$$B_n = \frac{2}{l}\int_0^l f(z)\sin\left(\frac{n\pi z}{l}\right)dz$$

Which is the required Solution.

Hence proved.

Unit - IV.

chapter - 5 :-

Hyperbolic Differential Equations :-

5.1. Occurrence of the wave equation :-

One of the most important and typical homogeneus hyperbolic differential equations is the wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \rightarrow (1)$$

where c is wave speed. The Solution of wave equation is called wave function.

Note :-

The differential equation (1) is used in ~~may~~ many branches of physics and engineering. and is seen many Situations Such as transverse vibrations in strings or membrane, longitudinal vibrations in a bar, Propagation of Sound waves, electromagnetic waves, sea waves, elastic waves in Solids and Surface waves in earthquakes.

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5.2. Derivation of one-Dimensional Wave equation:

To derive the partial differential equation describing the transverse vibrations of a string.

Suppose a flexibile string is stretched under tension T between two points at a distance L apart as shown in Figure. We assume the following!

1) The motion takes place in one plane only and in this plane each particle moves in a direction perpendicular to the equilibrium position of the string.

2) The tension T in string is constant.

3) The gravitational force is neglected as compared with tension T of the string.

4) The slope of the deflection curve is small.

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Let the two fixed ends O(0, 0) and A(L, 0) of the string lie along x-axis (153.) in its equilibrium position. Consider an infinitesimal segment PQ of the string.

Let P be the mass per unit length of the string.

If the string is set vibrating in the xy-plane, the subsequent displacement y from the equilibrium position of a point P of the string will be a function of x and time t, while an element of length dx is stretched into an element of length ds given by,

$$ds = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\, dx$$

$$ds = \left(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right) dx \quad [\because \text{ negelecting higher order terms}]$$

The elementary elongation is given by

$$dL = ds - dx$$

$$= \left[1+ 1/2\left(\frac{\partial y}{\partial x}\right)^2\right] dx - dx$$

$$dL = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

and the work done by the element against tension T is

$$\frac{1}{2}\, T\left(\frac{\partial y}{\partial x}\right)^2 dx \qquad \left[\because \text{Workdone} = \text{Tension} \times \text{displacement}\right] \quad (15)$$

$\therefore$ The total work done W for whole string is

$$W = \frac{1}{2}\int_0^l T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

If U is the potential energy of the string, then

$$U = W = \frac{1}{2}\int_0^l T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Also, the total kinetic energy K of the string is given by

$$K = \frac{1}{2}\int_0^l \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

Using Hamilton's principle, we have

$$\delta \int_{t_0}^{t_1} (K - U)\, dt = 0$$

$\int_{t_0}^{t_1} (K-U)\, dt$ is stationary.

In other words,

$$\frac{1}{2}\int_{t_0}^{t_1}\int_0^l \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - T\left(\frac{\partial y}{\partial x}\right)^2\right] dx\, dt$$

is stationary and is of the form

$$\iint F(x, t, y, y_x, y_t)\, dx\, dt.$$

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From Euler - ortrogradsky equation, (155)

we have

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) = 0$$

which gives

$$\frac{\partial}{\partial t}\left(P\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = 0$$

If the string is homogeneous, then P and T are constants and the governing equation representing the transverse vibration of a string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where } c^2 = T/P$$

Example - 5.2.1 :-

Consider Maxwell's equation of electromagnetic theory given by $\nabla\cdot\bar{E} = 4\pi\rho$, $\nabla\cdot\bar{H} = 0$, $\nabla\times\bar{E} = \frac{-1}{c}\frac{\partial\bar{H}}{\partial t}$, $\nabla\times\bar{H} = \frac{4\pi i}{c} + \frac{1}{c}\frac{\partial\bar{E}}{\partial t}$

where $\bar{E}$ is an electric field, ρ is electric charge density, $\bar{H}$ is the magnetic field, i is the current density and c is velocity of light. Show that in the absence of charge, (when $\rho = i = 0$), $\bar{E}$ and $\bar{H}$ satisfy the wave equations.

Solution :-

Given $\text{curl}\,\bar{E} = \nabla\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{H}}{\partial t}$

Taking curl on both sides, we get

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$$\nabla \times (\nabla \times \bar{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \bar{H}}{\partial t}\right)$$

$$\nabla \times (\nabla \times \bar{E}) = -\frac{\partial}{\partial t}\left(\frac{1}{c}\nabla \times \bar{H}\right)$$

Also we know the identity

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla . \bar{E}) - \nabla^2 \bar{E}$$

Using this identity, we get

$$\nabla(\nabla . \bar{E}) - \nabla^2 \bar{E} = -\frac{\partial}{\partial t}\left(\frac{1}{c}\nabla \times \bar{H}\right)$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{4\pi i}{c} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t}\right]$$

$$\nabla(4\pi\rho) - \nabla^2 \bar{E} = -\frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \bar{E} = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

Which is a wave equation.

Similarly, we can observe that the magnetric field $\bar{H}$ also satisfies

$\nabla^2 \bar{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \bar{H}}{\partial t^2}$ which is also a wave equation.

5.3. Reduction of one Dimensional wave equation to canonical form and its Solution :-

Consider the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 u_{xx} \rightarrow (1)$$

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Choosing the characteristic lines (157)

$$\xi = x - ct, \quad \eta = x + ct \rightarrow (2)$$

we get,

$$\frac{\partial u}{\partial x} = u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_x = u_\xi + u_\eta = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) u$$

$$\frac{\partial u}{\partial t} = u_t = u_\xi \xi_t + u_\eta \eta_t$$

$$= -c u_\xi + c u_\eta$$

$$u_t = c(u_\eta - u_\xi)$$

$$u_t = c\left[\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right].$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \left[\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right]\left[\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right] u.$$

[Since in operator notation $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$]

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u$$

$$\frac{\partial^2 u}{\partial x^2} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \rightarrow (3)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right]$$

$$= c\left[\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right]\left[c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\right] u$$

[Since in operator notation $\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$]

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$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right]^2 u$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi}\right] \rightarrow (4)$$

Substituting (3) and (4) in (1) we get,

$$c^2[u_{\eta\eta} - 2u_{\xi\eta} + u_{\eta\eta}] = c^2[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]$$

$$\Rightarrow 4u_{\xi\eta} = 0$$

$$\Rightarrow u_{\xi\eta} = 0 \rightarrow (5)$$

On Integration, we get

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta)$$

where ϕ and ψ are arbitary functions.

Replacing $\xi = x - ct$ and $\eta = x + ct$, we have general solution of the wave equation (1) in the form.

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \rightarrow (6)$$

The two terms in equation (6) can be interpreted as waves travelling to right and left respectively.

Let k be an arbitrary real parameter. Then

$$u(x, t) = \phi[k(x - ct)] + [\psi[k(x + ct)]] \rightarrow (7)$$

is also a solution.

Further if $\omega = kc$, then

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$$u(x,t) = \phi[kx - \omega t] + \psi[kx + \omega t] \rightarrow (8)$$

(159)

is also a solution.

5.4. D' Alembert Solution of one-Dimensional wave Equation :-

Consider the initial value problem of Cauchy type described as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t \geq 0 \rightarrow (1)$$

Subject to initial conditions

$$u(x,0) = \eta(x), \quad \frac{\partial u}{\partial t}(x,0) = v(x) \rightarrow (2)$$

where the curve on which the initial data $\eta(x)$ and $v(x)$ are prescribed in the x-axis.

The function $\eta(x)$ and $v(x)$ are assumed to be twice continuously differentiable and Velocity of the String be prescribed at $t=0$.

We know that the general solution of wave equations is given by

$$u(x,t) = \phi(x+ct) + \psi(x - xt) \rightarrow (3)$$

where ϕ and ψ are arbitrary functions.

Using the initial conditions

$u(x,0) = \eta(x)$ and $u_t = v(x)$ from (3)

we get $u(x,0) = \phi(x) + \psi(x)$

$$\eta(x) = \phi(x) + \psi(x)$$

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$u_t(x,0) = \phi'(x+ct)c + \psi'(x-ct)(-c)$ (1b)

$= c[\phi'(x)] - c[\psi'(x)]$

i.e.) $v(x) = c[\phi'(x) - \psi'(x)]$.

$\phi(x) + \psi(x) = \eta(x)$ and $\rightarrow (4)$

$c[\phi'(x) - \psi'(x)] = v(x)$ $\rightarrow (5)$

Integrating the equation

$c[\phi'(x) - \psi'(x)] = v(x)$, we have

$c[\phi(x) - \psi(x)] = \int_0^x v(\xi)\,d\xi$

$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x v(\xi)\,d\xi$ $\rightarrow (6)$

Adding equation (4) and (6) we get

$$2\phi(x) = \eta(x) + \frac{1}{c}\int_0^x v(\xi)\,d\xi$$

$$\Rightarrow \phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(\xi)\,d\xi \rightarrow (7)$$

Also Subtracting (6) from (4) we get

$$2\psi(x) = \eta(x) - \frac{1}{c}\int_0^x v(\xi)\,d\xi$$

$$\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(\xi)\,d\xi \rightarrow (8)$$

Substituting (7) and (8) in equation (3)

we get, $u(x,t) = \phi(x+ct) + \psi(x-ct)$ becomes

$u(x,t)$ t ·

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$$u(x,t) = \frac{1}{2}\left[\eta(x+ct) + \eta(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi \rightarrow (9)$$

This is known as D' Alembert's solution of the one dimensional wave equation.

If $v=0$, i.e) if the string is released from rest, the required solution is

$$u(x,t) = \frac{1}{2}\left[\eta(x+ct) + \eta(x-ct)\right] \rightarrow (10).$$

Example - 5.4.1 : -

Solve the cauchy problem, described by the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$ subject to the conditions $u(x,0) = \eta(x)$, $\frac{\partial u}{\partial t}(x,0) = u_t(x,0) = v(x)$.

Solution : -

To make easy, we shall set $u = u_1 + u_2$, so that u_1 is a solution of the homogeneous wave equation subject to the general initial conditions given above.

Then u_2 will be a solution of

$$\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} = f(x,t) \rightarrow (1)$$

Subject to the homogeneous initial

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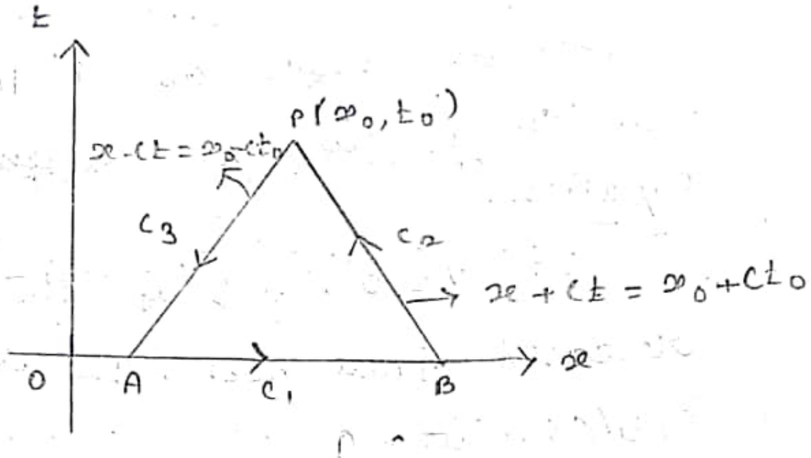
Conditions;

$$u_2(x,0)=0\,,\ \frac{\partial u_2}{\partial t}(x,0)=0 \longrightarrow (2)$$



Consider the xt - plane and a point $P(x_0,t_0)$. Draw two characteristics through P backwards until they intersect the initial line. the x-axis at $A(x_0-ct_0,0)$ and $B(x_0+ct_0,0)$. The equations of these two characteristics are

$$x \pm ct = x \pm ct_0.$$

To obtain the value of u at $P(x_0,t_0)$ we integrate the PDE (1) over the region R as shown in the figure.

We obtain

$$\iint_R \left[\frac{\partial^2 u_2}{\partial t^2} - c^2\frac{\partial^2 u_2}{\partial x^2}\right] dx\,dt = \iint_R f(x,t)\,dx\,dt \longrightarrow (3)$$

Using Green's theorem in a plane to the left hand side of the above equation to replace the surface integral over R by a line integral around the boundary

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∂-R of R.

∴ Equation (3) reduces to

$$-\iint_R \left[\frac{\partial}{\partial x}\left(c^2\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial t}\right)\right] dx\,dt = \iint_R f(x,t)\,dx\,dt$$

and finally to

$$\oint_{\partial R}\left(\frac{\partial u_2}{\partial t}dx + c^2\frac{\partial u_2}{\partial x}dt\right) = \iint_R f(x,t)\,dx\,dt \quad \to (4).$$

Now the boundary ∂R comprises three segments BP, PA and AB.

Along BP, $\frac{dx}{dt} = -c$,

Along PB, $\frac{dx}{dt} = c$

Using these results, equation (4) becomes,

Doubt

$$\int_{BP} c\left[\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right] - \int_{PA} c\left[\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right]$$

$$-\int_{AB}\left[\frac{\partial u_2}{\partial t}dx + c^2\frac{\partial u_2}{\partial x}dt\right] = \iint_R f(x,t)\,dx\,dt$$

The integrands of the first two integrals are simply the total differentials; while in the third integral, the first term vanishes on AB in view of the second initial condition in equation (2) and the second term vanishes because AB

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is directed along the x-axis on which (164)

$\frac{dt}{dx} = 0$. Then we get,

$$\int_{BP} c\,du_2 - \int_{PA} c\,du_2 = \iint_R f(c,t)\,dx\,dt$$

which can be written as

$$[cu_2(P) - cu_2(B)] - [cu_2(A) - cu_2(P)] = \iint_R f(x,t)\,dx\,dt$$

$$cu_2(P) - cu_2(B) + cu_2(P) - cu_2(A) = \iint_R f(x,t)\,dx\,dt \quad \rightarrow (5)$$

Using the first initial condition of equation (2), we get,

$u_2(A) = u_2(B) = 0$ and hence equation (5) becomes,

$$2c\,u_2(P) = \iint_R f(x,t)\,dx\,dt$$

$$\Rightarrow u_2(P) = \frac{1}{2c}\iint_R f(x,t)\,dx\,dt$$

Using the figure, we get

$$u_2(P) = \frac{1}{2c}\int_0^{t_0}\int_{x_0 - ct_0 + ct}^{x_0 + ct_0 - ct} f(x,t)\,dx\,dt \quad \longrightarrow (6)$$

Thus the required solution of the homogeneous wave equation, subject to the given initial conditions is given by

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$$u(x,t) = u_1 + u_2$$

$$= \frac{1}{2}\left[\eta(x+ct)+\eta(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi$$

$$+ \frac{1}{2c}\int_{0}^{t_0}\int_{x-c t_0+ct}^{x+ct_0-ct} f(x,t)\,dx\,dt$$

This solution is known as the Riemann-Volterra solution.

5.5. Separation of variables Method :-

Consider one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L, \; t > 0 \qquad \rightarrow (1)$$

Subject to the conditions

$$u(0,t) = 0, \quad t > 0, \quad u(L,t) = 0, \quad t > 0 \qquad \rightarrow (2)$$

and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

To obtain the variables separable Solution, we assume

$$u(x,t) = X(x)\,T(t) \rightarrow (3)$$

From this

$$\left.\begin{aligned} u_x &= X'T, \quad u_{xx} = X''T \\ u_t &= XT', \quad u_{tt} = XT'' \end{aligned}\right\} \rightarrow (4)$$

Using (4) in (1) we get,

$$XT'' = c^2 X'' T$$

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$$\Rightarrow \frac{1}{X} X'' = \frac{1}{c^2 T} T''$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = K$$

K is a separation constant.

Case-I : When $K > 0$

We have $K = \lambda^2$

Then $\frac{1}{X} \frac{d^2 X}{dx^2} = \lambda^2$, $\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \lambda^2$

$$\Rightarrow \frac{d^2 X}{dx^2} - \lambda^2 X = 0, \quad \frac{d^2 T}{dt^2} - c^2 \lambda^2 T = 0$$

Their solutions can be put in the form,

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x},$$

$$T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}.$$

$$\therefore u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \rightarrow (5)$$

Now, using boundary conditions;

$u(0,t) = 0$, we get

$$u(0,t) = (c_1 e^{\lambda 0} + c_2 e^{-\lambda \times 0})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t})$$

$$0 = (c_1 + c_2)(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t})$$

$$\Rightarrow c_1 + c_2 = 0 \longrightarrow (6)$$

Also $u(l,t) = 0$, gives us

$$u(l,t) = (c_1 e^{\lambda l} + c_2 e^{-\lambda l})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t})$$

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$$\Rightarrow 0 = c_1 e^{\lambda L} + c_2 e^{-\lambda L} \rightarrow (7) \quad (167)$$

Equations (6) and (7) posses a non-trivial solution ~~iff~~ if,

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = 0$$

$$\Rightarrow e^{-\lambda L} - e^{\lambda L} = 0$$

$$\Rightarrow 1 - e^{2\lambda L} = 0$$

$$\Rightarrow e^{2\lambda L} = 1 = e^{0}$$

$$\Rightarrow \lambda L = 0.$$

This implies that $\lambda = 0$, since $L \neq 0$. This is against our assumption that $k > 0$. Hence this solution is not acceptable.

Case II : Let $k = 0$.

Then we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = 0.$$

$$\Rightarrow \frac{d^2X}{dx^2} = 0, \quad \frac{d^2T}{dt^2} = 0$$

Their solutions can be in the form

$$X = Ax + B, \quad Y = Ct + D$$

The required solution is

$$u(x,t) = (Ax + B)(Ct + D)$$

Using the boundary conditions, we got

$$u(0,t) = 0$$

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$$\Rightarrow u(0,t) = (A\times 0 + B)(Ct+D)$$

$$0 = C(Ct+D)$$

$$\Rightarrow B = 0.$$

Also $u(L,t)=0$

$$\Rightarrow u(L,t) = (AL+D)(Ct+D)$$

$$0 = AL(Ct+D)$$

$$\Rightarrow A = 0.$$

This leads to a trivial solution. Since we are looking for a non-trivial solution, consider the following case.

Case - III : When $k<0$

Say $k = -\lambda^2$

Then we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad \frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

Their solutions can be in the form

$$X = (c_1 \cos\lambda x + c_2 \sin\lambda x) \text{ and}$$

$$Y = (c_3 \cos c\lambda t + c_4 \sin c\lambda t)$$

$\therefore$ the general solution is

$$u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos c\lambda t + c_4 \sin c\lambda t) \quad \text{(8)}$$

Using the boundary conditions,

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$u(0,t) = 0$, we have

$$u(0,t) = (c_1 + c_2 0)(c_3 \cos c\lambda t + c_4 \sin c\lambda t)$$

we have $c_1 = 0$.

Also using the boundary condition

$u(L,t) = 0$, we get $\sin \lambda L = 0$

$$\Rightarrow \lambda L = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

Which are eigen values. Hence the possible solution is

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right)\right] \quad (n = 1, 2, 3, \ldots) \rightarrow (A)$$

Using the superposition principle, we have,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right] \rightarrow (9)$$

The initial condition $u(x,0) = f(x)$ gives us

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

which is a half-range Fourier sine series, where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \rightarrow (10)$$

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Also, using the initial condition, (1"0)

$u_t(x,0) = g(x)$ gives us

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)$$

which is also a half-range Fourier Sine series, where

$$B_n = \frac{2}{L} \times \frac{L}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \rightarrow (11)$$

Hence the required physically meaningful solution is obtained from equation (9), where A_n and B_n are given by equations (10) and (11).

$u_n(x,t)$ given by equation (A) are called normal modes of vibration and $\omega_n = \frac{n\pi c}{L}$, $n = 1, 2, 3, \ldots$ are called normal frequencies.

5.6. Periodic Solutions:

5.6.1. Cylindrical Coordinates: -

To find the periodic solution of one-dimensional wave equation in cylindrical coordinates.

Solution: -

In cylindrical coordinates with u depending only on r, the one-dimensional

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wave equation assumes the form (141)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \rightarrow (1)$$

For periodic solution in time, we get

$$u = F(r)\, e^{i\omega t} \rightarrow (2)$$

Then,

$$\left.\begin{aligned}
\frac{\partial u}{\partial r} &= F'(r)\, e^{i\omega t} \\
\frac{\partial^2 u}{\partial r^2} &= F''(r)\, e^{i\omega t} \\
\frac{\partial u}{\partial t} &= F(r)\, i\omega\, e^{i\omega t} \\
\frac{\partial^2 u}{\partial t^2} &= F(r)\,(i\omega)(i\omega)\, e^{i\omega t} \\
\frac{\partial^2 u}{\partial t^2} &= F(r)\,(-\omega^2)\, e^{i\omega t}
\end{aligned}\right\} \rightarrow (3)$$

Using (3) in (1), we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\, e^{i\omega t}\right)=\frac{1}{c^2}\left(-\omega^2 F(r)\, e^{i\omega t}\right)$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF'(r)\right)\right]=-\frac{\omega^2}{c^2}F(r)$$

$$\frac{1}{r}\left[F'(r)+rF''(r)\right]=-\frac{\omega^2}{c^2}F(r)$$

$$F''(r)+\frac{1}{r}F'(r)+\frac{\omega^2}{c^2}F(r)=0 \rightarrow (4)$$

which has the form of Bessel's equation and hence, its solution can be written as

$$F = A J_0\left(\frac{\omega r}{c}\right)+B Y_0\left(\frac{\omega r}{c}\right) \rightarrow (5)$$

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In complex form we can write this (17) equation as

$$F = c_1\left[J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right)\right] + c_2\left[J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)\right]$$

It can be written as

$$F = c_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + c_2 H_0^{(2)}\left(\frac{\omega r}{c}\right).$$

where $H_0^{(1)}$, $H_0^{(2)}$ are Hankel functions defined by

$$H_0^{(1)} = J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right),$$

$$H_0^{(2)} = J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)$$

which behave as damped trigonometric functions for large r.

Thus the solution of one-dimensional wave equation becomes

$$u(r,t) = e^{i\omega t}\left[c_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + c_2 H_0^{(2)}\left(\frac{\omega r}{c}\right)\right] \quad (6)$$

Using asymptotic expressions for $H_0^{(1)}$ and $H_0^{(2)}$ defined by

$$H_0^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i(x-\pi/4)},$$

$$H_0^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)} \quad \text{for large } x.$$

The periodic solution to the given

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wave equation in cylindrical coordinates is

$$u(r,t) = \sqrt{\frac{2c}{\pi\omega}}\left[c_1 e^{-i\pi/4}\frac{\exp\left[i\left(\frac{\omega}{c}\right)(r+ct)\right]}{\sqrt{r}} + c_2 e^{i\pi/4}\frac{\exp\left[i\left(\frac{\omega}{c}\right)(r-ct)\right]}{\sqrt{r}}\right] \quad (173)$$

5.6.2. Spherical polar coordinates :-

To find the periodic solution of one-dimensional wave equation in spherical polar coordinates.

Solution :-

In spherical polar coordinates, the one-dimensional wave equation assumes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad r > 0 \longrightarrow (1)$$

For perodic solution in time, we take

$$u = F(r)e^{i\omega t} \longrightarrow (2)$$

From this,

$$\frac{\partial u}{\partial t} = F(r)e^{i\omega t}, \quad \frac{\partial u}{\partial t} = F(r)(i\omega)e^{i\omega t}$$

$$\frac{\partial^2 u}{\partial t^2} = F(r)(i\omega)(i\omega)e^{i\omega t}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= F'(r)e^{i\omega t} \\ \frac{\partial^2 u}{\partial t^2} &= -\omega^2 F(r)e^{i\omega t} \end{aligned}\right\} \longrightarrow (3)$$

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Using (3) in (1) we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F'(r) e^{i\omega t}\right) = \frac{1}{c^2}\left[F(r)(-\omega^2) e^{i\omega t}\right]$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 F'(r)\right)\right] = \frac{-\omega^2}{c^2}\left[F(r)\right]$$

$$\frac{1}{r^2}\left[2rF'(r) + r^2 F''(r)\right] = -\frac{\omega^2}{c^2} F(r)$$

$$F''(r) + \frac{2}{r}F'(r) + \frac{\omega^2}{c^2}F(r) = 0 \longrightarrow (4)$$

Let $F(r) = \left(\frac{\omega}{c} r\right)^{-1/2} \phi(r)$

Then

$$F'(r) = -\frac{1}{2}\left(\frac{\omega}{c}r\right)^{-3/2}\left(\frac{\omega}{c}\right)\phi(r) + \left(\frac{\omega}{c}r\right)^{-1/2}(\phi'(r))$$

$$F'(r) = -\frac{\omega}{2c}\left(\frac{\omega}{c}r\right)^{-3/2}\phi(r) + \left(\frac{\omega}{c}r\right)^{-1/2}\phi'(r)$$

$$F''(r) = -\frac{\omega}{2c}\left[\left[\left(-\frac{3}{2}\right)\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}r\right)^{-5/2}\cdot\phi(r)\right] + \left(\frac{\omega}{c}r\right)^{-3/2}\cdot\phi'(r)\right]$$

$$+ \left(-\frac{1}{2}\right)\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi'(r) + \left(\frac{\omega}{c}r\right)^{-1/2}\phi''(r)$$

$$F''(r) = \left(\frac{3}{4}\right)\left(\frac{\omega}{c}\right)^2\left(\frac{\omega}{c}r\right)^{-5/2}\phi(r) - \left(\frac{\omega}{2c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi'(r)$$

$$+ \left(-\frac{\omega}{2c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi'(r) + \left(\frac{\omega}{c}r\right)^{-1/2}\phi''(r)$$

$$F''(r) = \frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega r}{c}\right)^{-5/2}\phi(r) - \left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi'(r)$$

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$$+\left(\frac{\omega r}{c}\right)^{-1/2}\phi''(r) \qquad (17.5)$$

Substituting F' and F'' in (14) we get

$$\left.\begin{array}{l}\left[\frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega r}{c}\right)^{-5/2}\phi(r)-\left(\frac{\omega}{c}\right)\left(\frac{\omega r}{c}\right)^{-3/2}\phi'(r)+\right. \\ \left.\left(\frac{\omega r}{c}\right)^{-1/2}\phi''(r)\right]+\frac{2}{r}\left[\left(-\frac{\omega}{2c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi(r)+\right. \\ \left.\left(\frac{\omega r}{c}\right)^{-1/2}\phi'(r)\right]+\frac{\omega^2}{c^2}F(r)\end{array}\right\}=0$$

$$\left.\begin{array}{l}\frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega r}{c}\right)^{-5/2}\phi(r)-\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi'(r)+ \\ \left(\frac{\omega r}{c}\right)^{-1/2}\phi''(r)-\left(\frac{\omega}{rc}\right)\left(\frac{\omega}{c}r\right)^{-3/2}\phi(r)+ \\ \frac{2}{r}\left(\frac{\omega r}{c}\right)^{-1/2}\phi'(r)+\frac{\omega^2}{c^2}\left(\frac{\omega}{c}r\right)^{-1/2}\phi(r)\end{array}\right\}=0$$

$$\left.\begin{array}{l}\left(\frac{\omega}{c}r\right)^{-1/2}\left[\phi''(r)+\phi'(r)\left[-\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}r\right)^{-1}+\left(\frac{2}{r}\right)\right]\right. \\ \left.+\phi(r)\left[\frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega r}{c}\right)^{-3/2}-\left(\frac{\omega}{rc}\right)\left(\frac{\omega}{c}r\right)^{-1}+\frac{\omega^2}{c^2}\right]\right]\end{array}\right\}=0$$

$$\left.\begin{array}{l}\left(\frac{\omega}{c}r\right)^{-1/2}\left[\phi''(r)+\phi'(r)\left[-\frac{\omega}{c}\times\frac{c}{\omega r}+\frac{2}{r}\right]+\right. \\ \left.\phi(r)\left[\frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega r}{c}\right)^{-3/2}-\left(\frac{\omega}{rc}\right)\left(\frac{c}{\omega r}\right)+\frac{\omega^2}{c^2}\right]\right]\end{array}\right\}=0$$

$$\left(\frac{\omega}{c}r\right)^{-1/2}\left[\phi''(r)+\frac{1}{r}\phi'(r)+\left[\frac{\omega^2}{c^2}-\left(\frac{1}{2r}\right)^2\right]\phi(r)\right]=0$$

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Since $\left(\frac{\omega}{c} r\right) \neq 0$ we have

$$\phi''(r) + \frac{1}{r}\phi'(r) + \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{1}{2r}\right)^2\right]\phi(r) = 0$$

which is a form of Bessel's equation, whose solution is given by

$$\phi(r) = A' J_{1/2}\left(\frac{\omega}{c} r\right) + B' J_{-1/2}\left(\frac{\omega}{c} r\right)$$

where A' and B' are constants.

$$\therefore F(r) = \left(\frac{\omega}{c} r\right)^{-1/2}\left[A' J_{1/2}\left(\frac{\omega}{c} r\right) + B' J_{-1/2}\left(\frac{\omega}{c} r\right)\right]$$

$$F(r) = \frac{A}{\sqrt{r}} J_{1/2}\left(\frac{\omega}{c} r\right) + \frac{B}{\sqrt{r}} J_{-1/2}\left(\frac{\omega r}{c}\right) \rightarrow (5)$$

where $A = \sqrt{\frac{c}{\omega}}\, A'$ and $B = \sqrt{\frac{c}{\omega}}\, B'$

But we know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

$$\therefore F(r) = \sqrt{\frac{2c}{\pi \omega}}\left[\frac{A \sin\left(\frac{\omega r}{c}\right)}{r} + \frac{B \cos\left(\frac{\omega r}{c}\right)}{r}\right] \rightarrow (6)$$

In complex form

$$F(r) = c_1 \frac{e^{j\frac{\omega r}{c}}}{r} + c_2 \frac{e^{-j\frac{\omega r}{c}}}{r} \rightarrow (7)$$

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Then the required solution of (17). wave equation is

$$u(r,t) = c_1 \frac{e^{i\omega/c[r+ct]}}{r} + c_2 \frac{e^{-i\omega/c[r-ct]}}{r} \to (18)$$

5.9. Duhamel's Principle for wave equation:-

Let R' be the three dimensional Euclidean space and $x = (x_1, x_2, x_3)$ be any point, If $v = (x,t,\lambda)$ satisfies for fixed λ, the parial differential equation.

$$\frac{\partial^2 v}{\partial t^2}(x,t) - c^2 \nabla^2 v(x,t) = 0, \quad x \in R^3$$

with the conditions.

$$v(x,0,\lambda) = 0, \quad \frac{\partial v}{\partial t}(x,0,\lambda) = F(x,\lambda)$$

where $F(x,\lambda)$ denotes a continuous function defined for x in R^3 and if u satisfies

$$u(x,t) = \int_0^t v(x, t-\lambda, \lambda)\, d\lambda, \text{ then}$$

$u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = F(x,t), \quad x \in R^2, t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0.$$

Proof:-

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla u = F(x,t) \to (1)$$

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with $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$ (1.8)

Let us assume the solution of equation (1) in the form

$$u(x,t) = \int_0^t v(x, t-\lambda, \lambda)\, d\lambda \rightarrow (2)$$

where $v(x, t-\lambda, \lambda)$ is a one Parameter family solution of

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 v = 0 \text{ for all } \lambda \rightarrow (3)$$

Further, we assume that at $t = \lambda$,

$$v(x,0,\lambda) = 0 \text{ for all value of } \lambda \rightarrow (4)$$

Differentiating (2) with respect to t under the integral sign and using Leibnitz rule, we have

$$\frac{\partial u}{\partial t} = v(x,0,t) + \int_0^t v_t(x, t-\lambda, \lambda)\, d\lambda \rightarrow (5)$$

$$u_t = 0 + \int_0^t v_t(x, t-\lambda, \lambda)\, d\lambda$$

Differentiating (5) once again with respect to t, we have

$$\frac{\partial^2 u}{\partial t^2} = v_t(x,0,t) + \int_0^t v_{tt}(x, t-\lambda, \lambda)\, d\lambda$$

$$\frac{\partial^2 u}{\partial t^2} = v_t(x,0,t) + \int_0^t c^2 \nabla^2 v\, dv \rightarrow (6)$$

[Using equ (3)]

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Finally, using (2), the above equation (17.9) reduces to

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = V_t (x, 0, t) \longrightarrow (7)$$

Comparing equations (1) and (6), we get

$$\frac{\partial v}{\partial t}(x, 0, t) = F(x, t)$$

$\therefore$ If v satisfies the equation

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 v = 0$$ with the conditions

$v(x, 0, \lambda) = 0$,

$$\frac{\partial v}{\partial t}(x, 0, \lambda) = F(x, \lambda) \text{ at } t = \lambda,$$

then u defined by equation (2) satisfies the given inhomogeneous equation (1) and the specified conditions.

Hence the function $v(x, t)$ is called the pulse function or the force function.

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