Using (1) and (2), $\frac{d\mathbf{t}}{ds}\cdot\mathbf{a}+\mathbf{t}\cdot\frac{d\mathbf{a}}{ds}=\frac{d}{ds}(\mathbf{t}\cdot\mathbf{a})=0$ which proves that $\mathbf{t}\cdot\mathbf{a}$ is constant. Hence the geodesic γ cuts the generators at a constant angle and therfore it is a helix.

Example 3. Show that a curve on a sphere is a geodesic if and only if it is a great circle.

Let γ be a geodesic on a sphere with centre C having the position vector $\mathbf{a}$. Let P be any point on γ. Then the surface normal $\mathbf{N}$ at P passes through C. Let $\mathbf{r}$ be the position vector of P. Since the principal normal $\mathbf{n}$ to γ has the same direction as $\mathbf{N}$, we can write $\mathbf{r}=\mathbf{a}+\lambda\mathbf{n}$ ...(1)

Differentiating (1), $\frac{d\mathbf{r}}{ds}=\frac{d\mathbf{n}}{ds}\lambda+\mathbf{n}\frac{d\lambda}{ds}$ which gives

$$\mathbf{t}=(\tau\mathbf{b}-\kappa\mathbf{t})\lambda+\mathbf{n}\frac{d\lambda}{ds} \qquad ...(2)$$

Equating the coefficient of $\mathbf{b}$ on both sides we get $\lambda\tau=0$ Since $\lambda\neq 0$, $\tau=0$ showing that γ is a plane curve. Thus γ is a plane curve lying on the sphere and the principal normals at all points of γ pass through the centre of the sphere. Therefore γ is the section of the sphere by a plane passing through its centre. Hence γ is a great circle.

Conversely every great circle on the sphere is a geodesic. For if γ is a great circle on the sphere, then at each point P of γ, we have $\mathbf{N}=\mathbf{n}$ so that γ is a great circle.

3.6 DIFFERENTIAL EQUATIONS OF GEODESICS USING NORMAL PROPERTY

The normal property of geodesics is given by the identities $\mathbf{r}''\cdot\mathbf{r}_1=0$ and $\mathbf{r}''\cdot\mathbf{r}_2=0$. Using the equation of a surface $\mathbf{r}=\mathbf{r}(u,v)$, we shall express the normal property in terms of $\mathbf{r}$ and its partial derivatives and establish how the new equation derived from the normals property is equivalent to the canonical geodesic equations derived earlier. We also use the Christoffel symbols to express the new equations elegantly.

Theorem 1. The geodesic equations are

$$Eu''+Fv''+\frac{1}{2}E_1u'^2+E_2u'v'+\left(F_2-\frac{1}{2}G_1\right)v'^2=0 \qquad ...(I)$$

$$Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1u'v'+\frac{1}{2}G_2v'^2=0 \qquad ...(II)$$

Proof. Let the equation of the surface be $\mathbf{r}=\mathbf{r}(u,v)$ where $u=u(s)$ and $v=v(s)$.

Now $\mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u}\cdot\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\cdot\frac{dv}{ds}$ which gives

$$\mathbf{r}' = \mathbf{r}_1 u' + \mathbf{r}_2 v' \quad \ldots(1)$$

Differentiating (1) with respect to s, we have

$\mathbf{r}'' = (\mathbf{r}_{11}u' + \mathbf{r}_{12}v')\, u' + \mathbf{r}_1 u'' + (\mathbf{r}_{21}u' + \mathbf{r}_{22}v')\, v' + \mathbf{r}_2 v''$ which gives

$$\mathbf{r}'' = \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11}u'^2 + 2\,\mathbf{r}_{12}u'v' + \mathbf{r}_{22}v'^2 \quad \ldots(2)$$

From the normal property we have $\mathbf{r}''\cdot\mathbf{r}_1 = 0$ and $\mathbf{r}''\cdot\mathbf{r}_2 = 0$ ...(3)

Taking the scalar product of (2) with $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively and using (3), we obtain

$$\mathbf{r}_1\cdot\mathbf{r}_1 u'' + \mathbf{r}_2\cdot\mathbf{r}_1 v'' + \mathbf{r}_{11}\cdot\mathbf{r}_1 u'^2 + 2\,\mathbf{r}_{12}\cdot\mathbf{r}_1 u'v' + \mathbf{r}_{22}\cdot\mathbf{r}_1 v'^2 = 0 \quad \ldots(4)$$

$$\mathbf{r}_1\cdot\mathbf{r}_2 u'' + \mathbf{r}_2\cdot\mathbf{r}_2 v'' + \mathbf{r}_{11}\cdot\mathbf{r}_2 u'^2 + 2\mathbf{r}_{12}\cdot\mathbf{r}_2 u'v' + \mathbf{r}_{22}\cdot\mathbf{r}_2 v'^2 = 0 \quad \ldots(5)$$

We shall rewrite (4) and (5) using the first fundamental coefficients and their partial derivatives for the coefficients of u'', v'', u'^2, v'^2, and $u'v'$ as follows

Now $\mathbf{r}_1\cdot\mathbf{r}_1 = E,\ F = \mathbf{r}_1\cdot\mathbf{r}_2,\ \mathbf{r}_2\cdot\mathbf{r}_2 = G.$

$$\mathbf{r}_1\cdot\mathbf{r}_{11} = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_1^2) = \frac{1}{2}\frac{\partial E}{\partial u} = \frac{1}{2}E_1 \quad \ldots(6)$$

$$\mathbf{r}_1\cdot\mathbf{r}_{12} = \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_1^2) = \frac{1}{2}\frac{\partial E}{\partial v} = \frac{1}{2}E_2 \quad \ldots(7)$$

$$\mathbf{r}_2\cdot\mathbf{r}_{22} = \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_2^2) = \frac{1}{2}\frac{\partial G}{\partial v} = \frac{1}{2}G_2 \quad \ldots(8)$$

$$\mathbf{r}_2\cdot\mathbf{r}_{21} = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_2^2) = \frac{1}{2}\frac{\partial G}{\partial u} = \frac{1}{2}G_1 \quad \ldots(9)$$

Further $\frac{\partial}{\partial u}(\mathbf{r}_1\cdot\mathbf{r}_2) = \mathbf{r}_{11}\cdot\mathbf{r}_2 + \mathbf{r}_1\cdot\mathbf{r}_{21}$ which gives

$$\mathbf{r}_2\cdot\mathbf{r}_{11} = F_1 - \frac{1}{2}E_2 \quad \ldots(10)$$

$\frac{\partial}{\partial v}(\mathbf{r}_1\cdot\mathbf{r}_2) = \mathbf{r}_{12}\cdot\mathbf{r}_2 + \mathbf{r}_1\cdot\mathbf{r}_{22}$ which gives

$$\mathbf{r}_1\cdot\mathbf{r}_{22} = F_2 - \frac{1}{2}G_1 \quad \ldots(11)$$

Using (6), (7) and (11) in (4), we have

$$Eu'' + Fv'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \quad \ldots(\mathrm{I})$$

Using (8), (9) and (10) in (5), we obtain

$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1 u'v' + \frac{1}{2}G_2 v'^2 = 0 \qquad \text{...(II)}$$

Theorem 2. The equations (I) and (II) of Theorem 1 are the same as the canonical geodesic equations

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \qquad \text{...(1)}$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \qquad \text{...(2)}$$

in a different form.

Proof. Let $T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$ ...(3)

Then we have $\frac{\partial T}{\partial u'} = Eu' + Fv'$

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) = \left(\frac{\partial E}{\partial u}u' + \frac{\partial E}{\partial v}v'\right)u' + Eu'' + \left(\frac{\partial F}{\partial u}u' + \frac{\partial F}{\partial v}v'\right)v' + Fv''$$

$$= E_1 u'^2 + E_2 u'v' + Eu'' + F_1 u'v' + F_2 v'^2 + Fv'' \qquad \text{...(4)}$$

and $$\frac{\partial T}{\partial u} = \frac{1}{2}[E_1 u'^2 + 2F_1 u'v' + G_1 v'^2] \qquad \text{...(5)}$$

Further $$\frac{\partial T}{\partial v'} = Fu' + Gv' \qquad \text{...(6)}$$

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) = F_1 u'^2 + F_2 u'v' + Fu'' + G_1 u'v' + G_2 v'^2 + Gv'' \qquad \text{...(7)}$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}[E_2 u'^2 + 2F_2 u'v' + G_2 v'^2] \qquad \text{...(8)}$$

Hence using equations (4) and (5), we have

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = Eu'' + Fv'' + E_2 u'v' + F_1 u'v' + \frac{1}{2}E_1 u'^2 - F_1 u'v' +$$

$\left(F_2 - \frac{1}{2}G_1\right)v'^2$ so that equation (1) becomes

$$Eu'' + Fv'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0$$

In a similar manner using equation (7) and (8), we get

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right)-\frac{\partial T}{\partial v}=Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1u'v'+\frac{1}{2}G_2v'^2=0$$

Hence equation (2) becomes

$$Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1u'v'+\frac{1}{2}G_2v'^2=0$$

This proves that equations (I) and (II) are equivalent to (1) and (2)

Note. Equations (I) and (II) can also be written as

$$\frac{d}{ds}(Eu'+Fv')=\frac{1}{2}[E_1u'^2+2F_1u'v'+G_1v'^2]$$

$$\frac{d}{ds}(Fu'+Gv')=\frac{1}{2}[E_2u'^2+2F_2u'v'+G_2v'^2]$$

We obtain the above equation by straight way substituting equations (3), (5), (6) and (8) in the canonical geodesic equations (1) and (2) respectively.

In the following theorem, we shall obtain the formula for u'' and v''

Theorem 3. (a) $u''=-\dfrac{1}{2H^2}[lu'^2+2mu'v'+nv'^2]$

where $l=(GE_1-2FF_1+FE_2)$, $m=GE_2-FG_1$

and $n=(2GF_2-GG_1-FG_2)$

(b) $v''=-\dfrac{1}{2H^2}(\lambda u'^2+2\mu u'v'+\nu v'^2)$

where $\lambda=(2EF_1-EE_2-FE_1)$, $\mu=(EG_1-EE_2)$

and $\nu=(EG_2-2FF_2+FG_1)$

Proof. We solve for u'' and v'' from the geodesic equations (I) and (II) of Theorem 1.

$$Eu''+Fv''+\frac{1}{2}E_1u'^2+E_2u'v'+\left(F_2-\frac{1}{2}G_1\right)v'^2=0 \quad \text{...(1)}$$

$$Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1u'v'+\frac{1}{2}G_2v'^2=0 \quad \text{...(2)}$$

Now (1) G – (2) F gives

$$(EG-F^2)u''+\frac{1}{2}(GE_1-2FF_1+FE_2)u'^2+(GE_2-FG_1)u'v'$$
$$+\frac{1}{2}(2GF_2-GG_1-FG_2)v'^2=0 \quad \text{...(3)}$$

Replacing the coefficients of u'^2, $u'v'$ amd v'^2 by l, m and n respectively, we obtain $u'' = -\dfrac{1}{2H^2}(lu'^2 + 2mu'v' + nv'^2)$

In a similar manner let us solve for v''

Now (2)E – (1)F gives

$$H^2 v'' + \frac{1}{2}(2EF_1 - EE_2 - FE_1)\,u'^2 + (EG_1 - FE_2)\,u'v'$$

$$+\frac{1}{2}(EG_2 - 2FF_2 + G_1F)\,v'^2 = 0 \qquad \text{...(4)}$$

Using λ, μ and ν for coefficients of u'^2, $u'v'$ and v'^2,

We obtain $v'' = -\dfrac{1}{2H^2}(\lambda u'^2 + 2\mu u'v' + \nu v'^2)$ which completes the proof of the theorem.

In the subsequent part of this section, we use tensor notation involving Christoffel symbols of the first kind Γ_{ijk} and second kind Γ^i_{jk} and modify the geodesic equations (I) and (II) of Theorem 1 and other two equations obtained from (I) and (II).

Definition 1. Let $i, j, k = 1, 2$. Then Γ_{ijk} is defined as

$$\Gamma_{ijk} = \frac{1}{2}\{(\mathbf{r}_i \cdot \mathbf{r}_j)_k + (\mathbf{r}_i \cdot \mathbf{r}_k)_j - (\mathbf{r}_j \cdot \mathbf{r}_k)_i\}$$

where $(\ )_i$ and other two similar symbols stand for partial differentiation with respect to u or v according as $i = 1, 2$.

Using the above definition, the expression on the right hand side reduces to $\mathbf{r}_i \cdot \mathbf{r}_{jk}$ as shown below.

$$\Gamma_{ijk} = \frac{1}{2}\{\mathbf{r}_{ik} \cdot \mathbf{r}_j + \mathbf{r}_i \cdot \mathbf{r}_{jk} + \mathbf{r}_{ij} \cdot \mathbf{r}_k + \mathbf{r}_i \cdot \mathbf{r}_{kj} - \mathbf{r}_{ji} \cdot \mathbf{r}_k - \mathbf{r}_j \cdot \mathbf{r}_{ki}\}$$

$$= \frac{1}{2}\{2\mathbf{r}_i \cdot \mathbf{r}_{jk}\} = \mathbf{r}_i \cdot \mathbf{r}_{jk}$$

Hence $\Gamma_{ijk} = \mathbf{r}_i \cdot \mathbf{r}_{jk}$ for $i, j, k = 1, 2$ ...(1)

In particular, let us verify $\Gamma_{122} = \dfrac{1}{2}\,\mathbf{r}_1 \cdot \mathbf{r}_{22}$.

$$\text{Now} \quad \Gamma_{122} = \frac{1}{2}\{(\mathbf{r}_1 \cdot \mathbf{r}_2)_2 + (\mathbf{r}_1 \cdot \mathbf{r}_2)_2 - (\mathbf{r}_2 \cdot \mathbf{r}_2)_1\}$$

$$= \frac{1}{2}\left\{\frac{\partial}{\partial v}(\mathbf{r}_1 \cdot \mathbf{r}_2) + \frac{\partial}{\partial v}(\mathbf{r}_1 \cdot \mathbf{r}_2) - \frac{\partial}{\partial u}(\mathbf{r}_2 \cdot \mathbf{r}_2)\right\}$$

$$= \frac{\partial}{\partial v}(\mathbf{r}_1 \cdot \mathbf{r}_2) - \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_2 \cdot \mathbf{r}_2)$$

$= \mathbf{r}_1 \cdot \mathbf{r}_{22} + \mathbf{r}_2 \cdot \mathbf{r}_{12} - \mathbf{r}_2 \cdot \mathbf{r}_{21} = \mathbf{r}_1 \cdot \mathbf{r}_{22}$

Hence $\Gamma_{122} = \mathbf{r}_1 \cdot \mathbf{r}_{22}$. Thus we have verified the case when $i = 1$, and $j, k = 2$.

Theorem 4. If Γ_{ijk}, $i, j, k = 1, 2$ are the Christoffel symbols of the first kind, then the geodesic equations are

$$Eu'' + Fv'' + \Gamma_{111}u'^2 + 2\Gamma_{112}u'v' + \Gamma_{122}v'^2 = 0 \quad ...(1)$$

$$Fu'' + Gv'' + \Gamma_{211}u'^2 + 2\Gamma_{212}u'v' + \Gamma_{222}v'^2 = 0 \quad ...(2)$$

Proof. Using repeatedly $\Gamma_{ijk} = \mathbf{r}_i \cdot \mathbf{r}_{jk}$ proved after its definition, we identify the coefficients of the different derivatives in the following geodesic equations

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \quad ...(3)$$

$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \quad ...(4)$$

with the Christoffel symbols of the first kind Γ_{1jk} and Γ_{2jk} for $j, k = 1, 2$ as shown below.

$$\Gamma_{111} = \mathbf{r}_1 . \mathbf{r}_{11} = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_1^2) = \frac{1}{2}E_1 \quad ...(5)$$

$$\Gamma_{112} = \mathbf{r}_1 \cdot \mathbf{r}_{12} = \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_1^2) = \frac{1}{2}E_2 \quad ...(6)$$

Since $\quad \mathbf{r}_{12} = \mathbf{r}_{21}, \Gamma_{112} = \Gamma_{121} = \mathbf{r}_1 \cdot \mathbf{r}_{21} = \frac{1}{2}E_2 \quad ...(7)$

Now $\quad \Gamma_{122} = \mathbf{r}_1 \cdot \mathbf{r}_{22} = \mathbf{r}_1 \cdot \frac{\partial \mathbf{r}_2}{\partial v}$

To find $\mathbf{r}_1 \cdot \frac{\partial \mathbf{r}_2}{\partial v}$, let us consider

$$\frac{\partial}{\partial v}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \mathbf{r}_{12} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{22}, \text{ and also } \frac{\partial}{\partial u}(\mathbf{r}_2^2) = 2\mathbf{r}_2 \cdot \frac{\partial \mathbf{r}_2}{\partial u} = 2\mathbf{r}_2 \cdot \mathbf{r}_{21}$$

From the above two eqations, we obtain

$$\mathbf{r}_1 \cdot \mathbf{r}_{22} = \frac{\partial}{\partial v}(\mathbf{r}_1 \cdot \mathbf{r}_2) - \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_2^2) = F_2 - \frac{1}{2}G_1$$

This proves that $\Gamma_{122} = \mathbf{r}_1 \cdot \mathbf{r}_{22} = \mathbf{F}_2 - \frac{1}{2}\mathbf{G}_1$ ...(8)

Next let us find Γ_{2jk} for $j, k = 1, 2$ in terms of the derivatives of E, F and G.

$$\Gamma_{222} = \mathbf{r}_2 \cdot \mathbf{r}_{22} = \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_2^2) = \frac{1}{2} G_2$$

$$\Gamma_{221} = \mathbf{r}_2 \cdot \mathbf{r}_{21} = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_2^2) = \frac{1}{2} G_1 \quad \text{...(9)}$$

Since $\mathbf{r}_{12} = \mathbf{r}_{21}, \Gamma_{221} = \Gamma_{212} = \mathbf{r}_2 \cdot \mathbf{r}_{12} = \frac{1}{2} G_1$ ...(10)

Now $\Gamma_{211} = \mathbf{r}_2 \cdot \mathbf{r}_{11} = \mathbf{r}_2 \cdot \frac{\partial \mathbf{r}_1}{\partial u}$

$\frac{\partial}{\partial u}(\mathbf{r}_2 \cdot \mathbf{r}_1) = \mathbf{r}_2 \cdot \mathbf{r}_{11} + \mathbf{r}_1 \cdot \mathbf{r}_{21}$ so that

$$\mathbf{r}_2 \cdot \mathbf{r}_{11} = \frac{\partial}{\partial u}(\mathbf{r}_1 \cdot \mathbf{r}_2) - \mathbf{r}_1 \cdot \mathbf{r}_{21} = F_1 - \frac{1}{2} E_2 \text{ from (7)}$$

Thus we obtain $\Gamma_{211} = F_1 - \frac{1}{2} E_2$ ...(11)

Using (5), (6) and (8) in (3) we obtain equation (1) and in a similar manner using (9), (10) and (11) in (4) we obtain (2). This completes the proof of the theorem.

Note. We can obtain the equations (1) and (2) by straight away subsitituting the Christoffel symbols of the first kind in equations (4) and (5) of Theorem 1.

As a next step in the use of tensor notation, let us define Christoffel symbols of second kind denoted by Γ^i_{jk} for $i, j, k = 1, 2$ in terms of the symbols of first kind.

Definition 2. The Christoffel symbols of the second kind denoted by Γ^i_{jk} for $i, j, k = 1, 2$ are defined as

$$\Gamma^1_{jk} = H^{-2}(G\,\Gamma_{1jk} - F\,\Gamma_{2jk}),\ \Gamma^2_{jk} = H^{-2}(E\,\Gamma_{2jk} - F\,\Gamma_{1jk})$$

Since the Christoffel symbols of the first kind are function of E_1, F_1, G_1 and E_2, F_2, G_2, we can also express the Christoffel symbols of the second kind also in terms of these quantities.

Theorem 5. If Γ^1_{jk} and Γ^2_{jk} are Christoffel symbols of the second kind, the geodesic equations are

$$u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2 = 0 \quad \text{...(1)}$$

$$v'' + \Gamma^2_{11} u'^2 + 2\,\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2 = 0 \quad \text{...(2)}$$

Proof. As in the previous theorem, we identify the coefficients of the different derivatives of the following geodesic equations

$$H^2 u'' + \frac{1}{2}(GE_1 - 2FF_1 + FE_2)\,u'^2 + (GE_2 - FG_1)\,u'v'$$

$$+ \frac{1}{2}(2GF_2 - GG_1 - FG_2)\,v'^2 = 0 \quad \text{...(3)}$$

$$H^2 v'' + \frac{1}{2}(2EF_l - EE_2 - FE_1)\, u'^2 + (EG_1 - FE_2)\, u'v'$$

$$+ \frac{1}{2}(EG_2 - 2FF_2 + FG_1)\, v'^2 = 0 \qquad ...(4)$$

of Theorem 3 with the Christoffel symbols of the second kind as follows.

$$\Gamma^1_{11} = H^{-2}(G\Gamma_{111} - F\Gamma_{211}) = H^{-2}\left[\frac{1}{2}GE_1 - F\left(F_1 - \frac{1}{2}E_2\right)\right]$$

$$= H^{-2}\left[\frac{1}{2}GE_1 - FF_1 + \frac{1}{2}FE_2\right] \qquad ...(5)$$

$$\Gamma^1_{12} = H^{-2}[G\Gamma_{112} - F\Gamma_{212}] = H^{-2}\left(\frac{1}{2}GE_2 - \frac{1}{2}FG_1\right) \qquad ...(6)$$

$$\Gamma^1_{22} = H^{-2}[G\Gamma_{122} - F\Gamma_{222}] = H^{-2}\left[G\left(F_2 - \frac{1}{2}G_1\right) - \frac{1}{2}FG_2\right]$$

$$= H^{-2}\left[GF_2 - \frac{1}{2}GG_1 - \frac{1}{2}FG_2\right] \qquad ...(7)$$

$$\Gamma^2_{11} = H^{-2}[E\Gamma_{211} - F\Gamma_{111}] = H^{-2}\left[E\left(F_1 - \frac{1}{2}E_2\right) - \frac{1}{2}FE_1\right]$$

$$= H^{-2}\left[EF_1 - \frac{1}{2}EE_2 - \frac{1}{2}FE_1\right] \qquad ...(8)$$

$$\Gamma^2_{12} = H^{-2}(E\Gamma_{212} - F\Gamma_{112}) = H^{-2}\left(\frac{1}{2}EG_1 - \frac{1}{2}FE_2\right) \qquad ...(9)$$

$$\Gamma^2_{22} = H^{-2}(E\Gamma_{222} - F\Gamma_{122}) = H^{-2}\left(\frac{1}{2}EG_2 - F\left(F_2 - \frac{1}{2}G_1\right)\right)$$

$$= H^{-2}\left[\frac{1}{2}EG_2 - FF_2 + \frac{1}{2}FG_1\right]. \qquad ...(10)$$

Since $H^2 \neq 0$ using (5), (6) and (7) in (3), we get equations (1) and (8), (9) and (10) in (4), we get equation (2). This completes the proof of the Theorem.

Note. Equations involving Christoffel symbols of the first kind and second kind in the above two theorems are equivalent forms of geodesic equations (I) and (II) of Theorem 1. Hence all of them are equivalent to the canonical geodesic equations.

Example 1. Obtain the geodesics of the plane in polar coordinates using tensor equations of Theorems 4 and 5.

If (r, θ) are the polar coordinates of the plane, then its metric is given by

$$ds^2 = dr^2 + r^2\, d\theta^2$$

For this metric, we have the following

(i) $E = 1, E_1 = 0, E_2 = 0,$

(ii) $F = 0, F_1 = F_2 = 0$

(iii) $G = r^2, G_1 = 2r_1, G_2 = 0$

Using these values, the Christoffel symbols of the first kind are

$$\Gamma_{111} = 0, \Gamma_{112} = 0, \Gamma_{121} = \Gamma_{112} = 0, \Gamma_{122} = -r \quad \ldots(1)$$

$$\Gamma_{222} = 0, \Gamma_{221} = r, \Gamma_{212} = \Gamma_{221} = r, \Gamma_{211} = 0 \quad \ldots(2)$$

Taking $u = r$, and $v = \theta$, the geodesic equation of Theorem 4 are ...(3)

$$r'' - r\theta'^2 = 0 \text{ and } \theta'' + \frac{2}{r} r'\theta' = 0$$

...(4)

We shall show that the above equation are the same as the equation of Theorem 5.

Substituting the values in (1) or using (2) and (3), the Christoffel symbols of the second kind are

$$\Gamma_{11}^1 = 0, \Gamma_{12}^1 = 0, \Gamma_{22}^1 = -r, \Gamma_{11}^2 = 0, \Gamma_{12}^2 = \frac{1}{r}, \Gamma_{22}^2 = 0 \quad \ldots(5)$$

Hence the geodesic equation of Theorem 5 are

$$r'' - r\theta'^2 = 0 \text{ and } \theta'' + \frac{2}{r} r'\theta' = 0 \quad \ldots(6)$$

Equations (4) and (6) are one and the same. Hence the geodesic equations of Theorem 5 are the same as those of Theorem 4 in different form.

To find the geodesic of the plane, it is enough if we solve the equation

$$\theta'' + \frac{2}{r} r'\theta' = 0 \text{ or } \frac{\theta''}{\theta'} + \frac{2}{r} r' = 0 \quad \ldots(7)$$

Rewriting equation (7) as $\dfrac{d}{ds}\left(\log \dfrac{d\theta}{ds}\right) + \dfrac{d}{ds} \log r^2 = 0$

and integrating, we obtain $\dfrac{d\theta}{ds} r^2 = c$ ...(8)

where c is the constant of integration, From equation (8) we have

$$r^4\, d\theta^2 = c^2\, ds^2 = c^2 [dr^2 + r^2\, d\theta^2] \text{ giving}$$

$$r^2(r^2 - c^2)\, d\theta^2 = c^2\, dr^2 \text{ or } d\theta = \frac{c\, dr}{r\sqrt{r^2 - c^2}} \quad \ldots(9)$$

To integrate (9), let $r = c \sec t$ so that $dr = c \sec t \tan t\, dt$.

Hence $\int d\theta = c\int \frac{c \sec t \tan t\, dt}{c \sec t\; c \tan t} = t = \sec^{-1}\left(\frac{\mathbf{r}}{c}\right)$ giving $\sec^{-1}\left(\frac{\mathbf{r}}{c}\right) = \theta + \alpha,$

α being constant of integration.

Thus the plane geodesics are given by $\mathbf{r} = c \sec(\theta + \alpha)$.

Example 2. Using Christoffel symbols equations of geodesics, obtain the geodesic equation of surface of revolution and show that they can be solved by quadratures.

Let the surface of revolution be $\mathbf{r} = [u \cos v, u \sin v, f(u)]$

Then its metric is $ds^2 = (1 + f_1^2)\, du^2 + u^2\, dv^2$

For this metric, we have the following

(i) $E = 1 + f_1^2,\ E_1 = 2 f_1 f_{11},\ E_2 = 0$

(ii) $F = 0,\ F_1 = F_2 = 0$

(iii) $G = u^2,\ G_1 = 2u,\ G_2 = 0$ and

(iv) $H^2 = (1 + f_1^2)\, u^2$

Using these values the Christoffel symbols of the first kind are

$\Gamma_{111} = f_1 f_{11},\ \Gamma_{112} = \Gamma_{121} = 0,\ \Gamma_{122} = -u,\ \Gamma_{211} = 0,\ \Gamma_{212} = \Gamma_{221} = u,\ \Gamma_{222} = 0$ ...(1)

Hence the geodesic equation of Theorem 4,

$$(1 + f_1^3)\, u'' + f_1 f_{11} u'^2 - u v'^2 = 0,\ u v'' + 2 u' v' = 0 \quad \text{...(2)}$$

We also obtain the above equation using Theorem 5.

Using (i) to (iv) or the Christoffel symbols of the first kind, we have

$$\Gamma_{11}^1 = \frac{f_1 f_{11}}{1 + f_1^2},\ \Gamma_{12}^1 = 0,\ \Gamma_{22}^1 = \frac{-u}{1 + f_1^2},\ \Gamma_{11}^2 = 0,\ \Gamma_{12}^2 = \frac{1}{u},\ \Gamma_{22}^2 = 0$$

Hence using the geodesic equation of the Theorem 5, we get

$$(1 + f_1^2)\, u'' + f f_{11} u'^2 - u v'^2 = 0,\ v'' + \frac{2}{u} u' v' = 0 \quad \text{...(3)}$$

Hence equations (2) and (3) are one and the same. To find the geodesic on the surface of revolution, it is enough if we solve one of the equations (3).

Consider the equation $v'' + \frac{2}{u} u' v' = 0$ or $\frac{v''}{v'} + \frac{2}{u} u' = 0$ ...(4)

Rewriting the equation (4) as $\frac{d}{ds}\left(\log \frac{dv}{ds}\right) + \frac{d}{ds}(\log u^2) = 0$

and integrating we obtain $\frac{dv}{ds} u^2 = c$ ...(5)

where c is the constant of integration. From equation (5), we have

$u^4\, dv^2 = c^2\, ds^2 = c^2[(1 + f_1^2)\, du^2 + u^2\, dv^2]$ which gives

$$dv = \pm c \frac{\sqrt{1+f_1^2}}{u\sqrt{u^2-c^2}}\, du \text{ so that } v = \pm c \int \frac{\sqrt{1+f_1^2}}{u\sqrt{u^2-c^2}}\, du$$

Hence the geodesics on the surface of revolution can be found by quadratures.

3.7 EXISTENCE THEOREMS

From Theorem 1 of 3.6, we know that the geodesic equation on a surface are characterised by simultaneous second order differential equations (I) and (II) of two functions u and v. Thus the existence of geodesics on a surface is guaranteed by the solution of these differential equations.

Since $EG - F^2 \neq 0$, the geodesic equations (I) and (II) of Theorem 1 of 3.6 can be solved for u'' and v'' as

$$u'' = f(u, v, u', v'),\ v'' = g(u, v, u', v')$$

where f and g are quadratic functions of u' and v' with single valued functions u and v as coefficients. Instead of using these two simultaneous second order differential equations for obtaining the existence of solutions, we can obtain a single differential equation from (I) and (II) and ascertain the existence of geodesics on a surface with the help of the solution of the single differential equation. We adopt this method in the following existence theorem.

Theorem 1. A geodesic can be found to pass through any given point and have any given direction on a surface. The geodesic is uniquely determined by the initial conditions.

Proof. The method of proof is to derive a single second order differential equation from (I) and (II) of Theorem 1 of 3.6 and deduce the existence of a geodesic at a point from the uniqueness of solution of the initial value problem of such a differential equation.

Now we have $\dfrac{dv}{du} = \dfrac{dv}{ds}\cdot\dfrac{ds}{du}$

and

$$\frac{d^2v}{du^2} = \frac{d}{du}\left[\frac{dv}{ds}\cdot\frac{ds}{du}\right] = \frac{d^2v}{ds^2}\cdot\left(\frac{ds}{du}\right)^2 + \frac{dv}{ds}\cdot\frac{d}{du}\left(\frac{ds}{du}\right)$$

$$= \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 + \frac{dv}{ds}\cdot\frac{d}{du}\left(\frac{1}{u'}\right)$$

$$= \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 - \frac{1}{u'^2}\, u''\cdot\frac{ds}{du}\cdot\frac{dv}{ds}$$

Hence

$$\frac{d^2v}{du^2} = \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 - \left(\frac{ds}{du}\right)^2 \cdot u''\cdot\frac{dv}{du} \quad \ldots(1)$$

From Theorem 3 of 3.6, we have

$$u'' = -(l\,u'^2 + 2\,m\,u'v' + n\,v'^2) \quad ...(2)$$

$$v'' = -(\lambda\,u'^2 + 2\,\mu\,u'v' + \nu\,v'^2) \quad ...(3)$$

Now (3) $\left(\frac{ds}{du}\right)^2 - (2)\,\frac{dv}{du}\cdot\left(\frac{ds}{du}\right)^2$ gives

$$v''\left(\frac{ds}{du}\right)^2 - u''\,\frac{dv}{du}\cdot\left(\frac{ds}{du}\right)^2 = -\left[\lambda + 2\mu\frac{dv}{du} + \nu\left(\frac{dv}{du}\right)^2\right]$$

$$+\left[l\frac{dv}{du} + 2m\left(\frac{dv}{du}\right)^2 + n\left(\frac{dv}{du}\right)^3\right] \quad ...(4)$$

Using (1) in the left hand side of (4) and simplifying, we obtain

$$\frac{d^2v}{du^2} = n\left(\frac{dv}{du}\right)^3 + (2m - \nu)\left(\frac{dv}{du}\right)^2 + (l - 2\,\mu)\,\frac{dv}{du} - \lambda \quad ...(5)$$

From the existence and uniqueness of solution of the initial value problem of an ordinary differential equation of second order, there exists a unique solution for v of (5) with the initial conditions $v = v_0$ and $\frac{dv}{du} = v_1$ at $u = u_0$. Thus any solution u, v of (5) gives the direction coefficient of the tangent at P. Hence a geodesic is uniquely determined by the initial point P and the tangent at P under the given conditions.

Note. Identifying l, m, n, λ, μ and ν with the Christoffel symbols of the second kind given in Theorem 5 of 3.6, the differential equation (5) can also be written as

$$\frac{d^2v}{du^2} = \Gamma^1_{22}\left(\frac{dv}{du}\right)^3 + \left(2\Gamma^1_{12} - \Gamma^2_{22}\right)\left(\frac{dv}{du}\right)^2 + \left[\Gamma^1_{11} - 2\Gamma^2_{12}\right]\frac{dv}{du} - \Gamma^2_{11}$$

Since the uniqueness of the initial value problem asserts the existence of a geodesic at a given point in a given direction, it is possible to join the given point to a sufficiently near point Q by a geodesic arc. This has motivated us to look for a result giving the local existence of geodesics. In this connection, we have the following theorem for surfaces of class 3.

Theorem 2. Every point P on a surface has a neighbourhood N with the property that every point of N can be joined by a unique geodesic arc which lies wholly in N.

This theorem is known as second existence theorem whose proof depends upon the study of the differential equations of geodesics given in Theorems 4 and 5 of the previous section. We omit the proof and note the following fact alone.

$\dfrac{1}{2\sin\alpha}$ gives the number of revolutions on one side, we must have $\dfrac{1}{2\sin\alpha} > 1$ for more than one complete revolution. Hence $\sin\alpha < \dfrac{1}{2}$ or $\alpha < \dfrac{\pi}{6}$.

Further the number of geodesic arcs joining the point P to itself can be obtained one after another after each revolution on one side. As the maximum number of revolution on one side is given by $\dfrac{1}{2\sin\alpha}$, the maximum number of geodesic arcs joining this point P to itself is the greatest integer less than $(2\sin\alpha)^{-1}$.

To find the number of times a geodesic other than a generator intersect itself, we use the formula of a geodesic given in the previous example. Let the geodesic intersect itself n times. If u starts with the positive sense in the direction of s increasing, it returns to the initial position in the negative sense after each rotation. Hence equating the initial value of u and the value of u of the n-th rotation when it returns to its original position,

$$\cos[(2n\pi + v)\sin\alpha + \beta] = (-1)^{2n-1}\cos[v\sin\alpha + \beta]$$

Thus $\quad \cos[(2n\pi + v)\sin\alpha + \beta] = -\cos[v\sin\alpha + \beta]$ from which

we have $\quad \cos[(2n\pi + v)\sin\alpha + \beta] = \cos[v\sin\alpha + \beta + \pi]$

Hence $\quad (2n\pi + v)\sin\alpha + \beta = v\sin\alpha + \beta + \pi$ so that

$2n\sin\alpha = 1$ giving $n = \dfrac{1}{2\sin\alpha}$ where we take n to be the greatest integer less than $(2\sin\alpha)^{-1}$.

Example 7. Show that on a right circular cone, the number of goedesics joining two given points may be one or more but strictly limited.

Let us consider a right circular cone of semivertical angle α. As in Example 4, the number of turns needed to obtain a full circle instead of a sector in the plane is $\dfrac{1}{\sin\alpha}$. Hence if we unroll it continuously more than $\dfrac{1}{\sin\alpha}$ times, the resulting sector will exceed a circle and therefore any two points cannot always be joined by a straight line lying on this multiple circle. So the number of geodesics joining two given points may be one or more but strictly limited.

3.8 GEODESIC PARALLELS

Since geodesics on surfaces behave like straight lines in planes, we formulate a coordinate system on a surface with the help of geodesics. As a prelude to this, we introduce geodesic parallels in the following.

Theorem 1. For any given family of geodesics on a surface, a parametric system can be chosen so that the metric takes the form $ds^2 = du^2 + G(u, v)\,dv^2$

The given geodesics are the parametric curves v = constant and their orthogonal trajectories are given by u = constant, u being the distance measured along a geodesic from a fixed parallel.

Proof. Given a family of geodesic curves, let us choose a system of parameters such that the geodesics of the family are given by v = constant and their orthogonal trajetories are given by u = constant. Since v = constant and u = constant form an orthogonal parametric system, $F = 0$.

We know that v = constant is a geodesic if and only if

$$EE_2 + FE_1 - 2EF_1 = 0$$

Since $E \neq 0$ and $F = 0$, the above condition reduces to $E_2 = 0$ implying E is independent of v and it is a function of u only. Therefore the metric becomes

$$ds^2 = E(u)du^2 + G(u, v)\,dv^2 \quad ...(1)$$

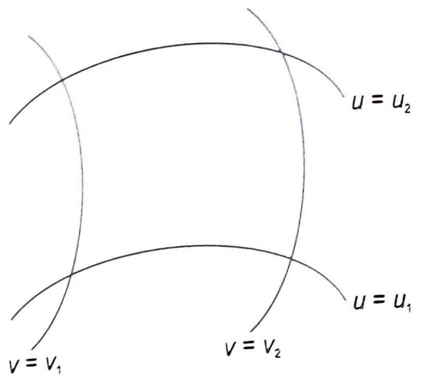

Fig. 12

Now let us consider the orthogonal trajectories $u = u_1$ and $u = u_2$ and find the distance between them along the geodesic v = constant v_1. Since $v = c$, $dv = 0$ so that (1) becomes

$$ds = \sqrt{E(u)}\,du \quad ...(2)$$

Integrating (2), we get

$$s = \int_{u_1}^{u_2} \sqrt{E(u)}\,du \quad ...(3)$$

Since s in (3) is independent of $v = c$, the distance between orthogonal trajectories is the same along any geodesic curve v = constant. Thus the orthogonal trajectories are parallel.

Let us find $E(u)$ in (1). Measuring the distance along a geodesic $v = c$, let the distance from some fixed parallel to the neighbouring parallel be du. Then $ds = du$ and $dv = 0$ so that we obtain from (1) $E(u) = 1$.

Thus (1) becomes $ds^2 = du^2 + G(u, v)\,dv^2$ which completes the proof of the theorem.

Note. If the parallel distance measured along the geodesic is du', then $du' = \sqrt{E}\, du$ from (1). Hence the mapping $u \to u'$ given by $du' = \sqrt{\mathbf{E}}\, du$ gives the simple form of the metric.

Definition 1. The orthogonal trajectories of the given family of geodesics v = constant on a surface are called geodesic parallels. u and v are called geodesic parameters.

Definition 2. $du^2 + \mathbf{G}(u, v)\, dv^2$ is called the geodesic form of ds^2.

Example 1. In the plane, we know that the straight lines are geodesics. Now consider a family of straight lines enveloping the given curve C. This family of straight lines envelops C so that C becomes the evolute and these family of straight lines are normal to the involute. Hence the geodesic parallels are the involutes of C.

Example 2. Let the family of geodesics be the straight lines concurrent at a point O. Then the geodesic parallels are the concentric circles with centre O. Since the concentric circles cut the family of straight line through O orthogonally, the concentric circles form a family of orthogonal trajectories which are the geodesic parallels.

Theorem 2. If a surface admits two orthogonal families of geodesics, then it is isometric with the plane.

Proof. v = constant be a family of geodesics. Then the family of orthogonal trajectories is u = constant.

If we take u = constant also as a family of geodesics, then v = constant is a family of orthogonal trajectories so that the surface admits the two orthogonal family of geodesics. Measuring the distance along a geodesic v = constant, let the distance from some fixed parallel to the neighbouring parallel be du. Hence $ds = du$ and $dv = 0$ so that $ds = \mathbf{E}\, du$ gives $du = \mathbf{E}\, du$ giving $\mathbf{E}(u) = 1$.

In a similar manner measuring the distance along the geodesic u = constant, let the distance from some fixed parallel to the neighbouring parallel be dv. Hence $ds = dv$ and $du = 0$ so that $ds = \mathbf{G}(u,v)\, dv$ gives $dv = \mathbf{G}(u, v)\, dv$ giving $\mathbf{G}(u, v) = 1$.

Thus the metric becomes $ds^2 = du^2 + dv^2$ which is the metric of the plane. This proves that the surface admitting two families of orthogonal geodesics is isometric with the plane.

3.9 GEODESIC POLAR COORDINATES

We shall introduce a coordinate system at a point on the surface with the help of geodesic, and the geodesic parallels at that point analagous to the polar coordinates of the plane.

As indicated in Example 2 of the previous section, all the straight lines θ = constant constitute geodesics through the origin and the concentric circles r = constant are the geodesic parallels in the plane.

Definition 1. Let O be a fixed point on the surface. Let us consider a family of geodesics at O. By the second existence theorem, there is a neighbourhood of O in which geodesics exist and we can take this to be the required family of geodesics at O. Let us take the orthogonal trajectories of this family as geodesic parallels u = constant where u is the distance of the orthogonal trajectory from O along any geodesics. The distances are measured from O along the geodesics in the neighbourhood of O. Let P be any point on the surface. Then u can be taken to be the distance measured from O along the geodesic joining P. Let v be the angle measured at O between a fixed geodesic and the geodesic along which u is measured. This is the same as the angle between the tangents to the geodesic through O joining P and the tangent to the fixed geodesic $v = 0$ at O. This angle v is the analogue of θ in the polar plane. The parameters (u, v) defined in this way are called the geodesic polar coordinates of P. Since we measure u along a geodesic, the metric on the surfac is

$$ds^2 = du^2 + G(u, v)\, dv^2 \quad ...(1)$$

by Theorem 1 of the previous section

Using (1), we note the following properties of metric approximation.

(i) $\lim_{u \to 0} \dfrac{\sqrt{G}}{u} = 1.$

Proof. A small neighbourhood of O is nearly a plane and the metric (1) should reduce to the polar form $d\mathbf{r}^2 + \mathbf{r}^2 d\theta^2$. Since u and v corresponds to polar coordinates, the metric on the surface becomes $du^2 + u^2 dv^2$. Identifying this metric with (1), G is approximately u^2 for points near O.

Hence we have

$$\lim_{u \to 0} \frac{\sqrt{G}}{u} = 1 \quad ...(2)$$

(ii) For the validity of the above metric approximation, $\sqrt{G}$ should have the power series expansion

$$\sqrt{G} = u + a_2 \frac{u^2}{2!} + a_3 \frac{u^3}{3!} \; ....$$

Definition 2. The parametric system (u, v) given above is called geodesic polar coordinate system. The curves u = constant are called geodesic circles.

3.10 GEODESIC CURVATURE

If $\mathbf{r} = \mathbf{r}(s)$ is the position vector of any point P on a curve on a surface, then the curvature vector $\mathbf{r}'' = \mathbf{t}' = \kappa \mathbf{n}$ where κ is the curvature and $\mathbf{n}$ is the principal normal at P to the curve. As we have already noted that at a point P on the surface, there exist three linearly independent non-coplanar vectors $\mathbf{N}$, $\mathbf{r}_1$ and $\mathbf{r}_2$ where $\mathbf{r}_1$ and $\mathbf{r}_2$ are tangential to the surface and $\mathbf{N}$ is the surface normal. Hence $\mathbf{r}''$ at P can be expressed as the linear combination of the vectors N, $\mathbf{r}_1$ and $\mathbf{r}_2$ so that we can write $\mathbf{r}'' = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ where κ_n is the normal component of $\mathbf{r}''$.

Definition 1. The normal component κ_n of $\mathbf{r}''$ is called the normal curvature at P.

Definition 2. The vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ with components (λ, μ) is tangential to the surface. The vector with components (λ, μ) of the tangential vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ to the surface is called the geodesic curvature vector at P. It is denoted by K_g.

Theorem 1. A curve on a surface is a geodesic if and only if the geodesic curvature vector is zero.

Proof. Let $\mathbf{r} = \mathbf{r}(s)$ be any curve on the surface with the principal normal $\mathbf{n}$ and surface normal $\mathbf{N}$ at P. If (λ, μ) is the geodesic curvature vector at P, then

$$\mathbf{r}'' = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2 \text{ or } \kappa \mathbf{n} = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2 \quad ...(1)$$

Let the curve be a geodesic. Then by the normal property of a geodesic $\mathbf{n} = \mathbf{N}$ ...(2)

Using (2) in (1), we get $\kappa \mathbf{N} = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2$.

Equating the coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$ on both sides we get $\lambda = \mu = 0$ so that $K_g = 0$.

Conversely, let $K_g = 0$. This implies $\lambda = 0$, $\mu = 0$. Hence from (1), we get $\kappa \mathbf{n} = \kappa_n \mathbf{N}$. Thus the principal normal to the curve is parallel to the surface normal. Therefore, the curve is a geodesic by the normal property.

Note. From the above theorem, we conclude that for any curve other than a geodesic on a surface, the geodesic curvature vector is not zero. This suggests that the magnitude measures in some sense its deviation from a geodesic at a point on the surface.

Theorem 2. The geodesic curvature vector of any curve is orthogonal to the curve.

Proof. If (λ, μ) is the curvature vector of the curve $\mathbf{r} = \mathbf{r}(s)$ at P, then as in Theorem 1.

$$\kappa \mathbf{n} = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2 \quad ...(1)$$

Since $\mathbf{t}$ is the tangent vector to the curve as well as to the surface, $\mathbf{n} \cdot \mathbf{t} = 0$ and $\mathbf{N} \cdot \mathbf{t} = 0$. ...(2)

Taking dot product with $\mathbf{t}$ on both sides of (1) and using (2), we obtain $(\lambda \mathbf{r}_1 + \mu \mathbf{r}_2) \cdot \mathbf{t} = 0$ which proves that (λ, μ) is orthogonal to the curve.

Theorem 3. For any curve on a surface, the geodesic curvature vector is intrinsic.

Proof. To prove that the vector (λ, μ) is intrinsic, we have to show that (λ, μ) can be found out from the metric of the surface. To prove this fact, first note that

$$\mathbf{r}'' \cdot \mathbf{r}_1 = U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u}$$

$$\mathbf{r}'' \cdot \mathbf{r}_2 = V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} \quad ...(1)$$

If (λ, μ) is the geodesic curvature vector at a point, on the surface, then $\mathbf{r}'' = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2$. ...(2)

Taking scalar product with $\mathbf{r}_1$ and $\mathbf{r}_2$ on both sies of (2) respectively, we obtain

$$\mathbf{r}''\cdot\mathbf{r}_1 = \kappa_n \mathbf{N}\cdot\mathbf{r}_1 + \lambda\mathbf{r}_1\cdot\mathbf{r}_1 + \mu\mathbf{r}_2\cdot\mathbf{r}_1$$

$$\mathbf{r}''\cdot\mathbf{r}_2 = \kappa_n \mathbf{N}\cdot\mathbf{r}_2 + \lambda\mathbf{r}_1\cdot\mathbf{r}_2 + \mu\mathbf{r}_2\cdot\mathbf{r}_2 \quad ...(3)$$

Using $\mathbf{N}\cdot\mathbf{r}_1 = 0$ and $\mathbf{N}\cdot\mathbf{r}_2 = 0$ and the first fundamental coefficients, we have from (1).

$$\mathbf{r}''\cdot\mathbf{r}_1 = U = E\lambda + \mu F,\ \mathbf{r}''\cdot\mathbf{r}_2 = V = \lambda F + \mu G \quad ...(4)$$

Solving for λ, μ in terms of U and V from (4), we obtain

$$\lambda = \frac{1}{H^2}(UG - VF),\ \mu = \frac{EV - FU}{H^2},\ H^2 = EG - F^2$$

which shows that the vector (λ, μ) is intrinsic.

Theorem 4. The condition of orthogonality of the geodesic curvature vector (λ, μ) with any vector (u, v) on a surface is

$$u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0$$

Proof. The tangential direction at a point (u, v) on a surface is (u', v'). Since (λ, u) and (u', v') are orthogonal, using $l = \lambda$, $m = \mu$, $l' = u'$ and $m' = v'$ in the condition of orthogonality $E\,ll' + F(lm' + l'm) + G\,mm' = 0$,

We obtain $E\lambda u' + F(\lambda v' + u'\mu) + G\mu v' = 0$ which can be written as

$$u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0.$$

Since $U = E\lambda + F\mu$ and $V = F\lambda + G\mu$, we rewrite the above condition as $Uu' + Vv' = 0$.

Theorem 5. In the notation of the Christoffel symbols, the components of the geodesic curvature vector are

$$\lambda = u'' + \Gamma^1_{11}\,u'^2 + 2\,\Gamma^1_{12}\,u'v' + \Gamma^1_{22}\,v'^2$$

$$\mu = v'' + \Gamma^2_{11}\,u'^2 + 2\,\Gamma^2_{12}\,u'v' + \Gamma^2_{22}\,v'^2$$

Proof. Taking $\mathbf{r} = \mathbf{r}(s)$, $\mathbf{r}' = \mathbf{r}_1 u' + \mathbf{r}_2 v'$ and hence

$$\mathbf{r}'' = \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11} u'^2 + 2\mathbf{r}_{12} u'v' + \mathbf{r}_{22} v'^2 \quad ...(1)$$

Taking dot product with $\mathbf{r}_1$ on both sides, we get

$$\mathbf{r}''\cdot\mathbf{r}_1 = \mathbf{r}_1^2 u'' + \mathbf{r}_1\cdot\mathbf{r}_2\, v'' + \mathbf{r}_{11}\cdot\mathbf{r}_1\, u'^2 + 2\,\mathbf{r}_{12}\cdot\mathbf{r}_1\, u'v' + 2\,\mathbf{r}_{22}\cdot\mathbf{r}_2\, v'^2 \quad ...(2)$$

Since $\mathbf{r}''\cdot\mathbf{r}_1 = U = E\lambda + F\mu$, using the fundamental coefficients in (2), we obtain

$$E\lambda + F\mu = Eu'' + Fv'' + \mathbf{r}_{11}\cdot\mathbf{r}_1\, u'^2 + 2\,\mathbf{r}_{12}\cdot\mathbf{r}_1\, u'v' + 2\,\mathbf{r}_{22}\cdot\mathbf{r}_1 v'^2 \quad ...(3)$$

In a similar manner, taking scalar product with $\mathbf{r}_2$ on both sides of (1) and Using $\mathbf{r}''\cdot\mathbf{r}_2 = V = F\lambda + G\mu$, we have

$$F\lambda + G\mu = Fu'' + Gv'' + \mathbf{r}_{11}\cdot\mathbf{r}_2\, u'^2 + 2\,\mathbf{r}_{12}\cdot\mathbf{r}_2\, u'v' + \mathbf{r}_{22}\cdot\mathbf{r}_2\, v'^2 \quad ...(4)$$

Solving for λ from (3) and (4), we get

$$\lambda(EG - F^2) = (EG - F^2)u'' + [G\,\mathbf{r}_{11}\cdot\mathbf{r}_1 - F\,\mathbf{r}_{11}\cdot\mathbf{r}_2]\,u'^2$$

$$+ 2[G\mathbf{r}_{12}\cdot\mathbf{r}_1 - F\mathbf{r}_{12}\cdot\mathbf{r}_2]\,u'v' + 2[G\mathbf{r}_{22}\cdot\mathbf{r}_1 - F\,\mathbf{r}_{22}\cdot\mathbf{r}_2]\,v'^2$$

Using (4) in (5), we get

$$\mu = -\frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'} = \frac{V}{H^2 u'}\frac{\partial T}{\partial u'}$$ by (2) which proves the theorem

Example 1. Obtain the geodesic curvature vector of a curve on a right helicoid $\mathbf{r} = (u \cos v, u \sin v, av)$ using different formulae for it.

We have derived three different formulae for the geodesic curvature vector. We obtain the geodesic curvature vector using these three different formulae.

Now $\quad \mathbf{r}_1 = (\cos v, \sin v, 0), \mathbf{r}_2 = (-u \sin v, u \cos v, a)$

Hence $\quad E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1, F = 0, G = a^2 + u^2, H = \sqrt{a^2 + u^2}$

(i) $$\lambda = \frac{1}{H^2}(UG - VF), \mu = \frac{1}{H^2}(EV - FU)$$

Let us find U and V in the above formula

Now $$T = \frac{1}{2}[u'^2 + (u^2 + a^2)v'^2]$$

$$\frac{\partial T}{\partial u'} = u', \quad \frac{\partial T}{\partial v'} = (u^2 + a^2)v'$$

$$\frac{\partial T}{\partial u} = uv'^2, \quad \frac{\partial T}{\partial v} = 0.$$

Hence $$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = u'' - uv'^2$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = (u^2 + a^2)v'' + 2uu'v'$$

Hence $$\lambda = \frac{1}{(a^2 + u^2)}(u'' - uv'^2)\cdot(a^2 + u^2) = u'' - uv'^2.$$

$$\mu = \frac{1}{(a^2 + u^2)}[(u^2 + a^2)v'' + 2uu'v']$$

(ii) $$\lambda = u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2$$

$$\mu = v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2.$$

First let us calculate the Christoffel symbols in the above formula for a curve on the right helicoid.

$$\Gamma_{111} = \frac{1}{2}E_1 = 0, \Gamma_{112} = \frac{1}{2}E_2 = 0, \Gamma_{122} = F_2 - \frac{1}{2}G_1 = -u$$

$$\Gamma_{211} = F_1 - \frac{1}{2}E_2 = 0, \Gamma_{212} = \frac{1}{2}G_1 = u, \Gamma_{222} = \frac{1}{2}G_2 = 0$$

Hence $\Gamma^1_{11} = \frac{1}{H^2}(G\Gamma_{111} - F\,\Gamma_{211}) = 0,\ \Gamma^1_{12} = \frac{1}{H^2}(G\,\Gamma_{112} - F\,\Gamma_{212}) = 0$

$$\Gamma^1_{22} = \frac{1}{H^2}(G\,\Gamma_{122} - F\,\Gamma_{222}) = \frac{1}{a^2 + u^2}(a^2 + u^2)\,(-u) = -u.$$

Therefore $\lambda = u'' - u\,v'^2$

To find μ, let us find other Christoffel symbols of the second kind.

$$\Gamma^2_{11} = \frac{1}{H^2}[E\,\Gamma_{211} - F\,\Gamma_{111}] = 0, \Gamma^2_{12} = \frac{1}{H^2}[E\,\Gamma_{212} - \Gamma_{112}F] = \frac{u}{a^2 + u^2}$$

and finally $\Gamma^2_{22} = \frac{1}{H^2}[E\,\Gamma_{222} - F\,\Gamma_{122}] = 0.$

Therefore $\mu = v'' + \frac{2u'v'u}{a^2 + u^2} = \frac{1}{(a^2 + u^2)}[v''(a^2 + u^2) + 2\,u\,u'v'].$

(iii) $$\lambda = \frac{1}{H^2}\,\frac{U}{v'}\,\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\,\frac{V}{u'}\,\frac{\partial T}{\partial v'}$$

$$\mu = \frac{1}{H^2}\,\frac{V}{u'}\,\frac{\partial T}{\partial u'} = -\frac{1}{H^2}\,\frac{U}{v'}\,\frac{\partial T}{\partial u'}$$

Using (1), (2) and (3), we obtain

$$\lambda = \frac{1}{a^2 + u^2}\cdot\frac{(u'' - u\,v'^2)}{v'}\,v'\,(a^2 + u^2) = u'' - u\,v'^2.$$

The other expression gives the formula for λ in terms of v'' as follows.

$$\lambda = -\frac{1}{a^2 + u^2}[(u^2 + a^2)\,v'' + 2\,u\,u'v']\,\frac{v'}{u'}(a^2 + u^2)$$

$$= -\frac{v'}{u'}[(u^2 + a^2)\,v'' + 2\,u\,u'v']$$

Further using (1), (2) and (3) in the formula for μ, we have

$$\mu = \frac{1}{a^2 + u^2}\,[(u^2 + a^2)\,v'' + 2\,u\,u'v']$$

and the alternate expression for μ is

$$\mu = -\frac{1}{a^2 + u^2}\frac{(u'' - uv'^2)}{v'}u'$$

Now we are in a position to define the geodesic curvature and derive the formula for it in terms of the parameters s and t.

Definition 3. The geodesic curvature at any point of a curve denoted by κ_g is defined as the magnitude of its geodesic curvature vector with proper sign. κ_g is considered to be positive or negative according as the angle between the tangent to the curve and the geodesic curvature vector is $\pi/2$ or $-\pi/2$.

So we have $\kappa_g = \pm\sqrt{\lambda^2 + \mu^2}$.

Note. It is to be noted that from Theorem 2, the geodesic curvature vector at a point of a curve on a surface is orthogonal to the curve. Hence κ_g is positive or negative according as the angle is $\pi/2$ or $-\pi/2$. In other words, κ_g is positive or negative according as (u', v'), the geodesic curvature vector (λ, μ) and the surface normal $\mathbf{N}$ form a right handed or left handed system.

From the very definition of κ_g, we note the following properties.

(i) If (λ, μ) is the geodesic curvature vector at a point (u, v) on a surface, then $\kappa_g = H(u'\mu - \lambda v')$.

Proof. Since (u, v) is a point P on the curve, (u', v') gives the unit vector along the tangential direction at P.

Now $\left(\frac{\lambda}{\sqrt{\lambda^2 + \mu^2}}, \frac{\mu}{\sqrt{\lambda^2 + u^2}}\right) = \left(\frac{\lambda}{\kappa_g}, \frac{\mu}{\kappa_g}\right)$ is the unit vector in the direction of (λ, μ).

If θ is the angle between the two directions, using the formula $\sin\theta = H(lm' - l'm)$, we obtain

$$\sin\theta = \frac{1}{\kappa_g}H(u'\mu - v'\lambda)$$

From Theorem 2, $\theta = \pi/2$ so that we get from the above step $\kappa_g = H(u'\mu - v'\lambda)$

(ii) Geodesic curvature of a geodesic on a surface is zero and conversely.

Proof. At every point P of a geodesic of a surface we show that $\kappa_g = 0$. At a point $\mathbf{r} = \mathbf{r}(s)$ on a geodesic we have $\mathbf{r}'' = \kappa\mathbf{n} = \kappa_n\mathbf{N} + \lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ ...(1)

Taking scalar product with $\mathbf{r}_1$ and $\mathbf{r}_2$ on both sides of (1), we obtain

$$\lambda E + \mu F = 0, \text{ and } \lambda F + \mu G = 0 \quad ...(2)$$

as $\quad \mathbf{r}_1\cdot\mathbf{N} = \mathbf{r}_2\cdot\mathbf{N} = 0$ and $\mathbf{r}_1\cdot\mathbf{n} = \mathbf{r}_2\cdot\mathbf{n} = 0$.

From (2) we obtain $(EG - F^2)\lambda = 0$ and $(EG - F^2)\mu = 0$ ...(3)

Since $EG - F^2 \neq 0$, (3) shows that $\lambda = 0$, $\mu = 0$ proving that

$$\kappa_g = \sqrt{\lambda^2 + \mu^2} = 0.$$

Conversely if the geodesic curvature at every point of a curve is zero, then the curve is a geodesic. $\kappa_g = 0$ implies $\lambda = 0, \mu = 0$ at every point of the curve. Thus the curvature vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ vanishes identically. Using this, (1) becomes $\kappa \mathbf{n} = \kappa_n \mathbf{N}$. Therefore the curve is a geodesic by the normal property.

In the following, we obtain the formula for finding κ_g in different forms.

Theorem 7. If $\mathbf{r} = \mathbf{r}(s)$ is the position vector of a point P of a curve on a surface, then

(i) $\kappa_g = [\mathbf{N}, \mathbf{r}', \mathbf{r}'']$ and

(ii) $\kappa_g = \dot{s}^{-3}[\mathbf{N}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}]$

Proof (i). From Theorem 2, the geodesic curvature vector is orthogonal to the unit tangent vector $\mathbf{r}' = \dfrac{d\mathbf{r}}{ds}$ at P. Since the geodesic curvature vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ lies in the tangent plane at P, it is orthogonal to this surface normal $\mathbf{N}$ at P. Thus the geodesic curvature vector is orthogonal to both $\mathbf{N}$ and $\mathbf{r}'$ and therefore it is parallel to the unit vector $\mathbf{N} \times \mathbf{r}'$. Since κ_g is the magnitude of the geodesic curvature vector, we can take the geodesic curvature vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ as $\kappa_g(\mathbf{N} \times \mathbf{r}')$ ...(1)

We know that $\mathbf{r}'' = \kappa_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ ...(2)

Using (1) in (2) we obtain $\mathbf{r}'' = \kappa_n \mathbf{N} + \kappa_g(\mathbf{N} \times \mathbf{r}')$ ...(3)

Taking scalar product with unit vector $\mathbf{N} \times \mathbf{r}'$ on both sides of (3), we obtain

$$(\mathbf{N} \times \mathbf{r}') \cdot \mathbf{r}'' = [\kappa_n \mathbf{N} + \kappa_g(\mathbf{N} \times \mathbf{r}')] \cdot (\mathbf{N} \times \mathbf{r}')$$

Since $\mathbf{N} \cdot (\mathbf{N} \times \mathbf{r}') = 0$ and $(\mathbf{N} \times \mathbf{r}') \cdot (\mathbf{N} \times \mathbf{r}') = 1$

we get from (4), $\kappa_g = [\mathbf{N}, \mathbf{r}', \mathbf{r}'']$ which proves (i)

(ii) We shall rewrite the formula (i) by using any parameter t.

Now $$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = \mathbf{r}'\dot{s} \text{ and also}$$

$$\ddot{\mathbf{r}} = \frac{d}{dt}(\mathbf{r}'\dot{s}) = \frac{d}{ds}(\mathbf{r}'\dot{s})\frac{ds}{dt} = \mathbf{r}''\dot{s}^2 + \mathbf{r}'\ddot{s}$$

Since $\mathbf{r}'\dot{\mathbf{r}} \times \mathbf{r}'\ddot{s} = 0$, we have

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \mathbf{r}'\dot{s} \times (\mathbf{r}''\dot{s}^2 + \mathbf{r}'\ddot{s}) = \mathbf{r}' \times \mathbf{r}''\dot{s}^3 \text{ so that we have}$$

$$\mathbf{r}' \times \mathbf{r}'' = \frac{1}{\dot{s}^3}(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})$$

Hence from the formula (i), we have

$$\kappa_g = \mathbf{N} \cdot (\mathbf{r}' \times \mathbf{r}'') = \frac{1}{\dot{s}^3}[\mathbf{N}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}]$$

Corollary. $\kappa_g = \dot{s}^3 H^{-1}[(\mathbf{r}_1\cdot\dot{\mathbf{r}})(\mathbf{r}_2\cdot\ddot{\mathbf{r}}) - (\mathbf{r}_2\cdot\dot{\mathbf{r}})(\mathbf{r}_1\cdot\ddot{\mathbf{r}})]$

Proof. Since $H\mathbf{N} = \mathbf{r}_1 \times \mathbf{r}_2$, we have $\mathbf{N} = H^{-1}(\mathbf{r}_1 \times \mathbf{r}_2)$

Using this value of **N** in the above formula.

$$\kappa_g = \dot{s}^3 H^{-1}(\mathbf{r}_1 \times \mathbf{r}_2)\cdot(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \quad ...(1)$$

Now $(\mathbf{r}_1 \times \mathbf{r}_2)\cdot(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = (\mathbf{r}_1\cdot\dot{\mathbf{r}})(\mathbf{r}_2\cdot\ddot{\mathbf{r}}) - (\mathbf{r}_2\cdot\dot{\mathbf{r}})(\mathbf{r}_1\cdot\ddot{\mathbf{r}})$ ...(2)

Using (2) in (1), we obtain

$$\kappa_g = \dot{s}^{-3} H^{-1}[(\mathbf{r}_1\cdot\dot{\mathbf{r}})(\mathbf{r}_2\cdot\ddot{\mathbf{r}}) - (\mathbf{r}_2\cdot\dot{\mathbf{r}})(\mathbf{r}_1\cdot\ddot{\mathbf{r}})]$$

As an application of the above corollary, we derive the formula for κ_g in the most simplest form in terms of the intrinsic quantities U and V of the surface.

Theorem 8. If U and V are the intrinsic quantities of a surface at a point (u, v), then

(i) $$\kappa_g = \frac{1}{H}\frac{V(s)}{u'} \text{ and}$$

(ii) $$\kappa_g = -\frac{1}{H}\frac{U(s)}{v'}$$

Proof. $$T = \frac{1}{2}\dot{\mathbf{r}}^2 = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2]$$

Hence $$\frac{\partial T}{\partial \dot{u}} = \dot{\mathbf{r}}\cdot\frac{\partial \dot{\mathbf{r}}}{\partial \dot{u}},\ \frac{\partial T}{\partial \dot{v}} = \dot{\mathbf{r}}\cdot\frac{\partial \dot{\mathbf{r}}}{\partial \dot{v}} \quad ...(1)$$

But $$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\dot{u} + \frac{\partial \mathbf{r}}{\partial v}\dot{v} = \mathbf{r}_1\dot{u} + \mathbf{r}_2\dot{v}$$

so that $$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{u}} = \mathbf{r}_1 \text{ and } \frac{\partial \dot{\mathbf{r}}}{\partial \dot{v}} = \mathbf{r}_2 \quad ...(2)$$

Thus from (1) and (2), we get

$$\frac{\partial T}{\partial \dot{u}} = \dot{\mathbf{r}}\cdot\mathbf{r}_1,\ \frac{\partial T}{\partial \dot{v}} = \dot{\mathbf{r}}\cdot\mathbf{r}_2 \quad ...(3)$$

We know that $U(t) = \ddot{\mathbf{r}}\cdot\mathbf{r}_1$, $V(t) = \ddot{\mathbf{r}}\cdot\mathbf{r}_2$ from Theorem 1 of 3.5 ...(4)

Using (3) and (4) in the corollary, we get

$$\kappa_g = \frac{1}{H\dot{s}^3}\left[V(t)\frac{\partial T}{\partial \dot{u}} - U(t)\frac{\partial T}{\partial \dot{v}}\right]$$

If we take s as parameter in place of t in the above equation, we obtain

$$\kappa_g = \frac{1}{H}\left[V(s)\frac{\partial T}{\partial u'} - U(s)\frac{\partial T}{\partial v'}\right] \text{ as } \dot{s} = 1 \quad ...(5)$$

Since $u' U(s) + v' V(s) = 0$, we have $U(s) = -\dfrac{v'}{u'} V(s)$ ...(6)

Using (6) in (5), we get $\kappa_g = \dfrac{V(s)}{Hu'}\left[u'\dfrac{\partial T}{\partial u'} + v'\dfrac{\partial T}{\partial v'}\right]$

From Euler's Theorem on Homogeneous functions

$u'\dfrac{\partial T}{\partial u'} + v'\dfrac{\partial T}{\partial v'} = 2T$ so that the above equation becomes

$$\kappa_g = \frac{V(s)}{Hu'} \cdot 2T \qquad ...(7)$$

Since s is the parameter $\mathbf{r}'^2 = 1$ so that $T = \dfrac{1}{2}\mathbf{r}'^2 = \dfrac{1}{2}$

Using this value of T in (7), $\kappa_g = \dfrac{V(s)}{Hu'}$

Similarly eliminating $V(s)$ in (5), we obtain

$\kappa_g = -\dfrac{1}{H}\dfrac{U(s)}{v'}$ which completes the proof of the theorem.

Corollary. If (λ, μ) is the geodesic curvature vector of a curve, then

$$\kappa_g = \frac{-H\lambda}{Fu' + Gv'} = \frac{H\mu}{Eu' + Fv'}$$

Proof. From Theorem 3, we have

$$\lambda = \frac{1}{H^2}(GU - FV), \mu = \frac{1}{H^2}(EV - FU) \qquad ...(1)$$

Since we take s as the parameter, $Uu' + Vv' = 0$

so that $U = -\dfrac{Vv'}{u'}$ and $V = -\dfrac{Uu'}{v'}$ ...(2)

Using (2) in (1), we obtain

$$\lambda = \frac{1}{H^2}\frac{U}{v'}(Gv' + Fu'), \mu = \frac{1}{H^2}\frac{V}{u'}(Eu' + Fv') \qquad ...(3)$$

Using the Theorem in (3), we obtain

$$\lambda = -\frac{\kappa_g}{H}(Gv' + Fu'), \mu = \frac{\kappa_g}{H}(Eu' + Fv')$$

so that we have $\kappa_g = \dfrac{-H\lambda}{Gv' + Fu'} = \dfrac{H\mu}{Eu' + Fv'}$

Example 2. ... geodesics.

$$\kappa_g = H[u'\{v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2\}$$

$$- v'\{u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2\}]$$

Simplifying the above, we find

$$\kappa_g = H[\Gamma_{11}^2 u'^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)u'^2 v' + (\Gamma_{22}^2 - 2\Gamma_{12}^1)u'v'^2$$

$$- \Gamma_{22}^1 v'^3 + (u'v'' - v'u'')] \quad \text{...(1)}$$

which gives formula for κ_g in Christoffel symbols.

By taking v = constant and u = constant respectively in the above formula (1), we have

$$\kappa_a = H\Gamma_{11}^2 u'^3. \text{ and } \kappa_b = -H\Gamma_{22}^1 v'^3 \quad \text{...(2)}$$

Under the conditions, we have $u' = \dfrac{1}{\sqrt{E}}$ and $v' = \dfrac{1}{\sqrt{G}}$ ...(3)

Using (3) in (2) we obtain

$\kappa_a = \Gamma_{11}^2 \dfrac{H}{E\sqrt{E}}$ and $\kappa_b = -\Gamma_{22}^1 \dfrac{H}{G\sqrt{G}}$ which prove the results

At each point of the curve C, there is a parametric curve v = constant cutting C at an angle θ which varies from point to point so that θ is a function of s. Liouville's formula aims at expressing κ_g in terms of θ and θ'.

Theorem 10. (Liouville's Formula). If θ is the angle which the curve C makes with the parametric curve v = constant,

then $\kappa_g = \theta' + Pu' + Qv'$

where $P = \dfrac{1}{2HE}(2EF_1 - FE_1 - EE_2),\ Q = \dfrac{1}{2HE}(EG_1 - FE_2).$

Proof. We make use of the formula $\kappa_g = -\dfrac{U(s)}{Hv'}$ in the proof.

The direction coefficients of the curve at (u, v) and the curve v = constant are respectively (u', v') and $\left(\dfrac{1}{\sqrt{E}}, 0\right)$. So if θ is the angle between the two directions $\left(\dfrac{1}{\sqrt{E}}, 0\right)$ and (u', v') we have, from the formulae,

$$\cos\theta = Ell' + F(lm' + ml') + Gmm',\ \sin\theta = H(lm' - l'm)$$

$$\cos\theta = \frac{1}{\sqrt{E}}(Eu' + Fv'),\ \sin\theta = \frac{Hv'}{\sqrt{E}} \quad \text{...(1)}$$

Now $$T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$$

$$\frac{\partial T}{\partial u'} = Eu' + Fv' \text{ and } \frac{\partial T}{\partial u} = \frac{1}{2}[E_1u'^2 + 2F_1u'v' + G_1v'^2] \quad ...(2)$$

Using (2) in (1), we obtain $\cos\theta = \dfrac{1}{\sqrt{E}}\dfrac{\partial T}{\partial u'}$ ...(3)

Differentiating (3) with respect to s, we have

$$-\sin\theta\,\theta' = \frac{1}{\sqrt{E}}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{1}{2E^{3/2}}\frac{\partial T}{\partial u'}\frac{dE}{ds} \quad ...(4)$$

But $$\frac{dE}{ds} = \frac{\partial E}{\partial u}\frac{du}{ds} + \frac{\partial E}{\partial v}\frac{dv}{ds} = E_1u' + E_2v' \quad ...(5)$$

Using (5) in (4) and rewriting it as

$$-\sqrt{E}\sin\theta\,\theta' = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{1}{2E}(E_1u' + E_2v')\frac{\partial T}{\partial u'}, \quad ...(6)$$

Let us substitute for $\dfrac{d}{ds}\left(\dfrac{\partial T}{\partial u'}\right)$ from $U = \dfrac{d}{ds}\left(\dfrac{\partial T}{\partial u'}\right) - \dfrac{\partial T}{\partial u}$,

Then $-\sqrt{E}\sin\theta\,\theta' = U + \dfrac{\partial T}{\partial u} - \dfrac{1}{2E}(E_1u' + E_2v')\dfrac{\partial T}{\partial u'}$,

Using the value of $\sin\theta$ in (1), we have

$$-Hv'\theta' = U + \frac{\partial T}{\partial u} - \frac{1}{2E}(E_1u' + E_2v')\frac{\partial T}{\partial u'},$$

Using (2) in the above equation, we have

$$-Hv'\theta' = U + \frac{1}{2}[E_1u'^2 + 2F_1u'v' + G_1v'^2] - \frac{1}{2E}(E_1u' + E_2v')(Eu' + Fv')$$

$$= U + \frac{1}{2E}[u'v'(2EF_1 - E_1F - EE_2) + v'^2(EG_1 - FE_2)] \quad ...(7)$$

Taking $$P = \frac{2EF_1 - E_1F - EE_2}{2HE},\ Q = \frac{1}{2HE}(EG_1 - FE_2)$$

We have from (7), $-\theta' = \dfrac{U}{v'H} + u'P + v'Q$ ...(8)

and $\delta\bar{s} = \delta s$ as it is of order $O(\delta s^2)$.

Also we have approximately, $\sin\theta = \delta\theta$, $\sin\delta\phi = \delta\phi$, ...(8)

From Theorem 5 of the curvature vector,

We have $\lambda = u_0'' - f(u_0, v_0, u_0', v_0')$, $\mu = v_0'' - g(u_0, v_0, u_0', v_0')$ ...(9)

Using (8) and (9) in (7), we have

$$\frac{\delta\theta + \delta\phi}{\delta s} = H_0(u_0'\mu - v_0'\lambda) + O(\delta s) \quad ...(10)$$

(10) is true at any point on the surface. Hence taking the limit as $\delta s \to 0$, we get

$$\lim_{\delta s \to 0} \frac{\delta\theta + \delta\phi}{\delta s} = H(u'\mu - v'\lambda) = \kappa_g$$ by the property (i) after the Definition 3.

This completes the proof of the theorem.

Note. If $\theta + \phi = \psi$, then $\frac{d\psi}{ds} = \kappa_g$ which is the plane analogue of the theorem.

In this connection, we have the following intrinsic generalisation giving the exact analogue of the curvature of plane curves. We state the theorem without proof.

Theorem 12. Let P be a point on a given curve C on the surface and Q be the neighbouring point of C at an arcual distance δs from P along C. Let the geodesics which are tangents to C at P and Q meet at R. If $\delta\psi$ is the angle between the tangents to the geodesics at R, then the geodesic curvature of C at P is $\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$.

3.11 GAUSS-BONNET THEOREM

So far we were dealing with the curvature κ of space curves and geodesic curvature κ_g of curves on surfaces and explained in the two theorems at the end of the previous section how κ_g is the analogue of curvature of plane curves. The next natural step is to create or find out intrinsic quantities so as to extend the notion of curvature to surfaces. This is what we do in the Gauss Bonnet theorem. This theorem deals with the surface integral of an intrinsic quantity leading to the definition of curvature of a surface. It is obtained as an application of Green's theorem on line and surface integrals in the plane. We shall introduce the following necessary preliminaries on the curvilinear polygons before taking up the theorem.

Definition 1. A region R of surface is said to be simply connected if every closed curve lying in the region R can be contracted or shrunk continuously into a point without leaving R.

In the plane, the interior of a circle is simply connected but the region between two concentric circles is not simply connected for a concentric circle between the two circles cannot be contracted into a point without leaving the region.

Definition 2. A closed curve on a surface is said to be described in the positive sense, if the sense of description of the curve is always on the left. This is nothing but the positive rotation of $\frac{\pi}{2}$ from the tangent to get the normal which points towards the interior of the region.

Definition 3. Let $R(u, v)$ be the given surface of class 3 and R be a simply connected region whose boundary is a closed curve of class 2. Let C consists of n arcs

$$A_0A_1, A_1A_2, \ldots. A_{n-1}A_n.$$

where n is finite.

Since C is a closed curve $A_0 = A_n$ and let each arc is of class 2. Let the vertices $A_0, A_1, \ldots. A_n$ are taken along the curve C described in the positive sense. At the vertex A_r, let α_r be the angle between the tangents to the arcs $A_{r-1}A_r$ and A_rA_{r+1} so that $-\pi < \alpha_r < \pi$. At A_n, let α_n be the angle between the tangents $A_{n-1}A_n$ and A_nA_1. Thus for the curvilinear polygon, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the exterior angles at the vertices $A_1, A_2, \ldots, A_n$.

Now the geodesic curvature κ_g exists at every point of C except possibly at the vertices. Hence $\int_C \kappa_g ds$ can be calculated as an integral along C by adding the integrals along separate arcs.

The excess of curve C denoted by $ex\ C$ is defined as

$$ex\ C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g ds$$

We first note the following simple properties of $ex\ C$.

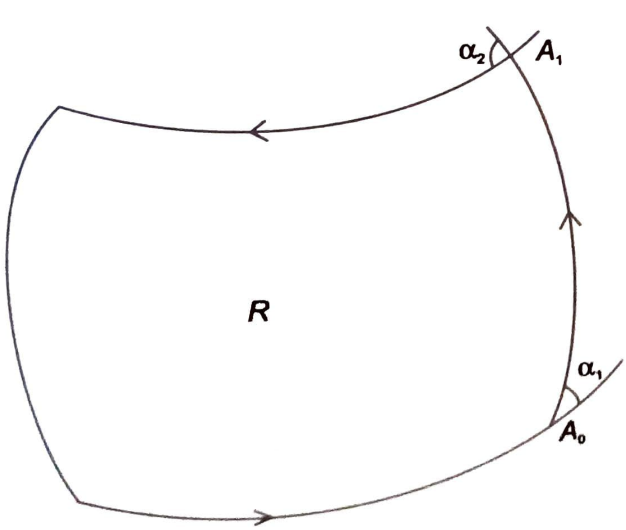

Fig. 13

(i) *ex C* is parametric invariant. That is, it is independent of the parametric system used for the representation of the surface. Since κ_g is the intrinsic property of the curve, it remains invariant for a parametric transformation.

Hence $\int_C \kappa_g ds$ in invariant. Since the parametric transformation at every point of the curve changes the sense in which angles are measured, it does

not affect $\sum_{r=1}^{n} \alpha_r$ and also κ_g. Hence *ex C* is invariant for parametric

transformations.

(ii) In the plane, if we consider a curvilinear polygon or rectilinear polygon enclosing a region of the plane, their *ex* $C = 0$ as shown below.

In the case of a curvilinear polygon C, κ_g is the ordinary curvature $\frac{d\psi}{ds}$ at

every point of the curve. Therefore $\int_C \kappa_g ds + \sum_{r=1}^{n} \alpha_r$ gives the total angle through

which the tangent turns in describing a closed curve C where the first term gives ψ and the second term gives the sum of the angles through which the tangents turned at the vertices of C. Since the total angle turned while describing a closed curve in the plane is 2π, we have

$$\int_C \kappa_g ds + \sum_{r=1}^{n} \alpha_r = 2\pi.$$

Hence the *ex* $C = 2\pi - \left(\int_C \kappa_g ds + \sum_{r=1}^{n} \alpha_r \right) = 0.$

In the case of a rectilinear polygon, the sides are straight lines which are the

geodesics of the plane so that $\kappa_g = 0$ at every point of the sides. Since $\sum_{r=1}^{n} \alpha_r$ is the

sum of the exterior angles, it is 2π. Hence *ex* $C = 2\pi - \sum_{r=1}^{n} \alpha_r = 0$. Hence whether

we take a curvilinear polygon or a rectilinear polygon in the plane, *ex* $C = 0$.

The above property (ii) leads to the following important conclusion

(iii) On any surface isometric with the plane, excess of a simple closed curve C is zero.

This follows from the fact that *ex C* is defined intrinsically in the sense that it is given in terms of the metric of the surface. Hence for the surfaces other than the plane, a closed curve enclosing a region R of a surface *ex C* is different from zero. So the question naturally arises whether for a surface which is not isometric with the plane the excess of a simple closed curve C on surface provides a satisfactory measure of intrinsic difference between R and a region of the plane. The answer to this question is in the affirmative and the *ex C* provides an intrinsic definition of curvature of a surface based on the convention that the plane and any surface isometric with the plane has zero curvature. The following theorem known as Gauss-Bonnet Theorem gives a complete answer to this question leading to the definition of Gaussian curvature of the surface.

Theorem (Gauss-Bonnet). For any curve C which encloses a simply connected region R on surface, *ex C* is equal to the total curvature of R.

Proof. We shall use Liouville's formula for κ_g and find $\int \kappa_g \, ds$ with the help of Green's theorem in the plane for a simply connected region R bounded by C. So we shall quote this Green's theorem as a lemma as applicable for a region R of a surface.

Lemma. If R is a simply connected region bounded by a closed curve C, then

$$\int_C P\,du + Q\,dv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv$$

where P and Q are differentiable functions of u and v in R.

Proof of the Theorem. From the Liouville's formula,

$$\kappa_g = \theta' + Pu' + Qv'$$

Integrating along the curve C, we have

$$\int_C \kappa_g\,ds = \int_C (\theta' + Pu' + Qv')\,ds = \int_C (d\theta + P\,du + Q\,dv) \qquad ...(1)$$

where θ is the angle between the curve C and the parametric curve v = constant and P and Q are differentiable functions of u, v.

Let us suppose the simple closed curve C contains a finite number of arcs starting from A. Then at each point of the arc there passes a curve v = constant making an angle θ with C. Hence when we describe the curve C, the tangents at various members of the family v = constant described in the positive sense returns to the starting point after increasing the angles of rotation by 2π.

This increase 2π after complete rotation in the positive sense also includes the angle between the tangents at the finite number of vertices. Hence we have

$$\int_C d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi \qquad ...(2)$$

From the definition $ex\, C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g\,dS$ ...(3)

Using (1) and (2) in (3), we obtain

$$ex\, C = 2\pi - \left[2\pi - \int_C d\theta\right] - \left[\int_C d\theta + P\,du + Q\,dv\right]$$

Thus $$ex\, C = -\int_C P\,du + Q\,dv \qquad \ldots(4)$$

Since R is a simply connected region and P and Q are differentiable functions of u, v, we have by Green's theorem,

$$\int_C (P\,du + Q\,dv) = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv \qquad \ldots(5)$$

Since the surface element $ds = H\,du.dv$, we rewrite (5) as

$$\int_C (P\,du + Q\,dv) = \frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dS \qquad \ldots(6)$$

Using (6) in (4), we get

$$ex\, C = -\frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dS \qquad \ldots(7)$$

If we take $K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$, we can rewrite (7) as

$$ex\, C = \iint_R K ds \qquad \ldots(8)$$

where K is a function of u and v and it is independent of the C and defined over the region R of the surface.

Next we shall show that the $ex\,C$ is uniquely determined by K. If K is not unique, let $\overline{K}$ be such that

$$\iint_R \overline{K} ds = ex\, C \qquad \ldots(9)$$

Using (8) and (9), we have $\iint_R (\overline{K} - K)\, ds = 0$ ...(10)

for every region R.

Now let $\overline{K} \neq K$ at some point A of R. Then we must have $\overline{K} > K$ or $\overline{K} < K$ at A. Let us first consider $\overline{K} > K$.

Since the given surface is of class 3, $\frac{\partial Q}{\partial u}$ and $\frac{\partial P}{\partial v}$ are continuous in R so that there exists a small region R_1 of R containinig the point A such that $\overline{K} - K > 0$ at every point of R_1. For this region R containing R_1, $\iint_{R_1} (\overline{K} - K)\, ds > 0$ which contradicts (10). We get a similar contradiction $\iint_{R_1} (\overline{K} - K)\, ds < 0$ at A when $\overline{K} < K$. These contradictions prove that $\overline{K} = K$ at every point of R. That is, K is uniquelly determined as a function of u and v.

Defining $\int_R KdS$ as the total curvature of R we have proved that the total curvature is exactly the ex C in any region R enclosed by C. This completes the proof of Gauss-Bonnet Theorem.

Note **1.** $ex\, c = \int_R KdS$ shows that there is a certain function K of u and v which is determined by E, F and G and that the excess of any curve C which encloses a simply connected region R is the integral of K over R. Also from the uniqueness of K and the form of the integral, K is invariant in the sense that it is independent of the parametric system. Since K can be found from the metric, K is an intrinsic geometrical invariant.

Definition 4. The invariant K as defined above is called the Gaussian curvature of the surface and $\int_R KdS$ is called the total curvature or integral curvature of R where R is any region whether simply connected or not.

Note **2.** We rewrite the excess of a curve C using the integral as follows. Since $ex\ C = \int_R Kds$, we have

$$\int_R KdS = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g\, ds$$

We give a few examples to illustrate Gauss-Bonnet theorem.

Example 1. Find the curvature of a geodesic triangle ABC enclosing a region R on the surface.

Since ex C gives total curvature of a region bounded by C, it is enough if we find ex C in the examples where C is known.

ABC is a geodesic triangle enclosing a region R on the surface with the interior angles A, B, C. So in this case the curve C is the geodesic triangle ABC.

Since AB, BC, and CA are geodesics on a surface $\kappa_g = 0$ at every point of AB, BC and CA so that $\int_{ABC} \kappa_g\, ds = 0$. Since A, B, and C are interior angles, the exterior angles at the vertices are $\pi - A$, $\pi - B$ and $\pi - C$.

Hence $\quad ex\, C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_C \kappa_g\, ds$

$$= 2\pi - [(\pi - A) + (\pi - B) + (\pi - C)] \text{ as } \int_{ABC} \kappa_g\, ds = 0.$$

$$= A + B + C - \pi.$$

Thus the $ex\, \Delta ABC$ is the excess of $A + B + C$ over the Euclidean value of π. In other words the total curvature of a geodesic triangle ABC on a surface is $A + B + C - \pi$.

Note. Let us consider the geodesic polygon $A_1, A_2, \ldots, A_n$ of n-sides with interior angles $A_1, A_2, \ldots A_n$. Since $\int \kappa_g\, ds$ is zero on every side of the polygon,

$$ex\, C = 2\pi - [(\pi - A_1) + (\pi - A_2) + \ldots + (\pi - A_n)]$$

$= (A_1 + A_2 + \ldots + A_n) - (n-2)\pi$ which gives the total curvature of the region bounded by a geodesic polygon of n-sides on a surface. It is nothing but the excess of the sum of the interior angles over $(n-2)\pi$ where n is the number of sides of the polygon.

Example 2. The total curvature of the whole surface of an anchor ring is zero.

To prove this, let us consider the region R bounded by two meridians and two parallels on the anchor ring. We know that the meridians and parallels are geodesics on the anchor ring and they are the parametric curves also. Since the parametric curves are orthogonal, the region bounded by two meridians and two parallels is a geodesic rectangle with interior angles $\frac{\pi}{2}$ each. Since the sides of the rectangle are geodesics, $\kappa_g = 0$ and consequently $\int_C \kappa_g\, ds = 0$. Hence we have

$$ex\, C = 2\pi - \left[\left(\pi - \frac{\pi}{2}\right) + \left(\pi - \frac{\pi}{2}\right) + \left(\pi - \frac{\pi}{2}\right) + \left(\pi - \frac{\pi}{2}\right)\right] + \int_C \kappa_g\, ds.$$

which is zero. Since $ex\, C = 0$ for the geodesic rectangle on the anchor ring, the total curvature of the surface bounded by the geodesic rectangle is zero on the anchor ring. Since every point of the anchor ring can be enclosed by a geodesic rectangle of the above type, the total curvature of the whole surface is zero.