ds^n} f(s)$	$(-1)^n \frac{d^n}{ds^n} f(s)$	$t^n f(t)$
19.	$erf(t)$	$\frac{1}{s} e^{\frac{s^2}{4}} erf_c\left(\frac{s}{2}\right)$	$\frac{1}{s} e^{\frac{s^2}{4}} erf_c\left(\frac{s}{2}\right)$	$erf(t)$
20.	$\int_0^t f(t-u)g(u)du$	$f(s)g(s)$	$f(s)g(s)$	$\int_0^t f(t-u)g(u)\,du$
21.	$\frac{f(t)}{t}$	$\int_s^\infty f(s)ds$	$\int_s^\infty f(s)ds$	$\frac{f(t)}{t}$

6.3 Solution of Partial Differential Equations

A large number of problems in science and engineering involve the solution of linear partial differential equations. A function of two or more variables may also have a Laplace transform. Suppose that $u(x, t)$ is a function of two independent real variables x and t. When Laplace transform is applied with respect to the variable t, the partial differential equation is reduced to an ordinary differential equation of the t-transform $\bar{u}(x, s)$. The general solution of ordinary differential equation is then subjected to the boundary conditions of the original problem. Finally, the solution $u(x, t)$, is obtained by using the complex inversion formula (6.2.2). Thus the Laplace transform is specially suited to solving initial boundary value problems (IBVP), when conditions are prescribed at $t = 0$.

6.3.1 Diffusion Equation

Example 6.3.1 : Use Laplace transform method to solve the initial value problem

$$k\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < l, \quad 0 < t < \infty$$

subject to the conditions

$$u(0,t) = 0, \; u(l,t) = g(t), \; 0 < t < \infty$$

and

$$u(x,0) = 0, \; 0 < x < l$$

Solution : Applying Laplace transform w.r.t. t to the given partial differential equation, we get

$$kL\left[\frac{\partial u}{\partial t}\right] = L\left[\frac{\partial^2 u}{\partial x^2}\right]$$

or

$$k[s\bar{u}(x,s) - u(x,0)] = \frac{d^2\bar{u}}{dx^2}$$

or

$$\frac{d^2 u}{dx^2} - k s\bar{u} = 0, \text{ as } \bar{u}(x,0) = 0 \tag{6.3.1}$$

Therefore,

$$\bar{u}(x,s) = A\cosh\left(\sqrt{k}sx\right) + B\sinh\left(\sqrt{k}sx\right) \tag{6.3.2}$$

Taking Laplace transform of boundary conditions, we get

$$\bar{u}(0, s) = 0, \; \bar{u}(l, s) = \bar{g}(s) \tag{6.3.3}$$

Using equation (6.3.3) in (6.3.2), we get

$$A = 0$$

Thus

$$\bar{g}(s) = B\sinh\left(\sqrt{ksl}\right)$$

or

$$B = \frac{\bar{g}(s)}{\sinh\left(\sqrt{ks}\,l\right)}$$

Taking the inverse Laplace transform, we obtain

$$u(x, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{g}(s)e^{st}\frac{\sinh\left(\sqrt{ks}\,x\right)}{\sinh\left(\sqrt{ks}\,l\right)}ds \tag{6.3.4}$$

To evaluate the integral on the right hand side of equation (6.3.4), we use the method of residues. The poles of the integral are given by

$$\sinh\left(\sqrt{ks}\,l\right)=0 \quad \text{or} \qquad e^{\sqrt{ks}\,l}-e^{-\sqrt{ks}\,l}=0$$

or
$$e^{2\sqrt{ks}\,l}=1=e^{2n\pi i}$$

or
$$s_n=-\frac{n^2\pi^2}{kl^2},\ n=0,\ \pm 1,\ \pm 2,\ \pm 3,...$$

Now by Cauchy Residue theorem, we have

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{g}(s)e^{st}\frac{\sinh\left(\sqrt{ks}\,x\right)}{\sinh\left(\sqrt{ks}\,l\right)}ds=\sum R_i$$

where R_i = residue at the ith pole.

Here residue at $s = 0$, is zero. The residue at $s = s_n$ is

$$\lim_{s\to s_n}\ \bar{g}(s)\frac{e^{st}\sinh\left(\sqrt{ks}\,x\right)}{\frac{d}{ds}\left[\sinh\left(\sqrt{ks}\,l\right)\right]}=\lim_{s\to s_n}\frac{2\sqrt{s}\,\bar{g}(s)}{l\sqrt{k}\left[\cosh\left(\sqrt{ks}\,l\right)\right]}e^{st}\sinh\left(\sqrt{ks}\,x\right)$$

$$=\frac{2\sqrt{-\frac{n^2\pi^2}{kl^2}}\,\bar{g}\left(-\frac{n^2\pi^2}{kl^2}\right)}{l\sqrt{k}\cosh\left(-\frac{n^2\pi^2}{kl^2}l\right)}e^{-\frac{n^2\pi^2}{kl^2}}\sinh\left(\sqrt{\left(-\frac{n^2\pi^2}{l^2}\right)}x\right),\ n=1,\ 2,\ 3,...$$

$$=\frac{2in\pi}{kl^2}\frac{\bar{g}\left(\frac{-n^2\pi^2}{kl^2}\right)}{\cos(in\pi)}\sinh\left(\frac{in\pi}{l}x\right)e^{\frac{-n^2\pi^2t}{kl^2}},\quad n=1,2,3,...$$

Therefore,

$$u(x,\ t)=\frac{2n\pi(-1)^n}{kl^2}\sin\left(\frac{n\pi}{l}x\right)\bar{g}\left(\frac{-n^2\pi^2}{kl^2}\right)e^{-\frac{n^2\pi^2}{kl^2}t},\ n=1,\ 2,\ 3,...$$

as
$$\cosh(in\pi)=\cos n\pi \text{ and } \sinh\left(\frac{in\pi}{l}x\right)=i\sin\left(\frac{n\pi}{l}x\right)$$

Thus, the required solution is

$$u(x,\ t)=\frac{2\pi}{kl^2}\sum_{n=1}^{\infty}(-1)^n n\bar{g}\left(\frac{n^2\pi^2}{kl^2}\right)\sin\left(\frac{n\pi}{l}x\right)e^{\frac{-n^2\pi^2}{kl^2}t}$$

Example 6.3.2 : Find the solution of the boundary value problem given by

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t},\quad 0\le x\le a,\ t>0$$

subject to the initial and boundary conditions

$$u(x,0)=0,\ u(0,t)=f(t),\ u(a,t)=0$$

by using the Laplace transform method.

Solution : Taking the Laplace transform of the given partial differential equation, we have

$$\frac{d^2\bar{u}}{dx^2}=\frac{1}{k}\left[s\bar{u}(x,s)-u(x,0)\right]$$

Using the initial condition $u(x,0)=0$, we get

$$\frac{d^2\bar{u}}{dx^2}-\frac{s}{k}\bar{u}(x,s)=0$$

Its general solution is found to be

$$\bar{u}(x,s)=Ae^{\sqrt{(s/k)}x}+B^{-\sqrt{(s/k)}x} \tag{6.3.5}$$

Now taking Laplace transform of the boundary conditions, we have

$$\bar{u}(0,s)=\bar{f}(s) \tag{6.3.6}$$

$$\bar{u}(a,s)=0 \tag{6.3.7}$$

Using equations (6.3.6) and (6.3.7) into (6.3.5), we get

$$\bar{f}(s)=A+B \tag{6.3.8}$$

and

$$\bar{u}(a,s)=A\exp\left[\sqrt{\left(\frac{s}{k}\right)}a\right]+B\exp\left[-\sqrt{\left(\frac{s}{k}\right)}a\right] \tag{6.3.9}$$

Combining equation (6.3.8) and (6.3.9), we have

$$\bar{f}(s)=A\left(1-e^{2a\sqrt{s/k}}\right)$$

Therefore
$$A=\bar{f}(s)/\left(1-e^{2a\sqrt{s/k}}\right)$$

and
$$B=\bar{f}(s)-\frac{\bar{f}(s)}{1-e^{2a\sqrt{s/k}}}=-\frac{\bar{f}(s)\,e^{2a\sqrt{s/k}}}{1-e^{2a\sqrt{s/k}}}$$

Which implies that

or
$$A=\bar{f}(s)\,e^{-a\sqrt{s/k}}\left[e^{-a\sqrt{s/k}}-e^{a\sqrt{s/k}}\right.$$

and $$B=\bar{f}(s)\, e^{a\sqrt{s/k}} \Big/ \left[e^{a\sqrt{s/k}} - e^{-a\sqrt{s/k}}\right]$$

Therefore $$\bar{u}(x,s)=\bar{f}(s)\frac{\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left[\sqrt{\frac{s}{k}}a\right]}$$

Taking inverse Laplace transform, we obtain

$$u(x,t)=L^{-1}\left[\bar{f}(s)\frac{\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left(\sqrt{\frac{s}{k}}a\right)}\right]$$

$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\bar{f}(s)\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left[\sqrt{\frac{s}{k}}a\right]}ds$$

6.3.2 Wave Equation

Example 6.3.3 : Using the Laplace transform method, solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\cos wt, \quad 0\le x\le\infty, \quad 0\le t\le\infty$$

subject to the initial and boundary conditions

$$u(0,t)=0, \qquad u \text{ is bounded as } x\to\infty$$

$$\frac{\partial u}{\partial t}(x,0)=u(x,0)=0$$

Solution : Taking the Laplace transform of the partial differential equation, we obtain

$$\frac{d^2\bar{u}}{dx^2}=\frac{1}{c^2}\left[s^2\bar{u}(x,s)-su(x,0)-\frac{\partial u(x,0)}{\partial t}\right]-\frac{s}{s^2+w^2}$$

Using the initial conditions, we get

$$\frac{d^2\bar{u}}{dx^2}-\frac{s^2}{c^2}\bar{u}(x,s)=-\frac{s}{s^2+\omega^2} \qquad (6.3.10)$$

The general solution of (6.3.10) is found to be

$$\bar{u}(x, s) = Ae^{(s/c)x} + Be^{-(s/c)x} + \frac{c^2}{s(s^2+\omega^2)} \tag{6.3.11}$$

As $x \to \infty$, the transform should also be bounded which is possible if $A = 0$, thus

$$\bar{u}(x, s) = Be^{-\frac{sx}{c}} + \frac{c^2}{s(s^2+\omega^2)}$$

Taking the Laplace transform of the boundary condition, we get

$$\bar{u}(0, s = 0)$$

Using this result in equation (6.3.11), we have

$$B = -\frac{c^2}{s(s^2+\omega^2)}$$

Hence

$$\bar{u}(x, s) = \frac{c^2}{s(s^2+\omega^2)}\left[1 - e^{-(s/c)x}\right]$$

Taking its inverse Laplace transform, we get

$$u(x, t) = c^2 L^{-1}\left[\frac{1}{s(s^2+\omega^2)}\right] - c^2 L^{-1}\left[\frac{e^{-(s/c)x}}{s(s^2+\omega^2)}\right]$$

$$= \frac{c^2}{\omega^2}(1 - \cos wt) - \frac{c^2}{\omega^2}\left[\left\{1 - \cos w\left(t - \frac{x}{c}\right)\right\} H\left(t - \frac{x}{c}\right)\right]$$

where H is the Heaviside unit function.

This is the required solution.

Example 6.3.4 : Solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \ t > 0$$

subject to the initial and boundary conditions

$$u(x, 0) = \sin \pi x, \ \frac{\partial u}{\partial t}(x, 0) = -\sin \pi x, \ 0 < x < 1$$

and

$$u(0, t) = u(1, t) = 0, \quad t > 0$$

Solution : Taking the Laplace transform of the partial differential equation, we get

$$\frac{d^2\bar{u}}{dx^2} = s^2\bar{u}(x, s) - su(x, 0) - \frac{\partial u}{\partial t}(x, 0)$$

Using the initial conditions, this equation becomes

$$\frac{d^2\bar{u}}{dx^2} - s^2\bar{u} = (1-s)\sin \pi x$$

Its general solution is given by

$$\bar{u}(x, s) = Ae^{sx} + Be^{-sx} + \frac{(s-1)\sin \pi x}{\pi^2 + s^2} \tag{6.3.12}$$

The Laplace transform of the boundary conditions gives

$$\bar{u}(0, s) = 0 = \bar{u}(1, s)$$

Using these equations (6.3.12), we find that

$$A = 0 = B$$

Hence, we obtain

$$\bar{u}(x, s) = \frac{(s-1)\sin \pi x}{\pi^2 + s^2}$$

Taking the inverse Laplace transform, we get

$$u(x, t) = \sin \pi x \; L^{-1}\left(\frac{s-1}{\pi^2 + s^2}\right) = \sin \pi x\left(\cos \pi x - \frac{\sin \pi x}{\pi}\right)$$

Hence the required solution of the given boundary value problem is

$$u(x, t) = \sin \pi x\left[\cos \pi x - \frac{\sin \pi x}{\pi}\right]$$

Example 6.3.5 : A string is stretched and fixed between two points (0, 0) and (l, 0). Motion is initiated by displacing the string in the form $u = \sin\left(\frac{\pi x}{l}\right)$ and released from rest at time $t = 0$. Find the displacement of any point on the string at any time t.

Solution : The displacement $u(x, t)$ of the string is governed by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

subject to the conditions

$$u(x,0)=\sin(\pi x/l),\ \frac{\partial u}{\partial t}(x,0)=0$$

$$u(0,t)=u(l,t)=0$$

Taking the Laplace transform of the given partial differential equation, we get

$$s^2\bar{u}(x,s)-su(x,0)-\frac{\partial u}{\partial t}(x,0)=c^2\frac{d^2\bar{u}}{dx^2}$$

Using the initial conditions, we get

$$\frac{d^2\bar{u}}{dx^2}-\frac{s^2}{c^2}\bar{u}=-\frac{s}{c^2}\sin\frac{\pi x}{l}$$

Its general solution is given by

$$\bar{u}(x,s)=Ae^{\frac{sx}{c}}+Be^{\frac{-sx}{c}}+\frac{s\sin(\pi x/l)}{s^2+\pi^2c^2/l^2} \qquad (6.3.13)$$

The Laplace transform of the boundary conditions is given by

$$\bar{u}(0,s)=0,\ \bar{u}(l,s)=0$$

Applying these conditions in equation (6.3.13), we get

$$A+B=0$$

$$A^{sl/c}+Be^{-sl/c}=0$$

Solving these equations, we obtain

$$A=B=0$$

Thus
$$\bar{u}(x,s)=\frac{s\sin(\pi x/l)}{s^2+\pi^2c^2/l^2}$$

Taking inverse of the Laplace transform, we get

$$u(x,t)=\cos\left(\frac{\pi c}{l}t\right)\sin\frac{\pi x}{l},$$

which is the required solution.

Example 6.3.6 : Find the solution of the partial differential equation

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},\quad 0<x<\infty,\quad t>0$$

subject to the initial and boundary conditions

$$u(x,0)=0,\ u(x,t)\to 0,\ as\ x\to\infty,$$

$$u(0, t) = g(t)$$

Solution : Taking the Laplace transform of the partial differential equation, we have

$$k\frac{d^2\bar{u}}{dx^2} = s\bar{u}(x, s) - \bar{u}(x, 0)$$

Using the initial condition $u(x, 0) = 0$, we get

$$k\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} = 0$$

Its general solution is given by

$$\bar{u}(x,s) = A\exp\left(\sqrt{\frac{s}{k}}x\right) + B\exp\left(-\sqrt{\frac{s}{k}}x\right)$$

The Laplace transform of the first boundary condition gives,

$$\bar{u} \to 0, \text{ as } x \to \infty \text{ and } \bar{u}(0, s) = \bar{g}(s)$$

Using these in the solution, we get

$$A = 0, \quad B = \bar{g}(s)$$

therefore,

$$\bar{u}(x,s) = \left\{\bar{g}(s)\exp\left(-\sqrt{\frac{s}{k}}x\right)\right\}$$

Taking the inverse Laplace transform, we obtain

$$u(x, t) = L^{-1}\left\{\bar{g}(s)\exp\left(-\sqrt{\frac{s}{k}}x\right)\right\}$$

$$= L^{-1}\left\{L\{g(t)\}L\left[\frac{x}{2\sqrt{k\pi t^3}}\exp(-x^2/4kt)\right]\right\}$$

since
$$L^{-1}\left\{\exp\sqrt{\frac{s}{k}}x = \frac{x}{2\sqrt{k\pi t^3}}\exp\left(\frac{-x^2}{4kt}\right)\right\}$$

Upon using convolution theorem, we arrive at the result

$$u(x, t) = \int_0^t \frac{x\exp\left[-x^2/4k(t-u)\right]}{2\sqrt{\pi k}(t-u)^{3/2}}\bar{g}(u)du$$

which is the required solution.

Example 6.3.7 : Using the Laplace transform method, solve

$$\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2}$$

subject to the initial and boundary conditions

$$u(x,0)=30\cos 5x,\ u\left(\frac{\pi}{2},0\right)=0, \frac{\partial u}{\partial x}(0,t)=0$$

Solution : Taking the Laplace transform of partial differential equation, we have

$$s\bar{u}(x,s)-u(x,0)=3\frac{d^2\bar{u}}{dx^2}$$

Using the initial condition, we get

$$\frac{d^2\bar{u}}{dx^2}-\frac{s}{3}\bar{u}=-10\cos 5x$$

Its general solution is given by

$$\bar{u}(x,s)=Ae^{\sqrt{\frac{s}{3}}x}+Be^{-\sqrt{\frac{s}{3}}x}+\frac{30\cos 5x}{75+s}$$

Taking Laplace Transform of boundary condition and using in the solution, we obtain $A=B=0$. Thus

$$\bar{u}(x,s)=\frac{30\cos 5x}{75+s}$$

Taking inverse Laplace transform, we get

$$u(x,t)=L^{-1}\left(\frac{30\cos 5x}{75+s}\right)=30e^{-75t}\cos 5x$$

which is the required solution.

6.4 Fourier Transforms and their Application to Partial Differential Equations

Joseph Fourier, a French mathematician, had invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light and it can also identify the regular contributions to a fluctuating signal, thereby helping to make sense of observations in astronomy, medicine and chemistry. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze mechanical vibrations, and to study wave propagation. Fourier transform techniques have been widely used to solve problems involving semi-infinite of totally infinite range of the variables or unbounded regions. In this section, we deal with some applications to diffusion, wave and Laplace equations. The Fourier and Hankel transforms of some important functions are given in Appendix A.

6.4.1 Diffusion Equation

Let us consider the problem of flow of heat in an infinite medium $-\infty < x < \infty$, when the initial temperature distribution $f(x)$ is known and no heat sources are present.

Example 6.4.1: Solve the heat conduction equation given by

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$$

subject to the initial and boundary conditions

$$u(x, t) \text{ and } \frac{\partial u}{\partial x}(x, t) \to 0, \text{ as } |x| \to \infty$$

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

Solution : Taking the Fourier transform of the partial differential equation, we get

$$\frac{\partial \bar{u}}{\partial t}(\alpha, t) + k\alpha^2 \bar{u}(\alpha, t) = 0 \tag{6.4.1}$$

since $\quad \bar{u}(\alpha, t) \text{ and } \frac{d}{dx}\bar{u}(\alpha, t) \to 0$

The Fourier transform of initial condition gives us

$$\bar{u}(\alpha, 0) = \bar{f}(\alpha), \quad -\infty < \alpha < \infty$$

The general solution of equation (6.4.1), is given by

$$\bar{u} = Ae^{-k\alpha^2 t}$$

Using the transformed initial condition in this solution, we get

$$.\bar{u}(\alpha, t) = \bar{f}(\alpha)\, e^{-k\alpha^2 t}$$

Taking inverse Fourier Transform, we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{-i\alpha x - k\alpha^2 t}\, d\alpha \tag{6.4.2}$$

We know that

$$\int_{-\infty}^{\infty} \exp\left(-ax^2 - 2bx\right) dx = \sqrt{\frac{\pi}{a}} \exp\left(b^2 / a\right)$$

Therefore

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k\alpha^2 t} e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k\alpha^2 t - i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{kt}} \exp\left(\frac{x^2}{4kt}\right) = \frac{1}{\sqrt{2kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

since here $a = kt$ and $b = \dfrac{ix}{2}$ and the Fourier transform of $g(x)$ is $e^{-k\alpha^2 t}$

Therefore, using convolution theorem to (6.4.2), we have

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x - \alpha)\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) \frac{1}{\sqrt{2kt}} \exp\left[\frac{-(x-\alpha)^2}{4kt}\right] d\alpha$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(\alpha) \exp\left[-\frac{(x-\alpha)^2}{4kt}\right] d\alpha$$

$$\therefore \quad u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(x + \sqrt{4kt}\, z\right) e^{-z^2} dz$$

where we have put $z = (\alpha - x) / \sqrt{4kt}$.

which is the required solution.

Example 6.4.2 : Solve the heat conduction problem described by

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \infty, \quad t > 0$$

subject to the initial and boundary conditions

$$u(x, 0) = 0, \quad 0 < x < \infty, u(0, t) = u_0, \quad t \geq 0,$$

u and $\dfrac{\partial u}{\partial t}$ both tend to zero as $x \to \infty$.

Solution : Because u is specified at $x = 0$, the Fourier sine transform is applicable to this problem. Taking Fourier sine transform of the partial differential equation and applying the boundary condition $u(0, t) = u_0,\ t \geq 0$, we get

$$\frac{\partial \bar{u}}{\partial t} + k\alpha^2 \bar{u} = \sqrt{\frac{2}{\pi}} k\alpha u_0, \quad \bar{u}(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, t) \sin \alpha x \, dx$$

Its general solution is given by

$$\bar{u}(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\alpha} \left(1 - e^{-k\alpha^2 t}\right) \tag{6.4.3}$$

Here, we have used the initial conditions and rest of the boundary conditions in order to derive the solution (6.4.3). Taking the inverse Fourier sine transform, we obtain

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}(\alpha, t) \sin \alpha x \, d\alpha$$

$$= \frac{2}{\pi} u_0 \int_0^\infty \frac{\sin \alpha x}{\alpha} \left(1 - e^{-k\alpha^2 t}\right) d\alpha$$

$$= \frac{2u_0}{\pi} \left[\frac{\pi}{2} - \frac{\pi}{2} erf\left(\frac{x}{\sqrt{2kt}}\right)\right]$$

where we have used the standard integral

$$\int_0^\infty e^{-\alpha^2} \frac{\sin(2\alpha y)}{\alpha} d\alpha = \frac{\pi}{2} erf(y)$$

and

$$erf(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-u^2)\, du.$$

Thus, the solution of the heat conduction equation is

$$u(x, t) = u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{2kt}}} \exp(-u^2)\, du\right] = u_0\, erf_c\left(\frac{x}{\sqrt{2kt}}\right)$$

6.4.2 Wave Equation

Wave motion that occur in nature, viz sound waves, surface waves, transverse waves of an infinite string, and of mechanical systems are governed by the wave equation.

Example 6.4.3 : Displacement of an infinite string is governed by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty$$

subjected to initial conditions

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}(x, 0) = 0$$

Solution : Taking the Fourier transform of partial differential equation, we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{i\alpha x} dx = \frac{c^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx$$

or

$$\frac{d^2 \bar{u}}{dt^2} + c^2 \alpha^2 \bar{u} = 0$$

Its general solution is given by

$$\bar{u}(\alpha, t) = a\cos(c\alpha t) + B\sin(c\alpha t) \qquad (6.4.4)$$

The Fourier transform of initial conditions gives

$$\frac{d\bar{u}(\alpha, t)}{dt} = 0, \quad \bar{u} = \bar{f}(\alpha)$$

Using these conditions in (6.4.4), we get

$$A = \bar{f}(\alpha) \quad \text{and} \quad B = 0$$

Therefore $$\bar{u}(\alpha, t) = \bar{f}(\alpha)\cos(c\alpha t)$$

Taking its inverse Fourier transform, we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha)\cos(c\alpha t) e^{-i\alpha x} d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,e^{i\alpha u}\,du\right]\frac{e^{ic\alpha t}+e^{-ic\alpha t}}{2}\,e^{-i\alpha x}\,d\alpha$$

$$=\frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\,\bar{e}^{isu}\,du\right\}\left(e^{-icst}+e^{icst}\right)e^{isx}\,ds\right]$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x-ct)}\,ds\right\}du\right.$$

$$\left.+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x+ct)}\,ds\right)du\right]$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}F(x-ct)\,du+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}F(x+ct)\,du\right]$$

Thus by using Fourier integral formula, we get

$$u(x,t)=\frac{1}{2}[\,f(x+ct)+f(x-ct)]$$

which is the well known D'Alemberts solution of the wave equation.

Example 6.4.4 : A uniform string of length L is stretched tightly between two fixed points at $x=0$ and $x=l$. If it is displaced a small distance d at a point $x=b,\ 0<b<l$, and released from rest at time $t=0$, find an expression for the displacement at subsequent times.

Solutions : Let $u(x,t)$ denote the displacement of the string. Then, the initial boundary value problem is described by

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

subjected to the initial and boundary conditions given by

$$u(x,0)=\begin{cases}\dfrac{dx}{b}, & 0\le x\le b\\[2mm] \dfrac{d(x-l)}{b-l}, & b\le x\le l\end{cases}$$

$$\frac{\partial u}{\partial t}(x,0)=0 \text{ and } u(0,t)=u(l,t)=0,\quad t\ge 0$$

Taking finite Fourier sine transform of the partial differential equation, we get

$$\int_0^l \frac{\partial^2 u}{\partial t^2} \sin\frac{n\pi x}{l}\,dx = c^2 \int_0^l \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{l}\,dx$$

which on using boundary conditions leads to

$$\frac{d^2\bar{u}}{dt^2} + \frac{n^2\pi^2c^2}{l^2}\bar{u} = 0$$

Its general solution is found to be

$$\bar{u}(n, t) = A\cos\frac{n\pi ct}{l} + B\sin\frac{n\pi ct}{l} \tag{6.4.5}$$

Now, taking finite Fourier sine transform of initial conditions, we obtain

$$u(n, 0) = \int_0^b \frac{dx}{b}\sin\frac{n\pi x}{l}\,dx + \int_b^l \frac{d(x-l)}{b-l}\sin\frac{n\pi x}{l}\,dx = \frac{dl^3}{n^2\pi^2 b(l-b)}\sin\frac{n\pi b}{l}$$

From equation (6.4.5) when $t = 0$, we have

$$\bar{u}(n, 0) = A = \frac{dl^3}{n^2\pi^2 b(l-b)}\sin\frac{n\pi b}{l}$$

Also, taking finite Fourier sine transform of the second initial condition, we get

$$\frac{d\bar{u}}{dt} = 0$$

From equation (6.4.5) it can be easily seen that $B = 0$.

Thus, we obtain

$$\bar{u}(n, t) = \frac{dl^3}{\pi^2 b(l-b)}\sin\frac{n\pi b}{l}\cos\frac{n\pi ct}{l}$$

Finally, inverting the Fourier transform, we get

$$u(x, t) = \frac{2dl^2}{n^2\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

which is the required solution.

6.4.3 Laplace Equation

One of the most important partial differential equation that occurs in many scientific and engineering applications is the Laplace equation. Steady flow of currents in solid conductors, the velocity potential of inviscid irrotational fluids, the gravitational potential at an exterior point due to ellipsoidal Earth and so on, are all governed by Laplace equation. We shall now consider a few related examples.

Example 6.4.5 : Solve the boundary value problem in the half-plane $y > 0$, described by

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, \qquad y > 0$$

subjected to the boundary conditions

u is bounded as $y \to \infty$, u and $\dfrac{\partial u}{\partial x}$ both vanish, as $|x| \to \infty$.

Solution : Since x has an infinite range of values, we take the Fourier exponential transform of partial differential equation w.r.t. the variable x and use the conditions that u and $\dfrac{\partial u}{\partial x}$ both vanish, as $|x| \to \infty$, to get

$$-\alpha^2 \bar{u}(\alpha, y) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{yy} e^{i\alpha x} dx = 0$$

or

$$\frac{d^2 \bar{u}(\alpha, y)}{dy^2} - \alpha^2 \bar{u}(\alpha, y) = 0 \qquad (6.4.6)$$

Its general solution is found to be

$$\bar{u}(\alpha, y) = Ae^{\alpha y} + Be^{-\alpha y} \qquad (6.4.7)$$

Because u must be bounded as $y \to \infty$, $\bar{u}(\alpha, y)$ is also bounded as $y \to \infty$, implying that $A = 0$, for $\alpha > 0$ and $B = 0$, for $\alpha < 0$. Thus for any α,

$$\bar{u}(\alpha, y) = Ce^{-|\alpha| y} \qquad (6.4.8)$$

C being a constant.

Now the Fourier transform of the boundary condition yields

$$\bar{u}(\alpha, 0) = \bar{f}(\alpha)$$

which upon using in (6.4.8), gives us

$$f(x) = C$$

Hence

$$\bar{u}(\alpha, y) = f(\alpha) e^{-|\alpha| y} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-|\alpha| y} e^{i\alpha x} dx$$

Taking inverse Fourier transform, we obtain

$$u(x, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\, e^{-|\alpha| y} e^{i\alpha\xi}\, d\xi\right] e^{-i\alpha x}\, d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty}\exp\left[\left\{\alpha\left[i(\xi-x)\right]-|\alpha| y\right\}\right] d\alpha$$

$$=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\, d\xi}{(\xi-x)^2+y^2}$$

since $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{\alpha[i(\xi-x)]-|\alpha| y\}\, d\xi=\frac{1}{\pi}\frac{y}{(\xi-x)^2+y^2}$$

Thus, $$u(x, y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\, d\xi}{(\xi-x)^2+y^2}$$

This is a well known Poisson integral formula and is valid for $y>0$, when $f(x)$ is bounded and piecewise continuous for all real x.

Example 6.4.6 : Find the steady state temperature distribution $u(x, y)$ in a long square bar of side π with one face maintained at constant temperature u_0 and the other faces at zero temperature.

Solution : Mathematically, this problem is described by

$$u_{xx}+u_{yy}=0, \quad 0<x<\pi, \quad 0<y<\pi$$

subjected to the boundary conditions

$$u(0, y)=u(\pi, y)=0,$$

$$u(x, 0)=u(x, \pi)=u_0$$

Taking the finite Fourier sine transform with respect to the variable x, we get

$$\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin nx\, dx+\int_0^{\pi}\frac{\partial^2 u}{\partial y^2}\sin nx\, dx=0$$

which after using the conditions $u(0, y)=u(\pi, y)=0$ implies that

$$\frac{d^2\bar{u}}{dy^2}-n^2\bar{u}=0$$

Its general solution is found to be (6.4.9)

$$\bar{u}=A\cosh ny+B\sinh ny$$

Taking the finite Fourier sine transform of the second set of boundary conditions, we have

$$\bar{u}(n, 0)=0$$

and
$$\bar{u}(n,\pi)=\int_0^{\pi} u_0 \sin nx\, dx = u_0\left(\frac{1-\cos n\pi}{n}\right)$$

Using these in equation (6.4.9), we get

$$A=0, \quad B=\frac{u_0(1-\cos n\pi)}{n\sinh n\pi}$$

Hence
$$\bar{u}=\frac{u_0}{\sinh n\pi}\left(\frac{1-\cos n\pi}{n}\right)\sinh ny$$

Finally, taking the inverse finite Fourier transform, we obtain

$$u(x, y)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{u_0}{\sinh(n\pi)}\left(\frac{1-\cos n\pi}{n}\right)\sinh ny \sin nx$$

Thus the required temperature distribution is

$$u(x, y)=\begin{cases}\dfrac{4u_0}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{\sinh(2k+1)\,y\sin(2k+1)x}{(2k+1)\sinh(2k+1)\pi}, & \text{for } n \text{ odd}\\ 0, & \text{for } n \text{ even}\end{cases}$$

6.4.4 Miscellaneous Examples

Example 6.4.7 : Derive the solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0$$

for the region $r \geq 0$, $z \geq 0$ satisfying the conditions

(i) $V \to 0$, as $z \to \infty$ and $r \to \infty$.

(ii) $V = f(r)$ on $z=0$, $r \geq 0$.

Solution : The given equation is

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0 \tag{6.4.10}$$

Applying Hankel transform of order zero, we get

$$\int_0^{\infty}\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right) rJ_0(\xi r)=-\xi^2\bar{V}$$

after using condition (i) and since $J_0(\xi r)$ is a solution of

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{df}{dr}+\xi^2 f=0$$

Therefore (6.4.10) implies that

$$\frac{d^2\bar{V}}{dz^2}-\xi^2\bar{V}=0$$

$\Rightarrow$

$$\bar{V}=Ae^{\xi z}+Be^{-\xi z}$$

Now $\bar{V}\to 0$ as $z\to\infty$ and $\bar{V}\to\bar{f}(\xi)$ on $z=0$, we have

$$A\to 0 \text{ and } B=\bar{f}(\xi)$$

Thus

$$\bar{V}=\bar{f}(\xi)\,e^{-\xi z}$$

$\therefore$

$$V(r,z)=\int_0^\infty \xi\bar{V}(\xi,z)\,J_0(\xi r)\,d\xi$$

$$=\int_0^\infty \xi\bar{f}(\xi)\,e^{-\xi z}J_0(\xi r)\,d\xi$$

which is the required solution.

Example 6.4.8 : Determine the solution of the equation

$$\frac{\partial^4 z}{\partial x^4}+\frac{\partial^2 z}{\partial y^2}=0,\quad -\infty<x<\infty,\quad y\ge 0$$

satisfying the conditions

(i) z and its partial derivative tends to zero as $x\to\pm\infty$

(ii) $z=f(x),\ \dfrac{\partial z}{\partial y}=0$ on $y=0$.

Solution : The given equation is

$$\frac{\partial^4 z}{\partial x^4}+\frac{\partial^2 z}{\partial y^2}=0 \qquad (6.4.11)$$

Applying Fourier transform with respect to x, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^4 z}{\partial x^2}e^{i\xi x}dx=\xi^4\bar{z}.$$

Therefore equation (6.4.11) implies that

$$\frac{d^2\bar{z}}{dy^2}+\xi^4\bar{z}=0$$

$\Rightarrow$
$$\bar{z}=A\cos\left(\xi^2 y\right)+B\sin\left(\xi^2 y\right)$$

Now
$$\bar{z}=\bar{f}(\xi),\ \frac{d\bar{z}}{dy}=0,\ \text{on } y=0$$

$\Rightarrow$
$$A=\bar{f}(\xi) \text{ and } B=0$$

Thus we have

$$\bar{z}=\bar{f}(\xi)\cos\left(\xi^2 y\right)$$

Taking inverse Fourier transform, we get

$$z=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{z}e^{-i\xi x}\,d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\xi)\cos\left(\xi^2 y\right)e^{-i\xi x}\,d\xi$$

which is the required solution.

Example 6.4.9 : The temperature θ in the semi-infinite rod $0\le x\le\infty$ is determined by the partial differential equation $\frac{\partial\theta}{\partial t}=k\frac{\partial^2\theta}{\partial x^2}$ and the conditions.

(i) $\theta=0$, when $t=0, x\ge 0$

(ii) $\theta=\theta_0$, when $x=0,\ t>0$

Making use of the sine transform show that

$$\theta(x,t)=\frac{2\theta_0}{\pi}\int_0^{\infty}\frac{\sin(\xi x)}{\xi}\left(1-e^{-k\xi^2 t}\right)d\xi.$$

Solution : The given equation is

$$k\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t} \tag{6.4.12}$$

Multiplying by $\sin(\xi x)$ and integrating w.r.t. x between the limits 0 to ∞, we get

$$\frac{d\bar{\theta}}{dt} = k\int_0^\infty \frac{\partial^2\theta}{\partial x^2}\sin(\xi x)\,dx = k\left[\sin(\xi x)\frac{\partial\theta}{\partial x}\right]_0^\infty - \int_0^\infty \xi\cos(\xi x)\frac{\partial\theta}{\partial x}dx$$

$$= k\left[\left|\sin(\xi x)\frac{\partial\theta}{\partial x} - \xi\cos(\xi x)\,\theta\right|_0^\infty - \xi^2\int_0^\infty \theta\sin(\xi x)dx\right]$$

$$= -k\left[\xi^2\bar{\theta} + \xi\theta_0\right], \quad \bar{\theta} = \int_0^\infty \theta\sin\xi x\,dx$$

after using the conditions $\theta \to 0, \frac{\partial\theta}{\partial x} \to 0$ as $x \to \infty$ and $\theta = \theta_0$, when $x = 0$.

Therefore,
$$\frac{d\bar{\theta}}{dt} - k\xi^2\bar{\theta} = k\xi\theta_0$$

$\Rightarrow$
$$\bar{\theta} = \frac{\theta_0}{\xi} + Ae^{-k\xi^2 t}$$

Now
$$\bar{\theta}(\xi, 0) = 0 \Rightarrow A = -\frac{\theta_0}{\xi}$$

$$\bar{\theta} = \frac{\theta_0}{\xi}\left[1 - e^{-k\xi^2 t}\right]$$

Inverting the sine transform, we get

$$\theta = \frac{2}{\pi}\int_0^\infty \bar{\theta}\sin(\xi x)d\xi$$

$$= \frac{2\theta_0}{\pi}\int_0^\infty \frac{\sin(\xi x)}{\xi}\left(1 - e^{-k\xi^2 t}\right)d\xi$$

which is the required result.

Example 6.4.10 : Show that the solution of the following equation

$$\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2}$$

which tends to zero as $y \to \infty$ and which satisfies the conditions

(i) $z \to \infty$, as $y \to \infty$

(ii) $z = f(x),\ y = 0,\ x > 0$ and

(iii) $z = 0,\ y > 0,\ x = 0$

may be written as $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(\xi)\, e^{x - y\sqrt{\xi}}\, dx$

If $f(x) = k$, k being constant, then evaluate the integral.

Solution : The given equation is

$$\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2} \tag{6.4.13}$$

Applying Laplace transform w.r.t. x, we get

$$\int_0^\infty \frac{\partial z}{\partial x} e^{-px}\, dx = \int_0^\infty \frac{\partial^2 z}{\partial y^2} e^{-px}\, dx$$

$$\Rightarrow \qquad \frac{d^2 \bar{z}}{dy^2} = z e^{-px}\Big|_0^\infty + \int_0^\infty p z e^{-px}\, dx = \bar{z} p$$

where $$\bar{z} = \int_0^\infty z e^{-px}\, dx$$

$$\therefore \qquad \frac{d^2 \bar{z}}{dy^2} + \bar{z} p = 0$$

$$\Rightarrow \qquad z = A \exp\left(\sqrt{p}\, y\right) + B \exp\left(-\sqrt{p}\, y\right)$$

Now $\quad z \to 0$ as $y \to \infty \Rightarrow \bar{z} \to 0$, as $y \to \infty$

Therefore, $A = 0$ and hence

$$\bar{z} = B \exp\left(-\sqrt{p}\, y\right)$$

Again $z = f(x)$, when $y = 0 \Rightarrow \bar{z} = \bar{f}(p)$, when $y = 0$

$$\Rightarrow \qquad \bar{f}(p) = B$$

$$\therefore \qquad \bar{z} = \bar{f}(p)\, e^{-\sqrt{p y}}$$

Taking inverse Laplace transform, we get

$$z = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{z} e^{px}\, dp = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(p)\, e^{px - y\sqrt{p}}\, dp$$

Hence the result in the required form.

Further, if $f(x)=k$, then $\bar{f}(p)=k\int_0^{\infty} e^{-px}dx=\frac{k}{p}$

$$\therefore \quad z=\frac{k}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{px-\sqrt{p}y}}{p}dx$$

This integrand has a simple pole at $p=0$, therefore by Cauchy residue theorem, we have

$$\int_{\gamma-i\infty}^{\gamma+i\infty} f(p)\,dp = 2\pi i \text{ (Residue at } p=0)$$

where
$$f(p)=\frac{e^{px-\sqrt{p}y}}{p}$$

$$\therefore \quad z=\frac{k}{2\pi i}2\pi i \lim_{p\to 0} p\frac{e^{px-\sqrt{p}y}}{p}$$

$$= k\ e^0 = k.$$

Hence $z(x, y)=k$,

which is the required solution.

Example 6.4.11 : The $V(r,\theta)$ satisfies the differential equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}=0$$

in the wedge-shaped region $|\theta|\le\alpha$ and the boundary conditions $V=f(r)$ when $\theta=\pm\alpha$. Show that it can be expressed in the form

$$V(r,\theta)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\cos(\xi\theta)}{\cos(\xi\alpha)}\bar{f}(\xi)r^{-\xi}d\xi, \text{ where } \bar{f}(\xi)=\int_0^{\infty} f(r)\,r^{\xi-1}dr.$$

Solution : The given equation is

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}=0,\ r\le 0,\ |\theta|\le\alpha \qquad (6.4.14)$$

Applying the Melline transform with respect to r, we get

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \right) r^{s+1} dr = 0$$

or
$$\int_0^\infty \frac{\partial^2 V}{\partial r^2} r^{s+1} dr + \int_0^\infty \frac{\partial V}{\partial r} r^s dr + \int_0^\infty \frac{\partial^2 V}{\partial \theta^2} r^{s-1} dr = 0$$

or
$$r^{s+1} \frac{\partial V}{\partial r}\Big|_0^\infty - (s+1) r^s V\Big|_0^\infty + s(s+1)\int_0^\infty r^{s-1} V dr + r^s V\Big|_0^\infty$$

$$- s\int_0^\infty r^{s-1} V dr + \frac{\partial^2}{\partial \theta^2}\int_0^\infty r^{s-1} V dr = 0$$

Now in all the physical problems, $V, \dfrac{\partial V}{\partial r} \to 0$ as $r \to \infty$, therefore

$$s^2 \bar{V} + \frac{d^2 \bar{V}}{d\theta^2} = 0, \quad \bar{V} = \int_0^\infty V r^{s-1} dr$$

$\Rightarrow$
$$\bar{V} = A\cos(s\theta) + B\sin(s\theta) \qquad (6.4.15)$$

where A and B are arbitrary constants.

Now
$$V = f(r), \text{ for } \theta = \pm\alpha$$

$\Rightarrow$
$$\bar{V} = \bar{f}(s), \text{ for } \theta = \pm\alpha, \ \bar{f}(s) = \int_0^\infty f(r) r^{s-1} dr$$

Therefore, $\bar{V}$ must be on even function of s, hence

$$B = 0 \text{ and } A = \frac{\bar{f}(s)}{\cos(s\alpha)}$$

Thus
$$\bar{V} = \frac{\bar{f}(s)}{\cos(s\alpha)}\cos(s\theta)$$

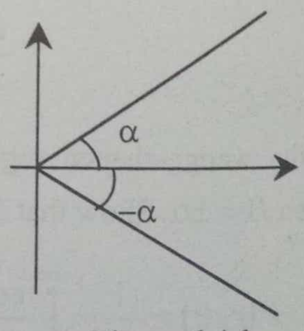

Figure 6.4.1

Inverting the transform, we get

$$V = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{V} r^{-s} ds$$

$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma - i\infty} \frac{\bar{f}(s) r^{-s} \cos(s\theta)}{\cos(s\alpha)} ds$$

which is the required result.

Example 6.4.12 : The variation of the function z over the xy-plane and for $t \geq 0$ is determined by the equation $\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$. If, when $t = 0$, $z = f(x, y)$ and $\frac{\partial z}{\partial t} = 0$, show that at any subsequent time

$$z(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(\xi,\eta)\left(\cos ct\sqrt{\xi^2+\eta^2}\right)e^{i(\xi x+\eta y)}d\xi d\eta$$

where
$$\bar{f}(\xi, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x, y)\,e^{i(\xi x+\eta y)}dx\,dy$$

Solution : The given equation is

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

or
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \tag{6.4.16}$$

Multiplying both sides by $\frac{1}{2\pi}e^{i\xi x+i\eta y}$ and integrating with respect to x and y from $-\infty$ to ∞, we get

$$\frac{1}{c^2}\frac{d^2\bar{z}}{dt^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)e^{i\xi x+i\eta y}dx\,dy$$

$$= -\left(\xi^2+\eta^2\right)\bar{z}, \quad \bar{z} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z e^{i\xi x+i\eta y}dxdy$$

Therefore,
$$\frac{d^2\bar{z}}{dt^2} + c^2\left(\xi^2+\eta^2\right)\bar{z} = 0$$

$\Rightarrow$
$$\bar{z} = A\cos\left(\sqrt{\xi^2+\eta^2}\,ct\right) + B\sin\left(ct\sqrt{\xi^2+\eta^2}\right)$$

Now at $t = 0$, $z = f(x, y)$,

$\Rightarrow$
$$\bar{z} = \bar{f}(\xi, \eta), \text{ at } t = 0$$

$\Rightarrow$
$$A = \bar{f}(\xi, \eta)$$

Also $$\frac{\partial z}{\partial t} = 0, \text{ at } t = 0 \Rightarrow \quad \frac{\partial \bar{z}}{\partial t} = 0, \text{ at } t = 0$$

$$\Rightarrow \quad Bc\sqrt{\xi^2 + \eta^2} = 0 \Rightarrow B = 0 \quad \left[\because c \neq 0 \neq \sqrt{\xi^2 + \eta^2}\right]$$

Thus $$\bar{z} = \bar{f}(\xi, \eta)\cos\left(ct\sqrt{\xi^2 + \eta^2}\right)$$

Taking inverse of the integral transform, we get

$$z(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{z}\, e^{i\xi x - i\eta y}\, d\xi\, d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\xi, \eta)\cos\left(ct\sqrt{\xi^2 + \eta^2}\right) e^{-i\xi x - i\eta y}\, d\xi\, d\eta$$

where $$\bar{f}(\xi, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{i\xi x + i\eta y}\, dx\, dy$$

6.5 Green's Function Method and its Applications

In order to find analytical solution of the boundary value problems, the Green's function method is one of the convenient techniques. In this section, we shall give definition of Green's function especially for Laplace, diffusion and wave equations and discuss some related problems through examples.

Consider the differential equation

$$Lu(x) = f(x) \tag{6.5.1}$$

where L is an ordinary linear differential operator, $f(x)$ is a known function, while $u(x)$ is an unknown function. Equation (6.5.1) provides us

$$u(x) = L^{-1} f(x) = \int G(x, \xi) f(\xi)\, d\xi \tag{6.5.2}$$

where $G(x, \xi)$ being the kernel of integral operator is called Green's function for the differential operator. Thus the solution of the non-homogeneous differential equation (6.5.1) can be written down ... the problem is known. ...