$$H^2 = EG - F^2 = u^2 + a^2 \text{ so that } H = \sqrt{u^2 + a^2}$$

Again $\mathbf{r}_1 \times \mathbf{r}_2 = (a \sin v, - a \cos v, u)$

Now $$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \left(\frac{a}{\sqrt{u^2 + a^2}} \sin v, - \frac{a}{\sqrt{u^2 + a^2}} \cos v, \frac{u}{\sqrt{u^2 + a^2}} \right)$$

Let the components of **N** be (N_1, N_2, N_3).

Using (ii), the direction coefficients of the parametric curves are

$$\left(\frac{1}{\sqrt{E}}, 0 \right) = (1, 0) \text{ and } \left(0, \frac{1}{\sqrt{G}} \right) = \left(0, \frac{1}{\sqrt{u^2 + a^2}} \right)$$

If γ is the angle made by **N** with the z-axis, then $\cos \gamma = N_3 = \dfrac{u}{\sqrt{u^2 + a^2}}$

If (l', m') is the direction coefficient orthogonal to the parametric direction v = constant, then by Theoem 2 we have $l' = -\dfrac{1}{H}(Fl + Gm)$, $m' = \dfrac{1}{H}(El + Fm)$.

Substituting for l, m, E, F, G and H in the above step, we have $l' = 0$ and $m' = \dfrac{1}{\sqrt{u^2 + a^2}}$ which is the direction of the parametric system u = constant. This is what we expect, since the parametric curves are orthogonal.

2.11 FAMILIES OF CURVES

So far, we were concerned with a single curve lying on a surface and associated tangential direction. Now we shall introduce families of curves on a surface and study some basic properties of such families.

Definition 1. Let $\phi(u, v)$ be a single valued function of u, v possessing continuous partial derivatives ϕ_1, ϕ_2 which do not vanish together. Then the implicit equation $\phi(u, v) = c$ where c is a real parameter gives a family of curves on the surface $\mathbf{r} = \mathbf{r}(u, v)$.

For different values of c, we get different curves of the family lying on the surfaces. From the very definition, we note the following properties.

(i) Through every point (u, v) of the surface, there passes one and only one member of the family.

Let $\phi(u_0, v_0) = c_1$ where (u_0, v_0) is any point on the surface. Then $\phi(u, v) = c_1$ is a member of the family passing through (u_0, v_0). Hence through every point (u_0, v_0) on the surface, there passes one and only one member of the family.

(ii) As noted in (vi) of 2.10, the direction ratios of the tangent to the curve of the family at (u, v) is $(-\phi_2, \phi_1)$.

Theorem 1. The curves of the family $\phi(u, v)$ = constant are the solutions of the differential equation

$$\phi_1\, du + \phi_2\, dv = 0 \qquad \ldots(1)$$

and conversely a first order differential equation of the form

$$P(u, v)du + Q(u, v)dv = 0 \qquad \ldots(2)$$

where P and Q are differentiable functions which do not vanish simultaneously defines a family of curves.

Proof. Since $\phi_1 = \dfrac{\partial \phi}{\partial u}$ and $\phi_2 = \dfrac{\partial \phi}{\partial v}$, we get from (1),

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0 \text{ giving } d\phi = 0$$

Hence we conclude that $\phi(u, v) = c$. Thus as the constant c varies, the curves of the family are the differnt solutions of the differential equation.

Conversely let us consider the equation (2). Unless the equation is exact, it is not in general possible to find a single function $\phi(u, v)$ such that $\phi_1 = P$ and $\phi_2 = Q$.

However we can find integrating factor $\lambda(u, v)$ such that $\lambda P = \phi_1$ and $\lambda Q = \phi_2$. Rewriting the equation (2) in the form. $\lambda P\, du + \lambda Q\, dv = 0$, we get $\phi_1\, du + \phi_2\, dv = 0$, so that the solution of the equation is $\phi(u, v) = c$.

Further from (2), $\dfrac{du}{dv} = -\dfrac{Q}{P}$ so that the direction ratios of the tangent to the curves of the family at the point P is $(-Q, P)$.

The next theorem gives the geometrical significance of the tangent vector $(-\phi_2, \phi_1)$.

Theorem 2. For a variable direction at P, $\left|\dfrac{d\phi}{ds}\right|$ is maximum in a direction orthogonal to the curve $\phi(u, v)$ = constant through P and the angle between $(-\phi_2, \phi_1)$ and the orthogonal direction in which ϕ is increasing is $\dfrac{\pi}{2}$.

Proof. Let $P(u, v)$ be any point on the surface. We shall show that ϕ increases most rapidly at P in a direction orthogonal to the curve of the family passing through P. For this, we prove that $\dfrac{d\phi}{ds}$ has the greatest value in such a direction.

Let (l, m) be any direction through P on the surface. Let μ be the magnitude of the vector $\phi = (-\phi_2, \phi_1)$. Let θ be the angle between (l, m) and the vector ϕ.

Let us take

$$\boldsymbol{a} = l\mathbf{r}_1 + m\mathbf{r}_2, \boldsymbol{b} = -\phi_2\mathbf{r}_1 + \phi_1\mathbf{r}_2$$

We shall find $a \times b$ expressing $\sin\theta$ in terms of H and μ where $\mu = |b|$. From the definition $|a| = 1$.

We have $$|a \times b| = \mu \sin\theta \qquad ...(1)$$

and $$a \times b = (l\phi_1 + m\phi_2)(\mathbf{r}_1 \times \mathbf{r}_2) \text{ so that}$$

$$|a \times b| = H(l\phi_1 + m\phi_2) \qquad ...(2)$$

Equating (1) and (2), we obtain

$$\mu \sin\theta = H(l\phi_1 + m\phi_2) \qquad ...(3)$$

Since (l, m) are the direction coefficients of any direction through P, we have

$$l = \frac{du}{ds}, m = \frac{dv}{ds} \qquad ...(4)$$

Using (4) in (3) and simplifying, we get $\mu \sin\theta = H\dfrac{d\phi}{ds}$

Now μ and H are always positive and do not depend on (l, m).

Hence $\dfrac{d\phi}{ds}$ has maximum value $\dfrac{\mu}{H}$ when $\sin\theta$ has maximum value in which case $\theta = \dfrac{\pi}{2}$. In a similar manner, $\dfrac{d\phi}{ds}$ has minimum value $-\dfrac{\mu}{H}$, when $\theta = -\dfrac{\pi}{2}$.

Since $H > 0$, and $\mu > 0$, the orthogonal direction for which $\dfrac{d\phi}{ds} > 0$ is such that $\theta = \dfrac{\pi}{2}$. Hence $\left|\dfrac{d\phi}{ds}\right|$ has maximum in a direction orthogonal to $\phi(u, v) = \text{constant}$.

2.12 ORTHOGONAL TRAJECTORIES

Among the families of curves on a surface, we are interested in those families which cut each other orthogonally. This leads to the definition of orthogonal trajectories.

Definition. Let $\phi(u, v) = c$ be the equation of the given family of curves on the surface $\mathbf{r} = \mathbf{r}(u, v)$. If there exists a second family of curves $\psi(u, v) = k$ lying on the surface such that at every point of the surface two curves one from each family are orthogonal, then the curves of the second family are called the orthogonal trajectories of the first family.

Note. The above definition of trajectories is a generalisation of orthogonal parametric curves. Each family of orthogonal parametric curves can be considered as an orthogonal trajectory of the other.

The following theorem gives the existence and parametrisation of orthogonal trajectories of the given family.

Theorem 1. Every family of curves on a surface possesses a family of orthogonal trajectories.

Proof. Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of the surface. Let the given family of curves be defined by

$$Pdu + Qdv = 0 \qquad \ldots(1)$$

Since $\dfrac{du}{dv} = -\dfrac{Q}{P}$, the tangent at any point of the curve has the direction ratios $(-Q, P)$. Let (du, dv) be the direction ratios of the tangent at (u, v) of a member of the orthogonal trajectories of the given family.

We know that the two directions (λ, μ) and (λ', μ') are orthogonal, then

$$E\lambda\lambda' + F(\lambda\mu' + \mu\lambda') + G\mu\mu' = 0$$

Applying this condition of orthogonality to the two directions $(-Q, P)$ and (du, dv), we obtain

$$E(-Q)du + F(-Qdv + Pdu) + GPdv = 0$$

which simplifies to $(FP - EQ)du + (GP - FQ)\,dv = 0$ ...(2)
which is the differential equation of the orthogonal trajectories of the given family of curves.

Since $EG - F^2 \neq 0$ and P and Q do not vanish simultaneously, the coefficients of du and dv are continuous and do not vanish together. Hence the differential equation of the orthogonal family exists and it is the solution of (2).

Theorem 2. The parameters on a surface can always be chosen so that the curves of the given family and the orthogonal trajectories become parametric curves.

Proof. Let the given family $\phi(u, v) = c$ of curves be given by the differential equation

$$P\,du + Q\,dv = 0 \qquad \ldots(1)$$

Then by Theorem 1 of 2.11, there exists an integrating factor $\lambda = \lambda(u, v) \neq 0$ such that $P = \lambda\phi_1$ and $Q = \lambda\phi_2$.

By Theorem 1, the orthogonal family $\psi(u, v) = k$ of the given family is the solution of

$$(FP - EQ)du + (GP - FQ)dv = 0$$

Hence there exists an integrating factor $\mu\,(u, v) \neq 0$ such that $FP - EQ = \mu\psi_1$ and $GP - FQ = \mu\psi_2$ where $\psi_1 = \dfrac{\partial\psi}{\partial u}$ and $\psi_2 = \dfrac{\partial\psi}{\partial v}$

Hence
$$\frac{\partial(\phi, \psi)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial\phi}{\partial u} & \dfrac{\partial\phi}{\partial v} \\ \dfrac{\partial\psi}{\partial u} & \dfrac{\partial\psi}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{P}{\lambda} & \dfrac{Q}{\lambda} \\ \dfrac{1}{\mu}(FP - EQ) & \dfrac{1}{\mu}(GP - FQ) \end{vmatrix}$$

$$= \frac{1}{\lambda\mu}[P(GP - FQ) - Q(FP - EQ)]$$

Let us derive the differential equation of the given family. From (1), we have

$$\frac{d\mathbf{r}}{d\theta} = -a \sin\theta \qquad ...(3)$$

Eliminating a between (1) and (3), we have

$$\frac{d\mathbf{r}}{d\theta} = -\mathbf{r}\tan\theta \text{ or } \frac{1}{\mathbf{r}}d\mathbf{r} + \tan\theta\, d\theta = 0$$

so that we have $\quad P = \frac{1}{\mathbf{r}},\ Q = \tan\theta \qquad ...(4)$

By Theorem 1, the differential equation of the orthogonal trajectory is

$$(FP - EQ)\,du + (GP - FQ)\,dv = 0 \qquad ...(5)$$

Using (2) and (4) in (5), we get

$$-\tan\theta\, d\mathbf{r} + \mathbf{r}^2 \frac{1}{\mathbf{r}} d\theta = 0 \quad \text{or} \quad \frac{d\mathbf{r}}{\mathbf{r}} = \frac{d\theta}{\tan\theta} \qquad ...(6)$$

Integrating (6) and simplifying we have $\mathbf{r} = c\sin\theta \qquad ...(7)$

where c is a constant to be found out.

From (1), when $\theta = 0$, $\mathbf{r} = a$. Using this initial condition, we have from (7), $c = a$. Hence the system of orthogonal trajectories of the given system is $\mathbf{r} = a\sin\theta$.

2.13 DOUBLE FAMILY OF CURVES

Earlier we have characterised a family of curves by a differential equation of the form $Pdu + Qdv = 0$. So it is natural to expect a quadratic equation in (du, dv) will give a double family of curves. In this connection, we have the following definition.

Definition. If P, Q and R are continuous functions of u and v which do not vanish together and if $Q^2 - PR > 0$, then the quadratic differential equation

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0 \qquad ...(1)$$

represents two families of curves on the given surface.

Since (du, dv) give the direction ratios of the tangential directions at a point on the surface, we obtain the two directions of the double family at a point on the surface by solving the quadratic

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \qquad ...(2)$$

Hence the question naturally arises under what conditions the two directions become orthogonal directions. The answer is contained in the following theorem.

Theorem 1. The two directions given by

$$Pdu^2 + 2Q\,du\,dv + Rdv^2 = 0 \qquad ...(1)$$

are orthogonal on a surface, if and only if $ER - 2QF + GP = 0$

Proof. If (l, m) and (l', m') are the direction coefficients of the two families of curves (1) at a point P on the surface, then $\frac{l}{m}$ and $\frac{l'}{m'}$ are the roots of the quadratic

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \qquad ...(2)$$

Hence $$\frac{l}{m} + \frac{l'}{m'} = -\frac{2Q}{P}, \quad \frac{ll'}{mm'} = \frac{R}{P} \qquad ...(3)$$

By Theorem 1 of 2.10, (l, m) and (l', m') are orthogonal if and only if

$$E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G = 0 \qquad ...(4)$$

Using (3) in (4), we obtain $ER - 2\,FQ + GP = 0$ ...(5)

Corollary. The parametric curves are orthogonal if and only if $F = 0$.

The differential equation of the parametric curves is $du\,dv = 0$. ...(6)

Hence (6) is a double family (1) with $P = 0$, $Q \neq 0$, $R = 0$. Using these the condition of orthogonality becomes $QF = 0$. Since $Q \neq 0$, $F = 0$ which proves the corollary.

Theorem 2. If θ is the angle between the two curves given by the double family

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 \qquad ...(1)$$

at a point (u, v) on the surfaces, then

$$\tan\theta = \frac{2\,H\,(Q^2 - PR)^{1/2}}{ER - 2\,FQ + GP}$$

Proof. If (l, m) and (l', m') are the direction coefficients of the tangential directions at a point of the double family (1), they are the roots of the quadratic

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0 \qquad ...(2)$$

so that $$\frac{l}{m} + \frac{l'}{m'} = -\frac{2Q}{P}, \quad \frac{ll'}{mm'} = \frac{R}{P} \qquad ...(3)$$

If θ is the angle between the two directions, then from Theorem 1 of 2.10, we know that

$$\tan\theta = \frac{H(lm' - l'm)}{E\,ll' + F(lm' + ml') + Gmm'} \qquad ...(4)$$

(4) can be rewritten as

$$\tan\theta = \frac{H\left[\left(\frac{l}{m}+\frac{l'}{m'}\right)^2 - 4\frac{ll'}{mm'}\right]^{1/2}}{E\frac{ll'}{mm'} + F\left(\frac{l}{m}+\frac{l'}{m'}\right) + G} \quad ...(5)$$

Using (3) in (5) and simplifying

$$\tan\theta = \frac{2H(Q^2 - PR)^{1/2}}{ER - 2FQ + GP}$$

Corollary. When $\theta = \frac{\pi}{2}$, the two curves of the double family at any point of the surface are orthogonal. Hence the above condition reduces to

$$ER - 2FQ + GP = 0.$$

Example. Show that the curves $du^2 - (u^2 + c^2)\,dv^2 = 0$ ...(1)
form an orthogonal system on the right helicoid

$$\mathbf{r} = (u\cos v,\ u\sin v,\ cv) \quad ...(2)$$

(1) is a double family of curves on (2) with $P = 1$, $Q = 0$ and $R = -(u^2 + c^2)$.
For the right helicoid (2), we have

$$\mathbf{r}_1 = (\cos v, \sin v, 0),\ \mathbf{r}_2 = (-u\sin v, u\cos v, c)$$

Hence $E = \mathbf{r}_1\cdot\mathbf{r}_1 = 1$, $F = \mathbf{r}_1\cdot\mathbf{r}_2 = 0$, $G = \mathbf{r}_2\cdot\mathbf{r}_2 = (u^2 + c^2)$

Now $ER - 2FQ + GP = 1[-(u^2 + c^2)] + (u^2 + c^2)\,1 = 0$ so that the condition of orthogonality is satisfied. Hence the family of curves (1) form an orthogonal system on (2).

2.14 ISOMETRIC CORRESPONDENCE

We are concerned with the parts of two different surfaces S and S' locally and a correspondence or a transformation between them. Two surfaces are considered to be equivalent if this correspondence or transformation preserves some geometrical structure of the surface.

We will assign a parametric system (u, v) to each portion of surface S so that if the point (u', v') on S' corresponds to a point (u, v) on S, then u', v' are single valued functions of u, v given by

$$u' = \phi(u, v),\ v' = \psi(u, v) \quad ...(1)$$

If S and S' are of class r, r' respectively, we assume that ϕ and ψ are functions of class min (r, r') with Jocabian $J = \frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0$ in the domain u, v. We assume that the correspondence is one-to-one throughout the domain. Thus we implicitly restrict the maps between a part of S and a part of S' to be differentiable homeomorphism of sufficiently high class regular at each point of the domain.

Theorem. To each direction of the tangent to a curve C at P in S, there corresponds a direction of the tangent to C' at P' in S' and vice-versa.

Proof. Let C be a curve of a class ≥ 1 passing through P and lying on S. Let it be parametrically represented by $u = u(t)$ and $v = v(t)$. If S' is the portion corresponding to S under the relation (1) in the preceding paragraph, then C on S will be mapped onto C' on S' passing through P' with the parametric equations

$$u' = \phi\{u(t), v(t)\}, \; v' = \psi\{u(t), v(t)\}$$

The direction ratios of the tangents at P to C are $(\dot{u}, \dot{v})$ where $\dot{u} = \frac{du}{dt}, \; \dot{v} = \frac{dv}{dt}$.

Now the direction ratios of the tangents at P' to C' are $(\dot{u}', \dot{v}')$ where

$$\dot{u}' = \frac{du'}{dt} = \frac{\partial \phi}{\partial u}\dot{u} + \frac{\partial \phi}{\partial v}\dot{v},$$

$$\dot{v}' = \frac{dv'}{dt} = \frac{\partial \psi}{\partial u}\dot{u} + \frac{\partial \psi}{\partial v}\dot{v}$$

Solving the above equations for $\dot{u}$ and $\dot{v}$, we get

$$\dot{u} = \frac{1}{J}\left(\dot{u}'\frac{\partial \psi}{\partial v} - \dot{v}'\frac{\partial \phi}{\partial v}\right),$$

$$\dot{v} = \frac{1}{J}\left(\dot{v}'\frac{\partial \phi}{\partial u} - \dot{u}'\frac{\partial \psi}{\partial u}\right) \quad \text{where } J \neq 0$$

which shows that a given direction to a curve C' at P' corresponds to a definite direction at P to C and vice-versa.

Note. Since the functions ϕ and ψ satisfy the conditions for a proper parametric transformation, after transforming the parameters of S' in this way, S' can be reparametrised with (u, v) as parameters. To each point P' of S', we can take the corresponding point P of S and assign the parameter (u, v) of P to P'. Thus P and P' have identical parametric values. We also assume that such a correspondence preserves some geometrical properties. These lead to the following definition of isometry between S and S'.

Definition. Two surfaces S and S' are said to be isometric or applicable if there exists a correspondence $u' = \phi(u, v), v' = \psi(u, v)$ between their parameters where ϕ and ψ are single valued and $\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0$ such that the metric of S is transformed into a metric of S'. The correspondence itself is called an isometry.

From the very definition of an isometry, we find that the length of an arc of a curve C in S must be equal to the length of the corresponding arc C' in S'. That is $ds = ds'$ where ds and ds' are the corresponding linear elements and this must be true for all u, v, du, dv and u', v', du', dv' given by the proper parametric transformation. This shows that an isometric transformation maps the first fundamental form of S into the first fundamental form of S'.

The following Examples 1 and 2 are some obvious cases of an isometry between surfaces.

1. If a plane sheet of paper is slightly bent, the length of the curve drawn on it is not altered. Thus the original plane sheet and the slightly bent sheet are isometric.
2. A plane sheet may be isometric with a portion of a cylinder of sufficiently large dimension, but the entire plane is not isometric with the cylinder. Thus the plane and the cylinder are locally isometric.

Example 3. Find the surface of revolution which is isometric with the region of the right helicoid.

Let S be $\mathbf{r} = (g(u)\cos v, g(u)\sin v, f(u))$ the surface isometric with the right helicoid S' given by

$$\mathbf{r}' = (u'\cos v', u'\sin v', av')$$

Using the fact that the isometry preserves the metrics, we determine $g(u)$ and $f(u)$ and then indicate the region of correspondence.

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u} = (g_1(u)\cos v, g_1(u)\sin v, f_1(u))$$

$$\mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v} = (-g(u)\sin v, g(u)\cos v, 0)$$

Now $\quad E = \mathbf{r}_1\cdot\mathbf{r}_1 = g_1^2(u) + f_1^2(u),\ F = \mathbf{r}_1\cdot\mathbf{r}_2 = 0,\ G = \mathbf{r}_2\cdot\mathbf{r}_2 = g^2$

Hence the metric on S is $(g_1^2 + f_1^2)\,du^2 + g^2\,dv^2$ ...(1)

For the surface S', we have

$$\mathbf{r}_1' = (\cos v', \sin v', 0),\ \mathbf{r}_2' = (-u'\sin v', u'\cos v', a)$$

Hence $\quad E' = \mathbf{r}_1'\cdot\mathbf{r}_1' = 1,\ F' = \mathbf{r}_1'\cdot\mathbf{r}_2' = 0,\ G' = \mathbf{r}_2'\cdot\mathbf{r}_2' = (u'^2 + a^2)$

Hence the metric on S' is $du'^2 + (u'^2 + a^2)\,dv'^2$ ...(2)

The problem is to find the transformation from S to S' such that (1) and (2) are identical. Without loss of generality let us take $u' = \phi(u),\ v' = v$.

Then we have $du' = \phi_1(u)\,du,\ dv' = dv.$ ...(3)

Using (3) in (2), we get $\phi_1^2\,du^2 + (\phi^2 + a^2)\,dv^2$ ...(4)

(4) is the metric after transformation. Hence (1) and (4) are identical so that we have

$$g^2 = (\phi^2 + a^2),\ g_1^2 + f_1^2 = \phi_1^2 \quad ...(5)$$

From the equations (5), we have to obtain f and g eliminating ϕ. However, we can guess the solution of (5) as follows.

Let us take $\phi(u) = a\sinh u$ and $g(u) = a\cosh u$. ...(6)

(6) satisfies $g^2 = \phi^2 + a^2$. Using (6) in the second equation of (5), we get

$$a^2\sinh^2 u + f_1^2(u) = a^2\cosh^2 u \text{ so that } f_1^2(u) = a^2.$$

Thus $f_1(u) = a$. Integrating and choosing the constant of integration to be zero, we get $f(u) = au$.

Hence the surface of revolution is generated by

$$x = a \cosh u, \; y = 0, \text{ and } z = au$$

where the generating curve lies in the XOZ plane and the curve in the XOZ plane is a catenary with parameter a and the directrix as z-axis. Such a surface of revolution is known as catenoid.

Let us briefly describe the region of correspondence. The correspondence $u' = a \sinh u$, $v' = v$ shows that the generators v' = constant on the helicoid corresponds to the meridian v = constant on the catenoid and helices u' = constant correspond to the parallels u = constant of the catenoid. Further u', v' take all real values on the helicoid, but on the catenoid we have $0 \leq v < 2\pi$. Hence the correspondence is an isometry only for the region of the helicoid for which $0 \leq v' < 2\pi$. Thus one period of right helicoid of pitch $2\pi a$ corresponds isometrically to the whole catenoid of parameter a. If we do not insist on this condition, the correspondence is locally isometric.

Note. Instead of taking $u' = \phi(u)$ and $v' = v$, let us take $v' = pv$ where $p > 1$, we prove that the region of the right holicoid given by

$$|u'| < \frac{a}{\sqrt{p^2 - 1}}, \; 0 < v' < 2p\pi$$

corresponds isometrically to the surface obtained by revolving that part of the curve

$$x = ap \cosh u, \; y = 0, \; z = \int_0^u (\cosh^2 t - p^2 \sinh^2 t)^{1/2} \, dt.$$

given by $|u| < \cosh^{-1}\left(\frac{p}{\sqrt{p^2 - 1}}\right)$ about the z-axis.

Making the transformation $u' = \phi(u)$, $v' = pv$, the metric of the right helicoid becomes

$$\phi_1^2 \, du^2 + (\phi^2 + a^2) \, p^2 \, dv^2 \qquad \ldots(7)$$

Comparing (7) and (1), we get

$$f_1^2 + g_1^2 = \phi_1^2, \text{ and } g^2 = (\phi^2 + a^2)p^2.$$

Now choose $\phi(u) = a \sinh u$, we find $g(u) = ap \cosh u$.

Hence the curve in the XOZ plane is

$x = ap \cosh u$, $y = 0$, and we determine $f(u)$ by integration,

$$f_1^2(u) = \phi_1^2 - g_1^2 = a^2(\cosh^2 u - p^2 \sinh^2 u)$$

so that

$$f(u) = a\int_0^u (\cosh^2 t - p^2 \sinh^2 t)^{1/2} \, dt$$

Using the variation of u' and v', let us find the variation of u, v.

From the hypothesis $u' \leq \dfrac{a}{\sqrt{p^2 - 1}}$, $0 < v' < 2p\pi$.

$$u' = \phi(u) = a \sinh u < \frac{a}{\sqrt{p^2 - 1}}$$

We have

From the basic relation $a^2 \cosh^2 u = a^2 + a^2 \sinh^2 u$,

we find $a^2 \cosh^2 u \leq a^2 + \dfrac{a^2}{p^2 - 1} = \dfrac{a^2 p^2}{(p^2 - 1)}$

so that $\cosh u \leq \dfrac{p}{\sqrt{p^2 - 1}}$ or $|u| \leq \cosh^{-1}\left(\dfrac{p}{\sqrt{p^2 - 1}}\right)$

about the z-axis and $0 < v' < 2p\pi$.

2.15 INTRINSIC PROPERTIES

We conclude this chapter by explaining what is meant by intrinsic properties of surfaces. In the study of surfaces so far, the metric or the first fundamental form

$$I = E\,du^2 + 2F\,du\,dv + G dv^2$$

played a very important role. In fact, there is an existence theorem which states that if E, F, G are any given singled valued functions with $E > 0$ and $EG - F^2 > 0$ in some domain D of uv plane, then every point of D has a neighbourhood D' of D in which $E\,du^2 + 2F\,du\,dv + G\,dv^2$ is the metric of the surface referred to u, v as parameters. The proof of this first fundamental existence theorem is beyond the scope of the book.

It is important to note that any two isometric surfaces have the same metric, when the corresponding points are assigned the same parameters. Thus the family of surfaces having a given metric is the class of surfaces isometric to one another. Thus any formula of a surface which is deducible from the metric alone applies to the whole class of isometric surfaces. The properties of this kind deducible from the metric alone without using the vector equation $\mathbf{r} = \mathbf{r}(u, v)$ of the surface and preserved by surfaces which are isometric are called intrinsic properties of surfaces. Also in the study of intrinsic properties of surfaces, we have never made use of the normal component $a_n = a \cdot \mathbf{N}$. The properties involved with the use of normal components of a vector at a point P on the surface are called non-intrinsic properties. We shall make a study of the local non-intrinsic properties in chapter 4.

3

Geodesics on a Surface

3.1 INTRODUCTION

Defining a geodesic on a surface, we shall obtain canonical geodesic equation and its normal property. Then after establishing the existence theorem of a geodesic on a surface, we shall explain what is meant by geodesic parallel and geodesic coordinates. We shall take up for detailed study the geodesic curvature κ_g leading to Gauss-Bonnet theorem which states that for any curve C which encloses a simply connected region R, the excess of C is equal to the total curvature of R where excess of the curve C is suitably defined. Motivated by these ideas, we shall briefly study the surfaces of constant curvature. We conclude this chapter by a short account of conformal mapping and geodesic mapping. All the properties of a geodesic on a surface are illustrated with the help of surfaces introduced in the previous chapter.

3.2 GEODESICS AND THEIR DIFFERENTIAL EQUATIONS

We know that the straight line joining two points A and B in a plane is the shortest distance between A and B. The extension of the notion of a straight line in a plane to curves on surface leads to a special class of curves called geodesics which we shall define precisely with the help of the notion, 'shortest distance' or 'stationary length'. So before proceeding further, we shall explain these two notions.

If A and B are two points on a surface S, we can find lengths of different arcs joining them. Though this collection of lengths has non-zero greatest lower bound, we cannot say that there is an arc AB on the surface corresponding to this greatest lower bound. Since the shortest distance between two points on a surface has this shortcoming, we choose the alternative method of defining a geodesic on a surface as an arc of stationary length on a surface. This method of definition of a geodesic on a surface is amenable to the treatment of differential calculus and enables us to use differential equations for the study of such intrinsic curves.

Definition 1. Let A and B be two given points on a surface S and let these points be joined by curves lying on S. Then any curve possessing stationary length for small variation over S is called a geodesic.

Let A and B be two points on a surface $\mathbf{r} = \mathbf{r}(u, v)$. On the surface, let us consider all the arcs joining A and B given parametrically as $u = u(t)$, $v = v(t)$ where $u(t)$, $v(t)$ are of class 2. For every arc α, let us assume that $t = 0$ at A and $t = 1$ at B so that the arcs are defined in $[0, 1]$.

Let α be one such arc and $S(\alpha)$ be the length of the arc α joining A and B on the surface. Then we know that

$$ds^2 = E du^2 + 2F du dv + G dv^2 \text{ which gives}$$

$$\dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \text{ so that we can take}$$

$$s(\alpha) = \int_0^1 \dot{s}\, dt = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

Keeping the end points A and B fixed, let us deform the arc α slightly and obtain α'. Then we can take the equation of the new arc α' parametrically as

$$u'(t) = u(t) + \varepsilon\lambda(t),\ v'(t) = v(t) + \varepsilon\mu(t).$$

where $\varepsilon > 0$ is small and $\lambda(t)$, $\mu(t)$ are such that $\lambda(0) = \mu(0) = 0$ and $\lambda(1) = \mu(1) = 0$. After the deformation let the length of the arc be $s(\alpha')$ which we can obtain by replacing u, v in $S(\alpha)$ by u', v'. The variation in $s(\alpha)$ is $s(\alpha') - s(\alpha)$ which is of order ε in general.

Definition 2. If α is such that the variation in $s(\alpha)$ is atmost of order ε^2 for all small variations in α for different $\lambda(t)$ and $\mu(t)$, then $s(\alpha)$ is said to be stationary and α is a geodesic.

Note. Thus the geodesics on a surface are intrinsic in the sense that it is defined by the metric on the surface independent of any particular parametric representation of the surface. The following theorem gives the differential equations of a geodesic.

Theorem 1. A necessary and sufficient condition for a curve $u = u(t)$, $v = v(t)$ on a surface $\mathbf{r} = \mathbf{r}(u, v)$ to be a geodesic is that $U\dfrac{\partial T}{\partial \dot{v}} - V\dfrac{\partial T}{\partial \dot{u}} = 0$ ...(1)

where
$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \quad ...(2)$$

Proof. Equations (2) are called geodesic equations and we use the usual method of calculus of variations to derive the equations (2). With the help of (2), we obtain (1) and then prove that the converse of (1) is also true.

To prove (2), we need the following lemma.

Lemma. If $g(t)$ is a continuous function for $0 < t < 1$ and if $\int_0^1 v(t)\, g(t)\, dt = 0$ ...(3)

for all admissible functions $v(t)$ as defined above, then $g(t) = 0$.

Proof of the Lemma. Suppose $\int_0^1 v(t)\, g(t) = 0$ for all admissible functions $v(t)$ and $g(t) \neq 0$. Then there exists a t_0 between 0 and 1 such that $g(t_0) \neq 0$. Let us take $g(t_0) > 0$. Since $g(t)$ is continuous in (0, 1) and $t_0 \in (0, 1)$, there exists a neighbourhood (a, b) of t_0 such that $g(t_0) > 0$ in (a, b) where $0 \leq a < t < b \leq 1$

Now let us define a function $v(t)$ as follows.

$$v(t) = \begin{cases} (t-a)^3\,(b-t)^3 \text{ for } a \leq t \leq b \\ 0 \text{ for } 0 \leq t < a \text{ and } b < t \leq 1 \end{cases}$$

The $v(t)$ is an admissible function in (0, 1) so that (3) can be rewritten as

$$\int_0^1 v(t)\, g(t)\, dt = \int_0^a v(t)\, g(t)\, dt + \int_a^b v(t)\, g(t)\, dt + \int_b^1 v(t)\, g(t)\, dt$$

Using $v(t)$ in [0, 1] in the above step,

$$\int_0^1 v(t)\, g(t)\, dt = \int_a^b (t-a)^3\,(b-t)^3\, g(t)\, dt \qquad ...(4)$$

Since $(t-a)^3\,(b-t)^3 > 0$ in (a, b) and $g(t) > 0$ for $a < t < b$, we get from (4) $\int_0^1 v(t)\, g(t)\, dt > 0$ contradicting the hypothesis $\int_0^1 v(t)\, g(t)\, dt = 0$ for all admissible functions $v(t)$.

Hence our assumption that there exists a t_0 such that $g(t_0) \neq 0$ is false. Consequently $g(t) = 0$ for all t in (0,1) and thus the lemma is proved.

Proof of the Theorem. To prove (2), we proceed as follows.

Let $\quad f(u, v, \dot{u}, \dot{v}) = \sqrt{2T}$ where

$$2T(u, v, \dot{u}, \dot{v}) = \dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

In terms of f, the arc length $s(\alpha)$ is

$$s(\alpha) = \int_0^1 \dot{s}\, dt = \int_0^1 \sqrt{2T}\, dt = \int_0^1 f(u, v, \dot{u}, \dot{v})\, dt$$

After a slight deformation the arc-length $s(\alpha')$ is

$$s(\alpha') = \int_0^1 f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}, \dot{v} + \varepsilon\dot{\mu})\, dt.$$

Hence the variation in $s(\alpha)$ is

$$s(\alpha') - s(\alpha) = \int_0^1 \{f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}, \dot{v} + \varepsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})\}\, dt \quad ...(5)$$

Using Taylor's theorem for several variables, we get

$$f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}, \dot{v} + \varepsilon\dot{\mu})$$

$$= f(u, v, \dot{u}, \dot{v}) + \varepsilon\lambda\frac{\partial f}{\partial u} + \varepsilon\mu\frac{\partial f}{\partial v} + \varepsilon\dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \varepsilon\dot{\mu}\frac{\partial f}{\partial \dot{v}} + O(\varepsilon^2) \quad ...(6)$$

Using (6) in (5), we obtain

$$s(\alpha') - s(\alpha) = \varepsilon\int_0^1 \left\{\lambda\frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} + \dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}}\right\} dt + O(\varepsilon^2) \quad ...(7)$$

We shall simplify the integral on the right hand side of (7) as follows.
Using integration by parts, we have

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}\, dt = \int_0^1 \frac{\partial f}{\partial \dot{u}}\, d(\lambda) = \left[\lambda\frac{\partial f}{\partial \dot{u}}\right]_0^1 - \int_0^1 \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt.$$

Since $\lambda = 0$ at $t = 0$ and $t = 1$, we have $\left[\lambda\frac{\partial f}{\partial \dot{u}}\right]_0^1 = 0$

Hence $$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}\, dt = -\int_0^1 \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt \quad ...(8)$$

In a similar manner, we have

$$\int_0^1 \dot{\mu}\frac{\partial f}{\partial \dot{v}}\, dt = -\int_0^1 \mu\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) dt \quad ...(9)$$

Using (8) and (9) in (7) and simplifying,

$$s(\alpha') - s(\alpha) = \varepsilon\int_0^1 \left\{\lambda\left[\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\right] + \mu\left[\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)\right]\right\} dt + O(\varepsilon^2)$$

$$= \varepsilon\int_0^1 (\lambda L + \mu M)\, dt + O(\varepsilon^2) \quad ...(10)$$

where $$L = \left[\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\right],\ M = \left[\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)\right]$$

For the arc α to be geodesic on S, $s(\alpha)$ should be stationary. It is stationary if and only if the variation $s(\alpha)-s(\alpha')$ is atmost of order ε^2 for all small variations. Since $\varepsilon>0$ and $s(\alpha)-s(\alpha')$ is of ε^2, equation (10) implies that

$$\int_0^1 (\lambda L+\mu M)\,dt = 0 \qquad \text{...(11)}$$

for all admissible functions λ, μ of class 2 in $0\le t\le 1$ such that $\lambda=\mu=0$ at $t=0$ and $t=1$.

Since E, F, and G are of class 1 and $\lambda(t)$, $\mu(t)$ are of class 2, functions L and M are continuous functions satisfying the conditions as that of $g(t)$ of the Lemma. Hence we can apply Lemma to (11) choosing λ, μ and g as follows.

(i) Let $v(t)=\lambda$, $\mu=0$, and $g(t)=L$

Then $\int_0^1 (\lambda L+\mu M)\,dt = \int_0^1 \lambda L\,dt = 0$ which implies $L=0$ by the Lemma.

(ii) Let $\lambda=0$, $v(t)=\mu$ and $g(t)=M$.

Then $\int_0^1 (\lambda L+\mu M)\,dt = \int_0^1 \mu M\,dt = 0$ which implies $M=0$ by the Lemma.

Hence $L=0$, $M=0$ are the differential equations for $u(t)$ and $v(t)$. Since these two equations do not involve these two points A and B explicitly, the equations $L=0$, $M=0$ are the same for all geodesics on the surface.

Let us rewrite $L=0$, $M=0$ in terms of T.

Since $\quad f=\sqrt{2T},\ L=\dfrac{\partial f}{\partial u}-\dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot u}\right)$ becomes

$$L = \frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u}-\frac{d}{dt}\left[\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial \dot u}\right]$$

$$= \frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u}-\left[\frac{1}{\sqrt{2T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)-(2T)^{\frac{-3}{2}}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot u}\right]$$

$$= \frac{1}{\sqrt{2T}}\left[\frac{\partial T}{\partial u}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)\right]+\left[\frac{1}{(2T)^{3/2}}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot u}\right]$$

Since $T\neq 0$, cancelling $\dfrac{1}{\sqrt{2T}}$ throughout, $L=0$ becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right)-\frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot u} \qquad \text{...(12)}$$

In a similar manner we obtain for M,

$$M = \frac{1}{\sqrt{2T}}\left\{\frac{\partial T}{\partial v} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)\right\} + \frac{1}{(2T)^{3/2}}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}$$ which gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \quad ...(13)$$

Equations (12) and (13) give the differential equations of a geodesic. (12) and (13) are usually written as

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} \quad ...(14)$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \quad ...(15)$$

where $T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}[E\,\dot{u}^2 + 2F\,\dot{u}\dot{v} + G\,\dot{v}^2]$. This completes the proof of (2)

To prove (1) as the necessary and sufficient condition for α to be a geodesic on the surface S, we proceed as follows.

To prove the necessity of the condition, let α be a geodesic on the surface so that $u(t)$, $v(t)$ satisfy the differential equations (2).

From the second expression of U and V in (14) and (15)

we have $\frac{U}{V} = \frac{\partial T}{\partial \dot{u}} \Big/ \frac{\partial T}{\partial \dot{v}}$ so that $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$ which proves the necessity of the condition.

To prove the sufficiency of the condition, we need the following lemma which is true for any curve whether it is a geodesic or not.

Lemma. If U and V are as in (1), then $\dot{u}\,U + \dot{v}\,V = \frac{dT}{dt}$ ...(16)

Proof of the Lemma. Since each of U and V have two equal expressions for it, we shall prove (16) by considering the following two cases.

Case (i) In this case, we prove (16) by considering the first expression for U and V.

Since T is a homogeneous function of degree two in $\dot{u}$ and $\dot{v}$, we have by Euler's Theorem.

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T \quad ...(17)$$

Since T is a function of $u, v, \dot{u}, \dot{v}$, we get

$$\frac{dT}{dt} = \frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v} \quad ...(18)$$

Using (17) and (18), we prove (16) .

Substituitng for U and V, we have

$$\dot{u}\,U+\dot{v}\,V=\dot{u}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)-\frac{\partial T}{\partial \dot{u}}\right]+\dot{v}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)-\frac{\partial T}{\partial v}\right] \quad \ldots(19)$$

Now consider, $\frac{d}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}}\right)=\dot{u}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)+\ddot{u}\left(\frac{\partial T}{\partial \dot{u}}\right)$

so that
$$\dot{u}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)=\frac{d}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}}\right)-\ddot{u}\left(\frac{\partial T}{\partial \dot{u}}\right) \quad \ldots(20)$$

In a similar manner,
$$\dot{v}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)=\frac{d}{dt}\left(\dot{v}\frac{\partial T}{\partial \dot{v}}\right)-\ddot{v}\left(\frac{\partial T}{\partial \dot{v}}\right) \quad \ldots(21)$$

Using (20) and (21) in (19), we obtain

$$\dot{u}\,U+\dot{v}\,V=\frac{d}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}}\right)-\ddot{u}\frac{\partial T}{\partial \dot{u}}-\dot{u}\frac{\partial T}{\partial u}$$

$$+\frac{d}{dt}\left(\dot{v}\frac{\partial T}{\partial \dot{v}}\right)-\ddot{v}\frac{\partial T}{\partial \dot{v}}-\dot{v}\frac{\partial T}{\partial v}$$

$$=\frac{d}{dt}\left[\dot{u}\frac{\partial T}{\partial \dot{u}}+\dot{v}\frac{\partial T}{\partial \dot{v}}\right]-\left[\frac{\partial T}{\partial u}\dot{u}+\frac{\partial T}{\partial v}\dot{v}+\frac{\partial T}{\partial \dot{u}}\ddot{u}+\frac{\partial T}{\partial \dot{v}}\ddot{v}\right]$$

$$=\frac{d}{dt}(2T)-\frac{dT}{dt}=\frac{dT}{dt} \text{ using (17) and (18).}$$

Case (ii) In this we take the second expressions for U and V.

Now $\dot{u}\,U+\dot{v}\,V=\dot{u}\left(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}\right)+\dot{v}\left(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}}\right)$

$$=\frac{1}{2T}\frac{dT}{dt}\left(\dot{u}\frac{\partial T}{\partial \dot{u}}+\dot{v}\frac{\partial T}{\partial \dot{v}}\right)=\frac{1}{2T}\frac{dT}{dt}2T=\frac{dT}{dt} \text{ by (17)}$$

Hence we have $\dot{u}\,U+\dot{v}\,V=\frac{dT}{dt}$.

The sufficiency is established by showing that a curve on the surface satisfying the condition (1) of the theorem is a geodesic. For this it is enough if we show the condition (1) implies that U and V satisfy the geodesic equations (2).

Let U and V satisfy the condition (1). Let us assume that $\dot{u}$ and $\dot{v}$ are not zero for the same value of t. For if $\dot{u}=\dot{v}=0$ simultaneously, then

$$\frac{\partial T}{\partial \dot{u}}=E\dot{u}+F\dot{v}, \text{ and } \frac{\partial T}{\partial \dot{v}}=F\dot{u}+G\dot{v}$$

implies $\frac{\partial T}{\partial \dot{u}} = 0$, $\frac{\partial T}{\partial \dot{v}} = 0$ simultaneously so that the condition is trivially satisfied.

From the given condition, we have $\frac{U}{\frac{\partial T}{\partial \dot{u}}} = \frac{V}{\frac{\partial T}{\partial \dot{v}}} = \theta \text{(say)}$

Then
$$U = \theta \frac{\partial T}{\partial \dot{u}} \text{ and } V = \theta \frac{\partial T}{\partial \dot{v}} \quad \text{...(22)}$$

We shall find θ using the Lemma.

From the Lemma, we have $U\dot{u} + V\dot{v} = \frac{dT}{dt}$

Using (22) in the above equation, we get

$$\frac{dT}{dt} = \left(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}\right)\theta = 2\,T\theta \text{ by (17)}$$

From the above equation, we find $\theta = \frac{1}{2T}\frac{dT}{dt}$.

Using this value of θ in (22), we get

$$U = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}},\ V = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}$$

which are the geodesic equations. Hence $(u(t), v(t))$ is a point on the geodesic of a surface which proves the theorem completely.

Note. By the lemma, U and V satisfy the equation $\dot{u}\,U + \dot{v}\,V = \frac{dT}{dt}$. Hence both the expressions satisfy the same equation so that the two expressions for U and V are not independent. Hence these two equations are equivalent to one equation for two functions $u = u(t)$ and $v = v(t)$ so that the curve can be defined by $v = f(u)$.

Specialising the above theorem for the parametric curves v = constant and u = constant on a surface, we have the following theorem.

Theorem 2. (i) When $v = c$ for all values of u, a necessary and sufficient condition that the curve $v = c$ is a geodesic is

$$EE_2 + FE_1 - 2EF_1 = 0$$

(ii) When $u = c$ for all values of v on a surface, a necessary and sufficient condition that the curve $u = c$ is a geodesic is $GG_l + FG_2 - 2GF_2 = 0$.

Proof. On the curve $v = c$, u itself can be taken as a parameter so that the equations of the curve are $u = t$ and v = constant.

By Theorem 1, we know that a curve on a surface is a geodesic if and only if

$$U\frac{\partial T}{\partial \dot v} - V\frac{\partial T}{\partial \dot u} = 0 \quad \dots(1)$$

So to obtain the condition for the parametric curve v = constant to be a geodesic.

We find U, V, $\frac{\partial T}{\partial \dot u}$ and $\frac{\partial T}{\partial \dot v}$ and use them in (1)

From the definition, we have

$$T = \frac{1}{2}(E\dot u^2 + 2F\dot u\dot v + G\dot v^2)$$ where E, F and G are function of u, v.

Now
$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[E_1\dot u^2 + 2F_1\dot u\dot v + G_1\dot v^2\right]$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left[E_2\dot u^2 + 2F_2\dot u\dot v + G_2\dot v^2\right]$$

$$\frac{\partial T}{\partial \dot u} = E\dot u + F\dot v,\ \frac{\partial T}{\partial \dot v} = F\dot u + G\dot v \quad \dots(2)$$

According to the choice of the parameters, $\dot u = 1$, $\dot v = 0$

Hence
$$\frac{\partial T}{\partial u} = \frac{1}{2}E_1,\ \frac{\partial T}{\partial v} = \frac{1}{2}E_2,\ \frac{\partial T}{\partial \dot u} = E,\ \frac{\partial T}{\partial \dot v} = F \quad \dots(3)$$

Using (3), we get

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) - \frac{\partial T}{\partial u} = \frac{dE}{dt} - \frac{1}{2}E_1$$

Using the formula for the derivative of E as a function of u, v, we obtain

$$U = \frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\frac{dv}{dt} - \frac{1}{2}E_1 = E_1\dot u + E_2\dot v - \frac{1}{2}E_1$$

Since $\dot u = 1$, and $\dot v = 0$, we have $U = \frac{1}{2}E_1$

and
$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{dT}{dv} = \frac{dF}{dt} - \frac{1}{2}E_2$$

$$= \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} - \frac{1}{2}E_2 = F_1 - \frac{1}{2}E_2$$

Using (4) and (5) in (1), we obtain

$$\frac{1}{2}E_1F - \left(F_1 - \frac{1}{2}E_2\right)E = 0$$ which gives $EE_2 - 2EF_1 + FE_1 = 0$

(ii) When u = constant, let v itself can be taken as a parameter so that $v = t$. Hence $\dot{u} = 0$, $\dot{v} = 1$. Using these two, (2) becomes

$$\frac{\partial T}{\partial u} = \frac{1}{2}G_1, \frac{\partial T}{\partial v} = \frac{1}{2}G_2, \frac{\partial T}{\partial \dot{u}} = F, \frac{\partial T}{\partial \dot{v}} = G. \quad ...(6)$$

Using (6), we get

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{dF}{dt} - \frac{1}{2}G_1$$

By the formula for the derivative of F as a function of u, v

we obtain $$U = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} - \frac{1}{2}G_1 = F_1\dot{u} + F_2\dot{v} - \frac{1}{2}G_1$$

Using $$\dot{u} = 0 \text{ and } \dot{v} = 1,\ U = F_2 - \frac{1}{2}G_1 \quad ...(7)$$

Also we have $$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{dG}{dt} - \frac{1}{2}G_2$$

$$= \frac{\partial G}{\partial u}\frac{du}{dt} + \frac{\partial G}{\partial v}\frac{dv}{dt} - \frac{1}{2}G_2 = \frac{1}{2}G_2 \quad ...(8)$$

Using (7) an (8) in equation (1), we get

$$\left(F_2 - \frac{1}{2}G_1\right)G - \frac{1}{2}G_2F = 0 \text{ giving}$$

$$GG_1 + FG_2 - 2GF_2 = 0$$

Converse follows by retracing the steps in both (i) and (ii)

Corollary. When the parametric curves are othogonal,

(i) v = constant is a geodesic if and only if $E_2 = 0$

(ii) u = constant is a geodesic if and only if $G_1 = 0$

Since the parametric curves are orthogonal, $F = 0$.

Taking $F = 0$ in the theorem, we get the above particular cases.

In the following, we obtain a generalised form of the above theorem by taking $u = t$ and $v = v(u)$ and this incidentally illustrate the complicated nature of the geodesic equation in general.

Theorem 3. If $\dot{u} \neq 0$ in the neighbourhood of a point on a geodesic, then taking $u(t) = t$, the curve $v = v(u)$ is a geodesic if and only if v satisfies the second order differential equation

$$\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0$$

where P, Q, R and S are functions of u and v determined by E, F and G.

Proof. Using the condition $U\dfrac{\partial T}{\partial \dot{v}} - V\dfrac{\partial T}{\partial \dot{u}} = 0$, we shall derive the differential equation of the geodesic.

Now $$T = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2]$$

$$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v} = E + \dot{v}F, \ \dot{u} = 1 \quad \text{...(1)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = \frac{dE}{dt} + \frac{dF}{dt}\dot{v} + F\ddot{v}$$

$$= \frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\frac{dv}{dt} + \left(\frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt}\right)\dot{v} + F\ddot{v}$$

As $$\dot{u} = 1, \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = E_1 + (E_2 + F_1)\dot{v} + F_2\dot{v}^2 + F\ddot{v}$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}[E_1 + 2F_1\dot{v} + G_1\dot{v}^2], \ \dot{u} = 1.$$

Hence $$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = F\ddot{v} + \left(F_2 - \frac{1}{2}G_1\right)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1 \quad \text{...(2)}$$

Let us find V.

$$\frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v} = F + G\dot{v} \quad \text{...(3)}$$

Hence $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = \left(\frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial v}\dot{v}\right) + \left(\frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial v}\dot{v}\right)\dot{v} + G\ddot{v}$$

As $$\dot{u} = 1, \text{ we get } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v}$$

So $$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{dT}{dv}$$

$$= F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v} - \frac{1}{2}[E_2 + 2F_2\dot{v} + G_2\dot{v}^2]. \quad \text{...(4)}$$

$$= G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2$$

Changing the sign, we use the condition of a geodesic in the form

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0$$

Substituting (1), (2), (3) and (4) in the above equation,

$$(E + F\dot{v})\left[G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2\right]$$

$$-(F + G\dot{v})\left[F\ddot{v} + \left(F_2 - \frac{1}{2}G_1\right)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1\right] = 0$$

Writing the above equation as a second order differential equation, we have

$$(EG - F^2)\ddot{v} + \frac{1}{2}(GG_1 + FG_2 - 2\,GF_2)\dot{v}^3$$

$$+ \frac{1}{2}[G_2E + 3\,G_1F - 2\,FF_2 - 2\,E_2G]\dot{v}^2$$

$$+ \frac{1}{2}[2\,G_1E + 2\,F_1F - 3\,E_2F - E_1G]\dot{v}$$

$$+ \frac{1}{2}(2\,EF_1 - E_2E - E_1F) = 0$$

which can be rewritten as $V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}}$

$= H^2\,[\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S] = 0$ where

$$P = \frac{1}{H^2}\cdot\frac{1}{2}\,(GG_1 + FG_2 - 2\,GF_2),$$

$$Q = \frac{1}{H^2}\,(G_2E + 3\,G_1F - 2\,FF_2 - 2\,E_2G)$$

$$R = \frac{1}{H^2}\cdot\frac{1}{2}\,(2\,G_1E + 2\,FF_1 - 3\,E_2F - E_1G)$$

$$S = \frac{1}{H^2}\cdot\frac{1}{2}\,(2\,EF_1 - E_2E - E_1F)$$

Hence the equation of the geodesic is given by

$$\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0$$

Example 1. Prove that the curves of the family $\frac{v^3}{u^2}$ = constant are geodesics on a surface with the metric

$$v^2du^2 - 2\,uv\,dudv + 2\,u^2\,dv^2,\ u > 0,\ v > 0.$$

$\frac{v^3}{u^2} = c$ is a geodesic at any point on the surface if and only if

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$$

Choosing t as a parameter, the parametric representation of the curve can be taken as $u = ct^3,\ v = ct^2$ ...(1)

Hence we have $\dot{u} = 3ct^2,\ \dot{v} = 2ct.$ ...(2)

From the given metric, we define T as

$$T = \frac{1}{2}[v^2\dot{u}^2 - 2\,uv\,\dot{u}\,\dot{v} + 2u^2\dot{v}^2]$$ ...(3)

Using (1) and (2), we obtain

$$\frac{\partial T}{\partial u} = -v\,\dot{u}\,\dot{v} + 2\,u\dot{v}^2 = 2\,c^3t^5$$

$$\frac{\partial T}{\partial \dot{v}} = v\dot{u}^2 - u\,\dot{u}\,\dot{v} = 3\,c^3t^6$$

$$\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6$$

$$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2\,u^2\dot{v} = c^3t^7$$

Hence $$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(c^3t^6) - 2\,c^3t^5 = 4\,c^3t^5$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(c^3t^7) - 3\,c^3t^6 = 4\,c^3t^6$$

Hence $$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 4\,c^3t^5 \cdot c^3t^7 - 4\,c^3t^6 \cdot c^3t^6 = 0$$

Therefore the curve $\frac{v^2}{u^3} = c$ is a geodesic on the surface for all values of c.

Example 2. Prove that the parametric curves on a surface are orthogonal, the curves v = constant is geodesic provided $\boldsymbol{E}$ is a function of u only and the curve u = constant is a geodesic if $\boldsymbol{G}$ is a function of v only.

The parametric curves v = constant are all geodesics if and only if

$$EE_2 + FE_1 - 2EF_1 = 0$$ ...(1)

for all values u and v by Theorem 2(i).

Since the parametric curves are orthogonal, $\boldsymbol{F} = 0$ and consequently $F_1 = 0$. If $\boldsymbol{E}$ is a function of u only, $E_2 = 0$ Thus E_2, $\boldsymbol{F}$ and F_1 are all zero so that the condition (1) is satisfied for all u, v. Hence v = constant is a geodesic.

Now the parametric curves u = constant are all geodesics if and only if

$$GG_1 + FG_2 - 2GF_2 = 0 \qquad ...(2)$$

for all values of u and v by Theorem 2(ii).

As in the previous case $F = 0$ and $F_2 = 0$. Since G is a function of v only, $G_1 = 0$. Thus F, F_2 and G_1 are all zero so that the condition (2) is satisfied. Hence u = constant is a geodesic.

3.3 CANONICAL GEODESIC EQUATIONS

Treating the parameter t to be arbitrary, we have derived the geodesic equations on the surface in the previous section. Hence in place of t, we can choose the arc length s of the curve measured from a fixed point on it as parameter and obtain the geodesic equations in a much simpler form and also modify the condition $\dot{u}\,U + \dot{v}\,V = \frac{dT}{dt}$ for a curve on a surface to be a geodesic in terms of s. We shall use prime to denote differentiation with respect to the parameter s.

Theorem 1. If the arc length s is the parameter of the curve, then the geodesic equations are

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \qquad ...(1)$$

The equations (1) are called canonical geodesic equations.

Proof. Since the geodesic equations are true for any arbitrary parameter t, it is equally true for the parameter s also. Hence if prime denotes the differentiation with respect to s, the geodesic equations become

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{ds}\frac{\partial T}{\partial u'} \qquad ...(2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{ds}\frac{\partial T}{\partial v'} \qquad ...(3)$$

where $$T = \frac{1}{2}\,[Eu'^2 + 2F\,u'v' + Gv'^2]$$

Since $u' = \frac{du}{ds} = l$, $v' = \frac{dv}{ds} = m$ are the directional coefficients at a point on the curve and $El^2 + 2Flm + Gm^2 = 1$,

$$T = \frac{1}{2}\,[El^2 + 2F\,lm + Gm^2] = \frac{1}{2} \text{ so that } \frac{dT}{ds} = 0 \qquad ...(4)$$

Using (4) in the right hand side of (2) and (3), we obtain the canonical geodesic equations (1).

Note. One should note that $T = \frac{1}{2}$ only along the curve and it is not equal to $\frac{1}{2}$ identically for all values of u, v, u' and v'. The partial derivatives $\frac{dT}{du'}$ and $\frac{dT}{dv'}$ are calculated from T before we substitute the values of u' and v' in T.

Theorem 2. (i) If the curves on a surface are not paramietric curves, then the sufficient condition for a curve to be a geodesic is either $U = 0$ or $V = 0$.

(ii) For a parametric curve u = constant to be a geodesic a sufficient condition is $U = 0$ and v = constant to be a geodesic, the sufficient condition is $V = 0$.

Proof. (i) When s is used as a parameter, then the second Lemma of Theorem 1 of 3.2 becomes

$$Uu' + Vv' = \frac{dT}{ds}$$

Since $\frac{dT}{ds} = 0$ by (4) of the previous theorem. We have

$$Uu' + Vv' = 0 \quad ...(1)$$

If the curves are not parametric curves, $u' \neq 0$ and $v' \neq 0$. This implies from (1) that U and V are not independent. Hence U is a scalar multiple of V and vice-versa so that either $U = 0$ or $V = 0$ is a sufficient condition for a curve to be geodesic.

(ii) For a curve to be a geodesic on a surface, it should satisfy the following canonical equations of Theorem 1.

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \quad ...(2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \quad ...(3)$$

If we take the parametric curve u = constant so that $u' = 0$ and $v' \neq 0$. Using this, (1) implies that $V = 0$ and conversely. Hence the equation (3) is automatically satisfied for all s. Hence the condition for u = constant to be a geodesic is $U = 0$. In a similar manner, $V = 0$ is a sufficient condition for the parametric curve v = constant to be a geodesic.

3.4 GEODESICS ON SURFACES OF REVOLUTION

Using canonical equations, we shall investigate the nature of geodesics on a surface of revolution in the following two theorems.

Theorem 1. Three types of geodesics on a surface of revolution

$$\mathbf{r} = (g(u) \cos v, g(u) \sin v, f(u)) \text{ are} \quad ...(1)$$

(i) $v = \alpha\phi(u, \alpha) + \beta$ where α and β are constants

(ii) Every meridian v = constant

(iii) A parallel u = constant is a geodesic if and only if its radius is stationary.

Proof. (i) From (1) we have

$$\mathbf{r}_1 = (g_1(u)\cos v, g_1(u)\sin v, f_1(u)),\ \mathbf{r}_2 = (-g(u)\sin v, g(u)\cos v, 0).$$

So $$E = \mathbf{r}_1\cdot\mathbf{r}_1 = g_1^2 + f_1^2(u),\ F = \mathbf{r}_1\cdot\mathbf{r}_2 = 0,\ G = \mathbf{r}_2\cdot\mathbf{r}_2 = g^2(u) \quad ...(2)$$

Let us consider $u(t) \neq 0$, $v(t) \neq 0$. Hence by Theorem 2(i) of 3.3, the canonical geodesic equations are given either by $U = 0$ or $V = 0$.

Without loss of generality, let us find $V = 0$.

Now $$T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$$

$$= \frac{1}{2}[(g_1^2 + f_1^2)u'^2 + g^2v'^2]$$

Since g and f are functions of u only, we get

$$\frac{\partial T}{\partial v} = 0 \text{ and } \frac{\partial T}{\partial v'} = g^2v'$$

Hence $$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{d}{ds}(g^2v')$$

Thus the canonical geodesic equation $\dot{V} = 0$ gives $\dfrac{d}{ds}(g^2v') = 0$ ...(3)

Integrating (3), we have $g^2v' = \alpha$ ...(4)

where α is an arbitrary constant. If the sense of the curve is in the direction of v increasing, v' is positive and $g^2 > 0$ so that α can be taken to be a positive constant. Hence (3) gives the differential equation of the geodesic on the surface of revolution.

Let us take $g^2(u) \neq \alpha^2$. Squaring both sides of (4),

$$g^4dv^2 = \alpha^2ds^2$$
$$= \alpha^2[E\,du^2 + 2F\,du dv + G\,dv^2]$$
$$= \alpha^2[(g_1^2 + f_1^2)du^2 + g^2dv^2] \text{ which gives}$$

$$g^2(g^2 - \alpha^2)dv^2 = \alpha^2(g_1^2 + f_1^2)du^2 \quad ....(5)$$

so that we have

$$dv = \pm\frac{\alpha}{g}\sqrt{\frac{g_1^2 + f_1^2}{g^2 - \alpha^2}}\,du$$

where we have taken both the signs, since the curve can change direction as u, v move on the curve. Integrating (4),