2

The First Fundamental Form and Local Intrinsic Properties of A Surface

2.1 INTRODUCTION

As in the case of a space curve introduced either as the intersection of two surfaces or with the parametric coordinates, we shall introduce surfaces in E_3 either implicitly by an equation of the type $F(x, y, z) = 0$ or parametrically by expressing x, y, z in terms of two parameters u, v varying over a domain. We shall make these two notions more explicit before defining a surface locally as equivalence class of surfaces by a suitable equivalence relation.

After defining the surface locally, we classify the points on a surface as ordinary points and singular points. Then we take up for study curves on surfaces and explain how the parametric curves on surfaces help us to study the properties of surfaces. Then with the help of the tangent plane at a point P and the surface normal at P, we introduce a coordinate system at every point of the surface. This system $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{N})$ at any point on the surface is analogous to the moving triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ at a point on the space curve. After introducing certain standard surfaces which we often come across in applications, we shall introduce a certain quadratic differential form on a surface and direction coefficients. This quadratic form is called the first fundamental form which enables us to study the local intrinsic properties of surfaces. We shall conclude this chapter with a brief study of the family of curves on surfaces and isometric transformations.

2.2 DEFINITION OF A SURFACE

We give the two different definitions of a surface and illustrate them with some simple examples.

Definition 1. A surface is the locus of a point $P(x, y, z)$ in E_3 satisfying some restrictions on x, y, z which is expressed by a relation of the type $F(x, y, z) = 0$.

The above definition implies that any point on the surface satisfies the equation and conversely. The equation $F(x, y, z) = 0$ is called the implicit or constraint equation of the surface. This implicit form of the equation describes the surface as a whole so that one can make a global study of the surface. But when we restrict ourselves to the local study of the surfaces which means the study of the properties of the surface in the neighbourhood of a point which is a small region, the constraint equation is not useful. So we are necessitated to use parametric representation of the surfaces in most of the cases.

Definition 2. If the parameters u, v take real values and vary over some domain D, a surface is defined parametrically as

$$x = f(u, v),\ y = g(u, v), \text{ and } z = h(u, v)$$

where f, g and h are single valued continuous functions possessing continuous derivatives of r-th order. Such surfaces are called surfaces of class r.

The parameters u and v are called curvilinear coordinates. (u, v) is used to represent the point determined by u and v.

Thus we have two methods of representation of a surface, one is the global representation by using a constraint equation and another by parameters u, v varying over a domain. Hence the question naturally arises whether the two methods are equivalent under a suitable equivalence relation. Before answering this question, we point out some disadvantages in these representations by a few examples.

(i) The parametric equations of a surface are not unique.

To see this, we produce a surface having two different parametric representations.

Now consider the following two sets of equations

$$x = u + v,\ y = u - v,\ z = 4uv$$

$$x = u,\ y = v,\ z = u^2 - v^2$$

Elimination of the parameters in both the representations lead to the same constraint equation $x^2 - y^2 = z$ which represents the whole of certain hyperbolic paraboloid.

(ii) Sometimes the constraint equation obtained by eliminating the parameters represents more than the given surface.

To see this consider the parametric equation

$$x = u \cosh v,\ y = u \sinh v,\ z = u^2$$

for all real values of u and v.

Eliminating u and v among the equations, the constraint equation of the surface is $x^2 - y^2 = z$. The constraint equation represents the whole of hyperbolic paraboloid, while the parametric equations give only that part of the hyperbolic paraboloid for which $z \geq 0$, since u takes only real values.

Definition 3. Let there be two parametric representations u, v and u', v' of the same surface. Any transformation of the form $u' = \phi(u, v)$ and $v' = \psi(u, v)$ relating these two representations is called a parametric transformation.

Definition 4. A parametric transformation is said to be proper if

(i) ϕ and ψ are single valued functions

and (ii) The Jacobian $\dfrac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0$ in some domain D.

Note. Let D' be the domain of u', v' corresponding to the domain D of the u, v plane. The conditions in the above definition are the necessary and sufficient conditions for the existence of the inverse in the neighbourhood of any point D' which means that the transformation is locally one to one. However it should be noted that the transformation ϕ, ψ may not have the inverse on the whole of D.

2.3 NATURE OF POINTS ON A SURFACE

To describe the nature of points on a surface, we introduce the following notation.

Let $\mathbf{r} = (x, y, z)$ be the position vector of a point on the surface. Since x, y, z are continuous functions of parameters u, v possessing partial derivatives of required order, we can take $\mathbf{r} = \mathbf{r}(u, v)$ as the paramatric equation of the surface. If the suffixes 1 and 2 are used for partial derivatives of $\mathbf{r}$ with respect to u and v respectively, let $\mathbf{r}_1 = \dfrac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_2 = \dfrac{\partial \mathbf{r}}{\partial v}$ ...(1)

and
$$\mathbf{r}_{11} = \frac{\partial^2 \mathbf{r}}{\partial u^2}, \mathbf{r}_{12} = \frac{\partial^2 \mathbf{r}}{\partial u\, \partial v}, \mathbf{r}_{21} = \frac{\partial^2 \mathbf{r}}{\partial v\, \partial u}, \mathbf{r}_{22} = \frac{\partial^2 \mathbf{r}}{\partial v^2}$$

Since $\mathbf{r}$ possesses continuous partial derivatives, we have $\mathbf{r}_{12} = \mathbf{r}_{21}$.

Since $\mathbf{r} = (x, y, z)$, we can express $\mathbf{r}_1$, $\mathbf{r}_2$ componentwise as

$$\mathbf{r}_1 = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (x_1, y_1, z_1)$$

$$\mathbf{r}_2 = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (x_2, y_2, z_2) \qquad ...(2)$$

and we have similar expressions for $\mathbf{r}_{22}$, $\mathbf{r}_{11}$, $\mathbf{r}_{12}$ and $\mathbf{r}_{21}$

Definition 1. If $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$ at a point on a surface, then the point is called an ordinary point. A point which is not an ordinary point is called a singularity.

From the very definition of an ordinary point, we note the following properties of a surface.

(i) using (2), we have

$$\mathbf{r}_1 \times \mathbf{r}_2 = \boldsymbol{i}(y_1 z_2 - z_1 y_2) + \boldsymbol{j}(z_1 x_2 - x_1 z_2) + \boldsymbol{k}(x_1 y_2 - y_1 x_2) \qquad ...(3)$$

$\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$ means that one of the coefficients in (3) is different from zero. That is at least one of the members

$$(y_1 z_2 - z_1 y_2), (z_1 x_2 - x_1 z_2), (x_1 y_2 - y_1 x_2) \neq 0 \qquad ...(4)$$

(ii) Let us consider the matrix

$$M = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

Since at least one of the members in (4) is different from zero at an ordin... point, there exists at least one determinant minor of order 2 of M which is diffe... from zero. In other words the rank of M is two at an ordinary point. ... consequence, if the rank of M is either zero or one, the point on the surfac... singular point.

(iii) If $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$ or equivalently if the rank of the matrix M is two, then x... uniquely determine the parameters u, v in the neighbourhood of an ordinary p...

Since the matrix M is of rank two, there exists at least one non-vani... Jacobian which we can take as $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0.$

As the condition of the inversion theorem is satisfied, there e... neighbourhoods $\mathbf{N}_\delta(x_0, y_0)$ and $\mathbf{N}_\varepsilon(u_0, v_0)$ such that for every $x, y \in \mathbf{N}_\delta(x...$ there exist $u, v \in \mathbf{N}_\varepsilon(u_0, v_0)$ such that $u = u(x, y)$ and $v = v(x, y)$. Hence u and ... determined by x, y, z in the neighbourhood of an ordinary point.

(iv) The points where the rank of M is 1 or zero are singular points.

If the rank of M is 1, then every determinant minor of order two of M is ... This implies

$(y_1z_2 - z_1y_2), (z_1x_2 - x_1z_2), (x_1y_2 - y_1x_2)$ are all zero so that $\mathbf{r}_1 \times \mathbf{r}_2 = 0$. Hence ... point where the rank of M is one is a singular point.

When the rank of M is zero, then all the determinant minors of order 1 or ... zero. This implies as in the previous case $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ at these points so that the ... where the rank of M is zero is a singular point.

Note. When only one determinant minor of M is zero, we cannot con... that the point is a singular point.

We shall illustrate the above properties by the following examples.

Example 1. Consider the surface given parametrically by

$$x = u + v,\ y = u + v,\ z = uv.$$

Now $\quad \dfrac{\partial x}{\partial u} = 1, \dfrac{\partial x}{\partial v} = 1, \dfrac{\partial y}{\partial u} = 1, \dfrac{\partial y}{\partial v} = 1, \dfrac{\partial z}{\partial u} = v, \dfrac{\partial z}{\partial v} = u$

Hence $\quad x_1y_2 - y_1x_2 = 0$ but $x_1z_2 - x_2z_1 = u - v \neq 0.$

Thus the rank of M is 2 at every point of the surface so that every poi... surface is an ordinary point.

(v) A proper parametric transformation transforms an ordinary poi... ordinary point.

Let $\quad \mathbf{r} = \mathbf{r}(u, v)$ be the equation of the surface.

and let $u' = \phi(u, v), v' = \psi(u, v)$ be the given proper parametric transfo...

Now $$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{r}}{\partial u'} \cdot \frac{\partial u'}{\partial u} + \frac{\partial \mathbf{r}}{\partial v'} \cdot \frac{\partial v'}{\partial u}$$

and so $$= \frac{\partial \mathbf{r}}{\partial u'} \frac{\partial \phi}{\partial u} + \frac{\partial \mathbf{r}}{\partial v'} \frac{\partial \psi}{\partial u}$$

Similarly $$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u'} \frac{\partial \phi}{\partial v} + \frac{\partial \mathbf{r}}{\partial v'} \frac{\partial \psi}{\partial v}$$

Since $$\frac{\partial \mathbf{r}}{\partial u'} \times \frac{\partial \mathbf{r}}{\partial u'} = 0, \quad \frac{\partial \mathbf{r}}{\partial v'} \times \frac{\partial \mathbf{r}}{\partial v'} = 0, \text{ being parallel vectors}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u'} \times \frac{\partial \mathbf{r}}{\partial v'} \left[\frac{\partial \phi}{\partial u} \cdot \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \frac{\partial \phi}{\partial v} \right]$$

Thus we have $$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u'} \times \frac{\partial \mathbf{r}}{\partial v'} \frac{\partial(\phi, \psi)}{\partial(u, v)}$$

Since the given parametric transformation is proper,

$$\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0.$$

Hence $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \neq 0$ implies $\dfrac{\partial \mathbf{r}}{\partial u'} \times \dfrac{\partial \mathbf{r}}{\partial v'} \neq 0$. That is $\mathbf{r}_1' \times \mathbf{r}_2' \neq 0$, proving that an ordinary point is invariant after proper parametric transformation.

***Note* 1.** Since $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$ at an ordinary point, $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ at a singularity.

Due to some geometrical nature of the surface, some singularities continue to be singularities, whatever may be the parametric representations. Such singularities are called essential singularities. There are other singularities depending upon the choice of parametric representation. Singularities of this type are called artificial singularities.

***Note* 2.** To find the nature of a point on the surface, we use either the matrix M or $\mathbf{r}_1 \times \mathbf{r}_2$. We shall illustrate the essential and artificial singularities in the following examples.

Example 2. Consider the circular cone represented by

$$x = u \sin \alpha \cos v,\ y = u \sin \alpha \sin v,\ z = u \cos \alpha$$

where α is the semivertical angle of the cone with the vertex 0 as origin and $OP = u$, P any point on the cone.

We show that the vertex of the cone is an essential singularity. Since u and v are parameters, we have

$$M = \begin{bmatrix} \sin \alpha \cos v & \sin \alpha \sin v & \cos \alpha \\ -u \sin \alpha \sin v & u \sin \alpha \cos v & 0 \end{bmatrix}$$

At $u = 0$, the determinant of every second order minor of M is zero. Hence rank of M is zero so that $u = 0$ is an essential singularity. This singularity ... since it arises as a result of the vertex of the cone.

Taking $\mathbf{r} = (u \sin\alpha \cos v, u \sin\alpha \sin v, u \cos\alpha)$, $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ at $u = 0$ shows ... $u = 0$ is an essential singularity.

Example 3. Taking any point O as the origin in the plane, $x = u \cos v$, $y = \ldots$ ... 0 is the representation of the plane in polar coordinates.

Now $\mathbf{r}_1 = (\cos v, \sin v, 0)$, $\mathbf{r}_2 = (-u \sin v, u \cos v, 0)$ then $\mathbf{r}_1 \times \mathbf{r}_2 = u\mathbf{k}$. Hence $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ when $u = 0$ so that $u = 0$ is a singularity. It is an artificial singularity since it arises due to the choice of the parametric coordinates and not due to the nature of the surface. It is to be noted that $u = 0$ is not a singularity in the coordinate system. We can arrive at the same conclusion by noting that the rank of the matrix M is zero at $u = 0$.

2.4 REPRESENTATION OF A SURFACE

In our study of surfaces, we shall consider only ordinary points on surface. This means that the domain of parameters u, v will be restricted so that every point of the surface is an ordinary point. Also we shall study the properties of the surface in the neighbourhood of an ordinary point. For such a study, the proper parametric transformation is very useful, since it is locally one-to-one. Since such a study depends upon a portion of a surface, we consider the entire surface as a collection of parts, each part being given a particular parameterisation and the adjacent parts are related by a proper parametric transformation. Using these ideas, we define the representation of a surface as follows.

Definition 1. A representation R of a surface S of class r in E_3 is a collection of points in E_3 covered by a system of overlapping parts $\{S_j\}$ where each part is given by a parametric equation of class r. Each point lying in the common portion of two prats S_i, S_j is such that the change of parameters from one part S_i to the adjacent part S_j is given by a proper parametric transformtion of class r.

Note. Since we cannot parametrise the whole surface without introducing artificial singularities, we resort to consider a surface composed of many overlapping parts. Since the points in the adjacent parts have two parametric representations one for S_i and another for its adjacent S_j, these two parametric representations are connected by a proper parametric transformation.

In the definition of the representation R of a surface, we are concerned with the system of overlapping parts S_j covering the whole surface. Hence it is possible to have many representations of the same surface by considering different systems of overlapping parts (S_j), each part is given by a parametric equation of class r. Since we have different representations of the same surface, it is but natural to state precisely, when the two representations R and R' behave alike. This leads to the notion of equivalence of representations of surfaces of class r and consequently definition of a surface as an equivalence class.

Definition 2. Let R an R' be two representations of class r of the surface S. Let (S_j) and (S_j') be two different systems of overlapping parts covering corresponding to R and R'. Then they are said to be equivalent, if the composite family of parts $\{S_j, S_j'\}$ satisfy the condition that each point P lying in the common portion of the overlap of two parts, the change of parameter of P considered as a point of S_j to the parameter of the same point considered as a point of S_j' is given by a proper parametric transformation of class r. That is if P is a point in the place of overlap, the change of parameter from S_j to S_j' at the point P is given by a proper parametric transformation of class r.

Theorem. The notion of r-equivalence of representations of a surface is an equivalence relation.

Proof. Let R be a representation of S and let S be composed of overlapping parts $\{S_j\}$. Since the change of parameters from S_i to S_j is given by a proper parametric transformation of class r, the relation of r-equivalence of representation R is reflexive.

Let the relation R be equivalent to R' and let S_j and S_j' be two overlapping parts in two representations with a point P in the overlapping portion. Since R and R' are equivalent, there exists a proper parametric transformation ϕ at P from S_j to S_j'. Since the proper parametric transformation is locally one-to-one and possesses inverse transformation, ϕ^{-1} exists at the point P of overlap of S_j and S_j'. In other words, there exists a proper parametric transformation ϕ^{-1} from S_j' to S_j. Thus R' is equivalent to R so that the relation of r-equivalence of class r is symmetric.

Let R, R' and R'' be any three representations of class r of a surface S and let them be r-equivalent such that $R \sim R'$ and $R' \sim R''$. We shall show that $R \sim R''$. Since R and R' are equivalent, there exists a proper parametric transformation ϕ at the common point P_1 in the overlap of the family $\{S_j, S_j'\}$. Since $R' \sim R''$, the change of parameter of a point in the overlap of S_j' and S_j'' is given by a proper parametric transformation ψ from S_j' to S_j''. Since ϕ and ψ are locally one-to-one, $\psi \circ \phi$ is locally one-to-one transformation giving the change of parameter from S_j to S_j''. Hence the representation R and R'' are equivalent so that the relation of equivalence of class r of surfaces is transitive.

Since the notion of the relation of equivalence of class r is reflexive, symmetric and transitive, it is an equivalence relation which completes the proof of the theorem.

This equivalence relation introduces a partition into the family of surfaces of class r splitting them into mutually disjoint equivalence classes, each class containing the surface equivalent to one another in the above equivalence relation. This leads to the formal definition of a surface as follows.

Definition 3. A surface S of class r in E_3 is an r-equivalence class of representations.

Thus a surface consists of different overlapping portions related to one another by proper parametric transformations and all other surfaces related to the given one by the equivalence relation of class r. We make a study of local properties without investigating the extent of the region of the surface in which the local properties are true.

2.5 CURVES ON SURFACES

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of a surface of class r where u, v vary over a domain in the uv-plane. Let $u = u(t)$ and $v = v(t)$ be a curve of class s lying in the domain D of the uv-plane.

Now consider $\mathbf{r} = \mathbf{r}[u(t), v(t)]$. Then $\mathbf{r}$ gives the position vector of a point in terms of a single parameter t so that $\mathbf{r} = \mathbf{r}[u(t), v(t)]$ is a curve lying on a surface with class equal to smaller of r and s. The equation $u = u(t)$ and $v = v(t)$ are called curvilinear equations of the curve on the surface.

Definition 1. (Parametric Curves). Let $\mathbf{r} = \mathbf{r}(u, v)$ be the given surface of class r. Let $v = c$ where c is an arbitrary constant. Then the position vector $\mathbf{r} = \mathbf{r}(u, c)$ is a function of a single parameter t and hence $\mathbf{r} = \mathbf{r}(u, c)$ represents a curve lying on the surface $\mathbf{r} = \mathbf{r}(u, v)$. This curve is called the parametric curve v = constant.

For every value of c, there is one such curve on the surface. Since $v = c$, where c is an arbitrary constant, we get a system of parametric curves for different constant values of c. In a similar manner, if we keep u constant and vary v, we get a system of parametric curves u = constant.

Since we are concerned with the ordinary points on the surface, we note the following basic properties of parametric curves. These properties are the consequences of the fact that we are concerned only with ordinary points on the surface.

(i) Through every point of the surface, there passes one and only one parametric curve of each system.

Let $P(x_0, y_0, z_0)$ be a point on the surface. Then as explained in (iii) of 2.3, (u_0, v_0) are uniquely determined by (x_0, y_0, z_0). Hence there are only two parametric curves $u = u_0$, $v = v_0$ passing through the point P.

(ii) No two curves of the same system intersect.

Let us consider the system at the point (u_0, v_0). Let $u = u_0$ and $u = u_1$ be two curves of the same system. If these two curves of the same system intersect, $u = u_1$ at the point of intersection so that the parametric coordinates determined by $P(x_0, y_0, z_0)$ is (u_1, v_0) contradicting the uniqueness of (u_0, v_0). This contradiction proves that no two curves of the same system intersect.

(iii) The curves of the systeem $u = u_0$ and $v = v_0$ intersect once but not more than once if $(u_0, v_0) \in D$.

Since the point of intersection (u_0, v_0) is uniquely determined by $P(x_0, y_0, z_0)$, they cannot intersect more than once.

(iv) The parametric curves of the system $u = c_1$ and $v = c_2$ cannot touch each other.

For a curve $v = c$, u serves as a parameter and determines a sense along the curve. The position vector of a point on the curve $v = c$ is $\mathbf{r} = \mathbf{r}(u, c)$. Hence the tangent to the curve $v = c$ in the direction of u increasing is $\mathbf{r}_1 = \dfrac{\partial \mathbf{r}}{\partial u}$. Similarly

$\mathbf{r}_2 = \dfrac{\partial \mathbf{r}}{\partial v}$ gives the direction of the tangent to the curve $u = c$ in the direction of v increasing. They do not vanish and have different directions. Since we consider $\mathbf{r}_1$ and $\mathbf{r}_2$ only at an ordinary point on the surface $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$. This shows that the two parametric curves are neither coincident nor parallel but cut at the point (u_0, v_0) determined by (u_0, v_0). Hence they do not touch each other.

Definition 2. Let $u = c_1$ and $v = c_2$. When the constants c_1 and c_2 vary, the whole surface is covered with a net of parametric curves, two of which pass through every point (u, v) are called the curvilinear coordinates of P. The parametric curves are called coordinate curves.

Definition 3. Two parametric curves through a point P are said to be orthogonal if $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ at P. If this condition is satisfied at every point (u, v) of the domain, then the two system of parametric curves are orthogonal.

2.6 TANGENT PLANE AND SURFACE NORMAL

Let $\mathbf{r} = \mathbf{r}[u(t), v(t)]$ be a general curve lying on the surface passing through $[u(t), v(t)]$. Then the tangent to the curve at any point P on the surface is

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} = \mathbf{r}_1 \frac{du}{dt} + \mathbf{r}_2 \frac{dv}{dt} \qquad (1)$$

Definition 1. Tangent to any curve drawn on a surface is called a tangent line to the surface.

From (1), we see that the tangent vector $\dfrac{d\mathbf{r}}{dt}$ is a linear combination of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Since $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$, $\mathbf{r}_1$ and $\mathbf{r}_2$ are non zero and independent. The tangents to different curves through P on a surface lie in a plane containing two independent vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ at P. This plane is called the tangent plane at P.

Theorem 1. The equation of a tangent plane at P on a surface with position vector $\mathbf{r} = \mathbf{r}(u, v)$ is either

$$\mathbf{R} = \mathbf{r} + a\mathbf{r}_1 + b\mathbf{r}_2, \text{ or } (\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = 0$$

where a and b are parameters.

Proof. Let $\mathbf{r} = \mathbf{r}(u, v)$ be the position vector of a point P on the surface. The tangent plane at P passes through $\mathbf{r}$ and contains the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. So if $\mathbf{R}$ is the position vector of any point on the tangent plane at P, then $\mathbf{R} - \mathbf{r}$, $\mathbf{r}_1$ and $\mathbf{r}_2$ are coplanar. Hence we have

$$\mathbf{R} - \mathbf{r} = a\mathbf{r}_1 + b\mathbf{r}_2$$

where a and b are arbitrary constants.

$\mathbf{r}_1 \times \mathbf{r}_2$ is perpendicular to the tangent plane at P. Hence $\mathbf{r}_1 \times \mathbf{r}_2$ is perpendicular to $\mathbf{R} - \mathbf{r}$ lying in the tangent plane so that $(\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = 0$ is another form of the equation of the tangent plane at P.

Definition 2. The normal to the surface at P is a line through P and perpendicular to the tangent plane at P.

Since $\mathbf{r}_1$ and $\mathbf{r}_2$ lie in the tangent plane at P and pass through P, the normal is perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$ and it is parallel to $\mathbf{r}_1 \times \mathbf{r}_2$ as in the adjoining Figure 6. The normal at P is fixed by the following convention.

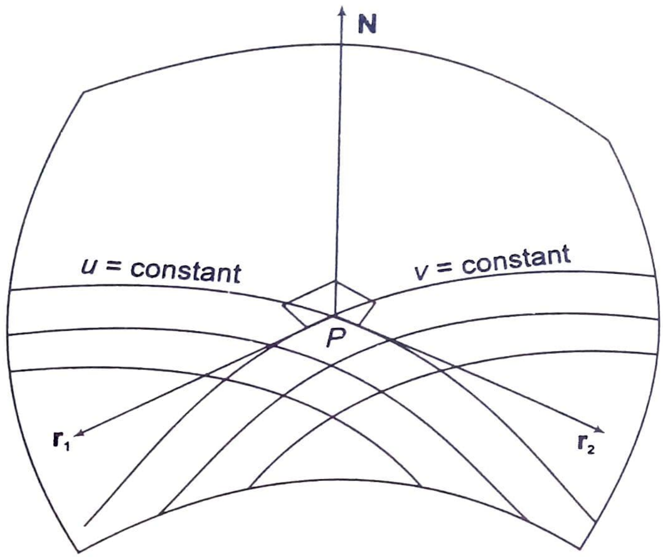

Fig. 6

If $\mathbf{N}$ denotes the unit normal at P, then $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{N}$ in this order should form a right handed system. Using this convention we have

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} \quad \text{where } H \text{ is } |\mathbf{r}_1 \times \mathbf{r}_2|.$$

Since $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$, $H = |\mathbf{r}_1 \times \mathbf{r}_2| \neq 0$ which shows that it is always a positive number and $\mathbf{N}H = \mathbf{r}_1 \times \mathbf{r}_2$.

Theorem 2. The equation of the normal $\mathbf{N}$ at a point P on the surface $\mathbf{r} = \mathbf{r}(u, v)$ is $\mathbf{R} = \mathbf{r} + a(\mathbf{r}_1 \times \mathbf{r}_2)$.

Proof. Let $\mathbf{R}$ be the position vector of any point on the normal to the surface at P whose position vector is $\mathbf{r} = \mathbf{r}(u, v)$. Since $\mathbf{r}_1 \times \mathbf{r}_2$ gives the direction of the normal and $(\mathbf{R} - \mathbf{r})$ lies along the normal, $\mathbf{r}_1 \times \mathbf{r}_2$ and $(\mathbf{R} - \mathbf{r})$ are parallel so that we have $\mathbf{R} - \mathbf{r} = a(\mathbf{r}_1 \times \mathbf{r}_2)$ where a is a parameter. Hence $\mathbf{R} = \mathbf{r} + a(\mathbf{r}_1 \times \mathbf{r}_2)$ gives the equation of the normal at P.

Using the convention that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{N}$ form a right handed system, we establish the following theorem.

Theorem 3. A proper parametric transformation either leaves every normal unchanged or reverses the direction of the normal.

Proof. $\mathbf{r} = \mathbf{r}(u, v)$ be the given surface and let the parametric transformation be

$$u' = \phi(u, v) \text{ and } v' = \psi(u, v)$$

Since the parametric transformation is proper,

$$J = \frac{\partial(u', v')}{\partial(u, v)} \neq 0$$

As in the case ($\ddot{v}$) of 2.3, we have

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial(u', v')}{\partial(u, v)} \left(\frac{\partial \mathbf{r}}{\partial u'} \times \frac{\partial \mathbf{r}}{\partial v'} \right)$$

Using H and H' in the above step, we get

$$H\mathbf{N} = \frac{\partial(u', v')}{\partial(u, v)} H'\mathbf{N}'$$

Since H and H' are always positive, $\mathbf{N}$ and $\mathbf{N}'$ are of the same sign if $J > 0$ and are of opposite sign if $J < 0$. Since J is a continuous function of the parameter u, v in the whole domain and J does not vanish in D, J retains the same sign in D. This proves that $\mathbf{N}$ and $\mathbf{N}'$ have the same sign.

Example 1. Obtain the surface equation of sphere and find the singularities, parametric curves, tangent plane at a point and the surface normal.

A sphere is a surface of revolution of a semi-circle lying in the XOZ plane about the z-axis. The curve meets the axis of revolution in two points. If P is any point on the circle lying in the XOZ plane, its equation can be taken as

$$x = a \sin u,\ y = 0,\ z = a \cos u$$

where u is the angle made by OP with the z-axis. u is called the co-latitude of the point P. After rotation through an angle v about z-axis, let PM be perpendicular on

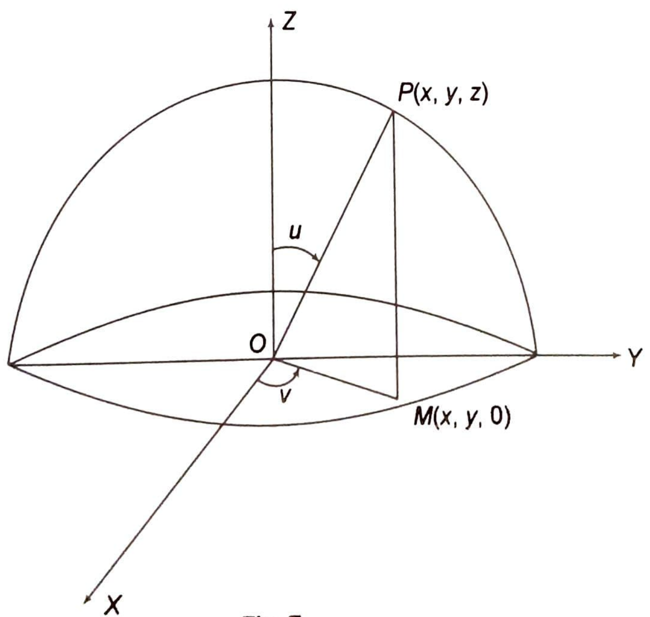

Fig. 7

the XOY plane. Then $X\hat{O}M$ is called the longitude of P and it is v. Hence the position vector of P on the sphere is

$$x = OM \cos v = OP \cos (90 - u) \cos v = a \sin u \cos v$$

$$y = OM \sin v = OP \sin u \sin v \text{ and } z = a \cos u.$$

Thus the surface equation of the sphere is

$$\mathbf{r} = (a \sin u \cos v, a \sin u \sin v, a \cos u)$$

where u and v are parameters and $0 \le u \le \pi, 0 \le v \le 2\pi$.

(i) We shall find the singularities

Now $\quad \mathbf{r}_1 = (a \cos u \cos v, a \cos u \sin v, - a \sin u)$

$$\mathbf{r}_2 = (- a \sin u \sin v, a \sin u \cos v, 0)$$

Hence the matrix

$$M = \begin{bmatrix} a \cos u \cos v & a \cos u \sin v & - a \sin u \\ - a \sin u \sin v & a \sin u \cos v & 0 \end{bmatrix}$$

At $u = 0$ and $u = \pi$, all the three determinant minors of M are zero so that the rank of M is zero. Thus $u = 0, u = \pi$ are singular points. Since these singularities are due to the choice of parameters, they are artificial singularities. The same conclusion may be arrived at by considering $\mathbf{r}_1 \times \mathbf{r}_2$ also.

(ii) *Parametric curves.* First let us find the parametric curves of the system $u =$ constant. When the colatitude u is a constant, $a \cos u$ is a constant. Let it be A. Then $z = A$ is a plane parallel to the XOY-plane. If P is the point of intersection of this plane and the sphere where u is constant, then the locus of P is a small circle. Hence the parametric curves of the system $u =$ constant is a system of parallel small circles which are called parallels.

When the longitude $v =$ constant, the plane ZOM is fixed and the point P where v is constant is the intersection of the sphere and this plane passing through the centre of the sphere. Hence the locus of P is a great circle. Thus the parametric curves of the system $v =$ constant is a system of great circles called meridians.

From (i) $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ so that the parametric curves are orthogonal.

(iii) Now $\mathbf{r}_1 \times \mathbf{r}_2 = a^2 (i \sin^2 u \cos v + j \sin^2 u \sin v + k \sin u \cos v)$

The equation of the tangent plane is $(\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = 0$

In the cartesian form, the above equation becomes

$$(X - x) \sin u \cos v + (Y - y) \sin u \sin v + (Z - z) \cos v = 0.$$

Now $\quad H = |\mathbf{r}_1 \times \mathbf{r}_2| = a^2 \sin u$

Hence $\quad \mathbf{N} = \dfrac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = (\sin u \cos v, \sin u \sin v, \cos u) = \dfrac{1}{a} \mathbf{r}$

where $\mathbf{r}$ is the position vector of a point on the surface so that the surface normal is the outward drawn normal.

Thus $\quad \mathbf{N} = \dfrac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = (-\cos v \cos \alpha, -\sin v \cos \alpha, \sin \alpha)$

2.7 THE GENERAL SURFACES OF REVOLUTION

We shall introduce some special standard surfaces of revolution which will be used in the illustrations for the study of local intrinsic properties of surfaces. To start with let us define a general surface.

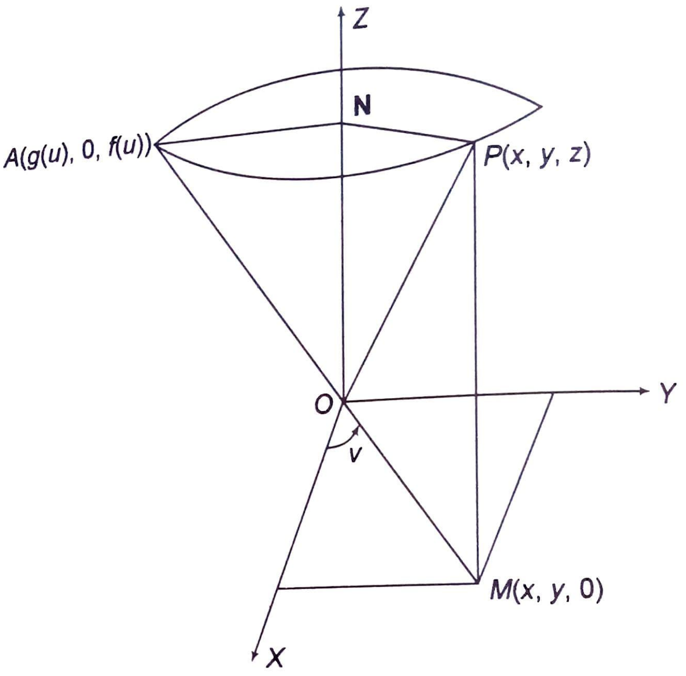

Fig. 9

Definition 1. A surface generated by the rotation of a plane curve about a[n] axis in its plane is called a surface of revolution.

Theorem 1. The position vector of any point on the surface of revolutio[n] generated by the curve $[g(u), o, f(u)]$ in the XOZ plane is

$$\mathbf{r} = [g(u) \cos v, g(u) \sin v, f(u)]$$

where v is the angle of rotation about the z-axis.

Proof. Let us take the z-axis as the axis of rotation and let $[g(u), o, f(u)]$ [be] the parametric representation of the generating curve in the XOZ plane. Let A [be] any point on the curve. Then its x-coordinate $g(u)$ gives the distance of A fr[om] the z-axis. When the curve revolves about the z-axis, A traces out a circle w[ith] radius $g(u)$. When the plane through the z-axis has rotated through an angle v, le[t P] be the position of the point corresponding to A on the curve after rotation. Dr[aw] PM and PN perpendicular to XOY and XOZ planes. Then $AN = PN = g(u)$ a[nd] $OM = PN$.

If (x, y, z) are the coordinates of P, then we have

$$x = OM \cos v = PN \cos v = g(u) \cos v$$

$$y = OM \sin v = PN \sin v = g(u) \sin v$$

$$z = PM = f(u)$$

Hence the position vector of a point P on the surface is

$$\mathbf{r} = [g(u) \cos v, g(u) \sin v, f(u)]$$

where the domain of (u, v) is $0 \leq v \leq 2\pi$ with a suitable range for u which depends on the surface.

Next we shall find the parametric curves.

Let P be a point on the surface with u = constant so that $g(u)$ is also a constant. Then the locus of the points like P is a circle with radius $g(u)$ for a complete rotation as v arises from 0 to 2π. Thus the parametric curves u = constant are circles parallel to the XOY plane as in the case of sphere we call them as parallels.

Let v = constant. Since v gives the angle of the plane of rotation in this position, the parametric curves are the curves of intersection of this plane of rotation with the surface. We call these curves meridians.

Further $\quad \mathbf{r}_1 = (g' \cos v, g' \sin v, f')$

$$\mathbf{r}_2 = (-g \sin v, g \cos v, 0)$$

and $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ so that the parametric curves are orthogonal. To find the direction of the normal, we find

$$\mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{i}(-gf' \cos v) - \mathbf{j}(f'g \sin v) + \mathbf{k}gg'$$

and $\quad |\mathbf{r}_1 \times \mathbf{r}_2|^2 = g^2(f'^2 + g'^2)$

Hence $\quad \mathbf{N} = \dfrac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \dfrac{(-f' \cos v, -f' \sin v, g')}{\sqrt{f'^2 + g'^2}}$

***Note* 1.** By specialising the curve to be a circle in the XOZ plane, we get the representation of a point on the sphere. The parametric representation of a point on the circle is $(a \sin u, 0, a \cos u)$ so that $g(u) = a \sin u$, and $f(u) = a \cos u$. Hence the representation of a point on the sphere is

$$\mathbf{r} = (a \sin u \cos v, a \sin u \sin v, a \cos u)$$

***Note* 2.** In the case of the cone, the curve in the XOZ plane is a generator. The parametric representation of a point on the generator is $(u, 0, u \cot \alpha)$.

Hence taking $g(u) = u$ and $f(u) = u \cot \alpha$, we obtain the representation of a point on the cone as,

$$\mathbf{r} = (u \cos v, u \sin v, u \cot \alpha)$$

Another important surface is anchor ring or torus which is defined as

Definition 2. The anchor ring is a surface generated by rotating a circle of radius a about a line in its plane at a distance $b > a$ from its centre.

This circle does not meet the axis of rotation, whereas in the case of a sphere, the curve is a semi-circle meeting the axis of rotation at two points.

Further $\mathbf{r}_1 = (-a \sin u \cos v, -a \sin u \sin v, a \cos u)$

$\mathbf{r}_2 = (-(b + a \cos u) \sin v, (b + a \cos u) \cos v, 0)$

Since $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$, the parametric curves are orthogonal.

$\mathbf{r}_1 \times \mathbf{r}_2 = -(b + a \cos u) [a \cos u \cos v, a \cos u \sin v, a \sin u]$

Since $b > a$, the above vector is negative for the range of values of u and v so that the normal is directed inside the anchor ring, since $|\mathbf{r}_1 \times \mathbf{r}_2|$ is always positive.

Note. The coordinates of a point A on the generating circle in XOZ plane is $(b + a \cos u, 0, a \sin u)$. Hence taking $g(u) = b + a \cos u$, $f(u) = a \sin u$ in Theorem 1, we can obtain the representation of a point on an anchor ring.

2.8 HELICOIDS

In the above examples, we considered surfaces obtained only by rotation about an axis in its plane such as spheres, cone and anchor ring. But there are surfaces which are generated not only by rotation alone but by a rotation followed by a translation. Such a motion is called a screw motion. The simplest case of a screw motion is the motion of the x-axis through a rotation about the z-axis and translation in the positive direction of the z-axis. Usually we take the angle v through which the positive x-axis rotated is proportional to the distance λ in the upward direction so that $\frac{\lambda}{v}$ is constant. The surface generated by the screw motion of the x-axis about the z-axis is called a right helicoid. So we shall derive the equation of the right helicoid before taking up the general case.

(i) *Representation of a right helicoid.* This is the helicoid generated by a straight line which meets the axis at right angles. If we take the x-axis as the generating line, it rotates about the z-axis and moves upwards. Let $O'P$ be the translated position of the x-axis after rotating through an angle v. Let (x, y, z) be the coordinates of P. Draw PM perpendicular to the XOY plane and let $OM = u$. Then $x = u \cos v$, $y = u \sin v$, and $z = PM$.

By assumption the distance $PM = z$ translated by the x-axis is proportional to the angle v of rotation. Taking the constant of proportionality to be a, let $\frac{z}{v} = a$.

Hence the position vector of any point on the right helicoid is

$$\mathbf{r} = (u \cos v, u \sin v, av)$$

Now $\mathbf{r}_1 = (\cos v, \sin v, 0)$, $\mathbf{r}_2 = (-u \sin v, u \cos v, a)$

Since $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$, the parametric curves are orthogonal. When u = constant c (say), then the equation of the helicoid becomes $r = (c \cos v, c \sin v, av)$ which are circular helices on the surface. The parametric curves v = constant are the generators at the constant distance from the XOY plane.

Further $\mathbf{r}_1 \times \mathbf{r}_2 = (a \sin v, -a \cos v, u)$ and $H = \sqrt{a^2 + u^2}$

Hence the unit normal $\mathbf{N} = \dfrac{1}{\sqrt{a^2 + u^2}} (a \sin v, -a \cos v, u)$

Definition. If $v = 2\pi$, then $2\pi a$ is the distance translated after one complete rotation. This is called the pitch of the helicoid.

(ii) Representation of the general helicoid. The general helicoid with the z-axis as the axis is generated by the curve of intersection of the surface with any plane containing z-axis. Since the section of the surface by such planes are congruent curves, without of loss of generality, we can take the plane to be XOZ plane and generate the helicoid. Thus the equation of the generating curve in the XOZ plane can be taken as $x = g(u), y = 0, z = f(u)$. Let the curve in the XOZ plane rotate about the z-axis through an angle v and let it have the translation proportional to the angle v of rotation which we can take it as av. Since any point on the generating curve traces a circle with centre on the z-axis and radius $g(u)$ and z-coordinate is translated through av, the position vector of any point $\mathbf{r}$ on the general helicoid is

$$\mathbf{r} = (g(u)\cos v, g(u)\sin v, f(u) + av)$$

Now $$\mathbf{r}_1 = (g'(u)\cos v, g'(u)\sin v, f'(u))$$

$$\mathbf{r}_2 = (-g(u)\sin v, g(u)\cos v, a)$$

Further $\mathbf{r}_1 \cdot \mathbf{r}_2 = f'(u)\, a.$

Hence when the parametric curves are orthogonal, then either $f'(u) = 0$ or $a = 0$. If $f'(u) = 0, f(u)$ is constant so that the surface is a right helicoid. If $a = 0$, we do not have screw motion and we have only rotation about z-axis so that the helicoid is a surface of revolution.

When v = constant, the parametric curves are the various positions of the generating curve on the plane of rotation. When u = constant, it follows from the equation of the helicoid, the parametric curves are helices on the surfce.

2.9 METRIC ON A SURFACE—THE FIRST FUNDAMENTAL FORM

Analogous to the arcual length ds^2 in the case of a space curve, we shall introduce a metric on a surface called the first fundamental form.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the given surface. Let the parameters u, v be functions of a single parameter t. Then $\mathbf{r} = \mathbf{r}[u(t), v(t)]$ is a function of a single variable t and hence it represents a curve on the surface with t as parameter. The arc length in terms of the parameter t is given by

$$\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \left(\frac{d\mathbf{r}}{dt}\right)^2 \quad ...(1)$$

But $$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} \quad ...(2)$$

Using (2) in (1), we get

$$\left(\frac{ds}{dt}\right)^2 = \left(\mathbf{r}_1\frac{du}{dt} + \mathbf{r}_2\frac{dv}{dt}\right)^2$$

$$= \mathbf{r}_1 \cdot \mathbf{r}_1 \left(\frac{du}{dt}\right)^2 + 2\,\mathbf{r}_1 \cdot \mathbf{r}_2 \frac{du}{dt} \cdot \frac{dv}{dt} + \mathbf{r}_2 \cdot \mathbf{r}_2 \left(\frac{dv}{dt}\right)^2 \quad \text{...(3)}$$

Let $E = \mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}_1^2,\ F = \mathbf{r}_1 \cdot \mathbf{r}_2$ and $G = \mathbf{r}_2 \cdot \mathbf{r}_2 = \mathbf{r}_2^2$ ...(4)

Using the above notation, (3) can be rewritten in terms of the differentials as

$$ds^2 = E du^2 + 2F\, du\, dv + G\, dv^2 \quad \text{...(5)}$$

Definition 1. The differential quadratic form (5) is called the first fundamental form or metric on the surface. It is usually denoted by *I*.

***Note* 1.** The expression for ds^2 in (5) is independent of t and so it can be considered as the infinitesimal distance between two points with parameters (u, v) and $(u + du, v + dv)$ on the surface.

Let P and Q be two neighbouring points on the surface with position vectors $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ corresponding to the parameters u, v and $u + du,\ v + dv$.

Now $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv = \mathbf{r}_1 du + \mathbf{r}_2\, dv \quad \text{...(1)}$$

Since P and Q are two neighbouring points, the length ds of the element of the arc joining them is equal to $|d\mathbf{r}|$. Using (1), we get

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = d\mathbf{r}^2 = (\mathbf{r}_1 du + \mathbf{r}_2 dv)^2$$

$$= \mathbf{r}_1^2 du^2 + 2\mathbf{r}_1 \cdot \mathbf{r}_2\, du\, dv + \mathbf{r}_2^2 dv^2$$

$$= \boldsymbol{E}\, du^2 + 2\boldsymbol{F}\, du\, dv + \boldsymbol{G}\, dv^2$$

Thus if ds denotes the length of the elementary arc joining (u, v) and $(u + du, v + dv)$ lying on the surface, then

$$ds^2 = \boldsymbol{E}\, du^2 + 2\boldsymbol{F}\, du\, dv + \boldsymbol{G}\, dv^2 \quad \text{...(2)}$$

From (2), we get $\left(\frac{ds}{dt}\right)^2 = \boldsymbol{E}\left(\frac{du}{dt}\right)^2 + 2\boldsymbol{F}\frac{du}{dt}\frac{dv}{dt} + \boldsymbol{G}\left(\frac{dv}{dt}\right)^2.$

Hence $$s = \int_{t_0}^{t} \sqrt{\boldsymbol{E}\left(\frac{du}{dt}\right)^2 + 2\boldsymbol{F}\frac{du}{dt}\frac{dv}{dt} + \boldsymbol{G}\left(\frac{dv}{dt}\right)^2}\, dt$$

***Note* 2.** ds is no longer a perfect differential in the sense that there exists no function $\phi(u, v)$ such that $ds = d\phi$.

***Note* 3.** Since the square root of the first fundamental form gives the length $|d\mathbf{r}|$, it is called the metric of the surface. Though the metric is usually employed for calculation of the arc length of a curve on the surface, the coefficients $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{G}$ are used to study many important properties of the surfaces. They are functions of parameters u, v and called first fundamental coefficients.

***Note* 4.** On the parametric curve v = constant, we have $dv = 0$ and the metric reduces to $ds^2 = E\, du^2$. In a similar manner, on the parametric curve u = constant, $ds^2 = G\, dv^2$.

We have the vector area of the parallelogram is $\mathbf{r}_1\, du \times \mathbf{r}_2\, dv$.

so that $dS = |\mathbf{r}_1\, du \times \mathbf{r}_2\, dv| = |\mathbf{r}_1 \times \mathbf{r}_2|\, du\, dv = H\, du\, dv$.

This proves that $H\, du\, dv$ gives the elementary area dS on a surface.

Example 1. Find E, F, G and H for the paraboloid $x = u,\ y = v,\ z = u^2 - v^2$.

Any point on the paraboloid has position vector $\mathbf{r} = (u, v, u^2 - v^2)$.

Hence $\mathbf{r}_1 = (1, 0, 2u)$, and $\mathbf{r}_2 = (0, 1, -2v)$.

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + 4u^2,\ F = \mathbf{r}_1 \cdot \mathbf{r}_2 = -4uv,\ G = \mathbf{r}_2 \cdot \mathbf{r}_2 = 1 + 4v^2.$$

Further $\mathbf{r}_1 \times \mathbf{r}_2 = (-2u, +2v, 1)$.

Hence $H = |\mathbf{r}_1 \times \mathbf{r}_2| = \sqrt{4u^2 + 4v^2 + 1}$, which is also equal to $\sqrt{EG - F^2}$.

Example 2. Calculate the first fundmental coefficients and the area of the anchor ring corresponding to the domain $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$

The position vector of any point on the anchor ring is

$$\mathbf{r} = \{(b + a \cos u) \cos v,\ (b + a \cos u) \sin v,\ a \sin u\}$$

Hence $\mathbf{r}_1 = \{-a \sin u \cos v,\ -a \sin u \sin v,\ a \cos u\}$

$$\mathbf{r}_2 = \{-(b + a \cos u) \sin v,\ (b + a \cos u) \cos v,\ 0\}$$

Now $E = \mathbf{r}_1^2 = \mathbf{r}_1 \cdot \mathbf{r}_1 = a^2 \sin^2 u\, (\cos^2 v + \sin^2 v) + a^2 \cos^2 v = a^2$ ...(1)

As we have already noted $F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ ...(2)

$$G = \mathbf{r}_2^2 = \mathbf{r}_2 \cdot \mathbf{r}_2 = (b + a \cos u)^2 \sin^2 v + (b + a \cos u)^2 \cos^2 v \quad \text{...(3)}$$

Hence we have $G = (b + a \cos u)^2$

(1), (2) and (3) give the first fundamental coefficients.

To find the area, let us find H.

$$H^2 = EG - F^2 = a^2(b + a \cos u)^2 \text{ so that } H = a(b + a \cos u) \quad \text{...(4)}$$

By Theorem 4, the elementary area of the surface is $H\, du\, dv$. Using (4), the entire surface area is given by

$$S = \int_0^{2\pi}\int_0^{2\pi} H\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} a(b + a \cos u)\, du\, dv$$

$$= 2\pi a \int_0^{2\pi} (b + a \cos u)\, du = 4\pi^2 ab$$

2.10 DIRECTION COEFFICIENTS ON A SURFACE

In the case of curves in space, we are able to obtain a moving triad of mutually perpendicular unit vectors (**t**, **n**, **b**) with the help of which we are able to express any vector at a point on the curve linearly in terms of (**t**, **n**, **b**). Though we cannot have an exact analogue of (**t**, **n**, **b**) at a point on the surface, we are trying to have something similar to this triad at any point on the surface. This leads to the notion of tangential and normal components of a vector at a point P on the surface.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of a surface and let P be any point on th surface. Then we know that the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are tangents to the parametr curves v = constant and u = constant passing through P. Let $\mathbf{N}$ be the surfa normal at P. Since $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$, neither $\mathbf{r}_1$ nor $\mathbf{r}_2$ can be a scalar multiple of the oth so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly independent. Further $\mathbf{N}$ cannot be a scalar multiple either $\mathbf{r}_1$ or $\mathbf{r}_2$. For if $\mathbf{N} = a\mathbf{r}_1$, then $\mathbf{N}\cdot\mathbf{N} = a\mathbf{r}_1\cdot\mathbf{N} = 0$ which is absurd, since it giv $1 = 0$. Thus at any point P on the surface, there are three linearly independe vectors $\mathbf{N}, \mathbf{r}_1, \mathbf{r}_2$.

Hence every vector $\boldsymbol{a}$ through P can be expressed uniquely as a line combination of three vectors $\mathbf{N}, \mathbf{r}_1$ and $\mathbf{r}_2$. Thus there exist unique scalars a_n, λ, ar μ such that $\boldsymbol{a} = a_n\mathbf{N} + \lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ ...(

Thus (1) expresses any vector through P as the sum of two vectors $a_n\mathbf{N}$ norm to the surface and $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ lying in the tangent plane to the surface at P.

On taking dot product with $\mathbf{N}$ on both sides of (1), we obtain $\boldsymbol{a}\cdot\mathbf{N} = a_n\mathbf{N}\cdot\mathbf{N} = a_n$ as $\mathbf{N}\cdot\mathbf{r}_1 = \mathbf{N}\cdot\mathbf{r}_2 = 0$. The scalar a_n is called the norm component of $\boldsymbol{a}$. From this it is easily seen that the vector lies in the tangent pla of $\boldsymbol{a}$ if and only if the normal component a_n is zero.

The vector $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ lying in the tangent plane at P of the surface is called t tangential part of $\boldsymbol{a}$ and λ, μ are called the tangential components of $\boldsymbol{a}$. The components λ, μ depend only on the tangential part of $\boldsymbol{a}$ and λ, μ are zero if a only if the vector $\boldsymbol{a}$ is normal to the surface.

Definition 1. The direction of any tangent line to the surface at the point P called a direction on the surface at the point P.

From the very definition of a direction on the surface, we see that there a infinitely many directions at each point of the surface.

In the remaining part of this chapter, we shall make a study of the tangent vectors to the surface. These are the vectors whose normal components are ze As noted in the previous paragraph, such a vector $\boldsymbol{a}$ is of the form $\boldsymbol{a} = \lambda\mathbf{r}_1 + \mu$ The components of the tangential vector $\boldsymbol{a}$ at P are (λ, μ) so that we write $\boldsymbol{a}$ $\boldsymbol{a} = (\lambda, \mu)$. (λ, μ) is a direction on the surface at P means that $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ w represent a vector at P along a tangent to the surface at P. In all our discussi components will mean tangential components and the vector (λ, μ) stands for tangential vector with components (λ, μ). From the definition, we note following property.

If $\boldsymbol{a} = (\lambda, \mu)$ is tangential vector at P on a surface, then its magnitude

$$|\boldsymbol{a}| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}.$$

From the definition, we have $\boldsymbol{a} = \lambda\mathbf{r}_1 + \mu\mathbf{r}_2$.

Hence
$$|\boldsymbol{a}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} = (\lambda\mathbf{r}_1 + \mu\mathbf{r}_2)\cdot(\lambda\mathbf{r}_1 + \mu\mathbf{r}_2)$$
$$= \lambda^2\mathbf{r}_1^2 + 2\mathbf{r}_1\cdot\mathbf{r}_2\,\lambda\mu + \mathbf{r}_2^2\,\mu^2.$$

Since $E = \mathbf{r}_1^2$, $F = \mathbf{r}_1\cdot\mathbf{r}_2$ and $G = \mathbf{r}_2^2$, we get

$|a|^2 = (E\lambda^2 + 2F\lambda\mu + G\mu^2)$ which gives

$$|a| = \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2} \quad ...(1)$$

Note. The above formula expresses the magnitude of the tangential vector in terms of the components and the first fundamental coefficients.

Definition 2. Let b be the unit vector along the tangential vector a at P. Let the components of b be (l, m) so that $b = l\mathbf{r}_1 + m\mathbf{r}_2$. The components (l, m) of the unit vector b at P along the direction a are called the direction coefficients of a. These direction coefficients are written as (l, m). From the definition of (l, m) $(-l, -m)$ gives the direction opposite to (l, m).

Since $b = l\mathbf{r}_1 + m\mathbf{r}_2$ and $|b| = 1$, we have from the property following Definition 1,

$$El^2 + 2Flm + Gm^2 = 1 \quad ...(2)$$

Hence the direction coefficients satisfy the above identity.

Note **1.** The direction coefficients (l, m) are analogous to the direction cosines (l, m, n) satisfying the identity $l^2 + m^2 + n^2 = 1$ in the Cartesian geometry of three dimensions.

Note **2.** In the case of the plane with rectangular cartesian coordinates, a direction is determined by the angle ψ made by the line with the positive direction of the x-axis. The direction coefficients are $\cos\psi$, $\sin\psi$. The metric becomes $dx^2 + dy^2$ and the above identity (2) becomes $\cos^2\psi + \sin^2\psi = 1$.

We use the following convention in measuring the angle between two tangential directions at the same point. The sense of rotation of the angles in the tangent plane is from the direction $\mathbf{r}_1$ to that of $\mathbf{r}_2$ through angle between 0 and π which means the smaller of the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$. This is also the positive sense of rotation about N.

Theorem 1. If (l, m) and (l', m') are the direction coefficients of two directions at a point P on the surface and θ is the angle between the two direction at P, then

(i) $\cos\theta = Ell' + F(lm' + l'm) + Gmm'$

(ii) $\sin\theta = H(lm' - l'm)$

Proof. If (l, m) and (l', m') are the direction coefficients of the two directions at the same point P on the surface $\mathbf{r} = \mathbf{r}(u, v)$, then the corresponding unit vectors along these directions at P are

$$a = l\mathbf{r}_1 + m\mathbf{r}_2, \quad a' = l'\mathbf{r}_1 + m'\mathbf{r}_2 \quad ...(1)$$

Let θ be the angle between the two directions. Measuring θ from the direction $\mathbf{r}_1$ to $\mathbf{r}_2$ through the smaller angle, we have

$$a \cdot a' = \cos\theta, \; a \times a' = \sin\theta\, \mathbf{N} \quad ...(2)$$

Now
$$\begin{aligned} a \cdot a' &= (l\mathbf{r}_1 + m\mathbf{r}_2) \cdot (l'\mathbf{r}_1 + m'\mathbf{r}_2) \\ &= ll'\mathbf{r}_1^2 + (lm' + l'm)\,\mathbf{r}_1 \cdot \mathbf{r}_2 + mm'\,\mathbf{r}_2^2 \\ &= Ell' + F(lm' + l'm) + Gmm' \quad ...(3) \end{aligned}$$

$$\cos\theta_1 = \frac{1}{\sqrt{E}}(El + Fm),\ \sin\theta_1 = \frac{H|m|}{\sqrt{E}}$$

In a similar manner, if θ_2 is the angle between (l, m) and the parametric direction $\left(0, \frac{1}{\sqrt{G}}\right)$ corresponding to u = constant, we have

$$\cos\theta_2 = \frac{1}{\sqrt{G}}(Fl + Gm),\ \sin\theta = \frac{H|l|}{\sqrt{G}}$$

Theorem 2. If (l', m') are the direction coefficients of a line which makes an angle $\frac{\pi}{2}$ with the line whose direction coefficients are (l, m), then

$$l' = -\frac{1}{H}(Fl + Gm),\ m' = \frac{1}{H}(El + Fm)$$

Proof. If (l, m) and (l', m') are two directions at a point on the surface, then by Theorem 1, we have

$$\cos\theta = Ell' + F(lm' + l'm) + Gmm' \quad \text{...(1)}$$

$$\sin\theta = H(lm' - l'm) \quad \text{...(2)}$$

When $\theta = \frac{\pi}{2}$, we have from (1)

$$Ell' + F(lm' + l'm) + Gmm' = 0$$

That is $$l'(El + Fm) + m'(Fl + Gm) = 0$$

The above equation is satisfied for

$$l' = -\alpha(Fl + Gm),\ m' = \alpha(El + Fm) \quad \text{...(3)}$$

for some scalar α

We shall find α with the help of (2).

When $\theta = \frac{\pi}{2}$, we have from (2), $H(lm' - l'm) = 1$ ...(4)

Using (3) in (4), we obtain

$$Hl[\alpha(El + Fm)] + Hm[\alpha(Fl + Gm)] = 1$$

which gives $\frac{1}{\alpha} = H[El^2 + 2Fml + Gm^2]$

Since (l, m) are direction coefficients, we have

$$El^2 + 2Fml + Gm^2 = 1 \text{ so that } \alpha = \frac{1}{H}$$

Using this value of α in (3), we obtain

$$l' = -\frac{1}{H}(Fl + Gm),\ m' = \frac{1}{H}(El + Fm)$$

(v) If (l, m) are the direction coefficients of the tangential direction to the curve $u = u(t),\ v = v(t)$ at a point on the surface $\mathbf{r} = \mathbf{r}(u, v)$, then $l = \frac{du}{ds},\ m = \frac{dv}{ds}$.

The unit tangent vector at any point P on the curve is

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u}\cdot\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\cdot\frac{dv}{ds} = \mathbf{r}_1\frac{du}{ds} + \mathbf{r}_2\frac{dv}{ds}.$$

Since $\frac{d\mathbf{r}}{ds}$ represents the unit tangent vector at P along the tangential direction to the curve, its components $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ give the direction coefficients of the tangent at P on the surface. Hence $l = \frac{du}{ds},\ m = \frac{dv}{ds}$.

As (du, dv) are proportional to $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$, (du, dv) give the direction ratios of the tangential direction to the curve at P.

Note. Using (iii), the angle between the tangential directions (du, dv) and $(\delta u, \delta v)$ is given by

$$\sin\theta = \frac{H(du\,\delta v - dv\,\delta u)}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\sqrt{E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2}}$$

$$\cos\theta = \frac{E\,du\,\delta u + F(du\,\delta v + \delta u\,dv) + G\,dv\,\delta v}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\sqrt{E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2}}$$

(vi) If the equation of the curve on the surface $\mathbf{r} = \mathbf{r}(u, v)$ is given in the implicit form $\phi(u, v) = 0$, then $(-\phi_2, \phi_1)$ are the direction ratios of the tangent at any point on the curve.

Differentiating the equation of the curve $\phi(u, v) = 0$, we obtain

$$\frac{\partial\phi}{\partial u}du + \frac{\partial\phi}{\partial v}dv = 0 \text{ so that } \frac{du}{dv} = -\frac{\phi_2}{\phi_1}$$

Hence (du, dv) are proportional to $(-\phi_2, \phi_1)$. Using (v), we see that $(-\phi_2, \phi_1)$ are the direction ratios of the tangent to the curve.

Example 1. Find the parametric directions and the angle between the parametric curves.

For the parametric curve v = constant, the parametric direction has the direction ratio $(du, 0)$ by (v), Using (i), its direction coefficients $(l, m) = \frac{(du, 0)}{\sqrt{E\,du^2}} = \frac{(1, 0)}{\sqrt{E}}$

In a similar manner, the direction ratios of the curve v = constant are $(0, dv)$, so that its direction coefficients are $(l', m') = \frac{(0, dv)}{\sqrt{G\,dv^2}} = \frac{(0, 1)}{\sqrt{G}}$

Let θ be the angle between the parametric curves. Then by Theorem 1,

$$\cos\theta = \frac{F}{\sqrt{EG}} \text{ and } \sin\theta = \frac{H}{\sqrt{EG}}.$$

When $\theta = \frac{\pi}{2}$, $\cos\theta = 0$ so that the condition of orthogonality of parametric curves is $F = 0$.

It should be noted that we have obtained the angle between the parametric curves in Theorem 3 of 2.9 by considering $\mathbf{r}_1 \cdot \mathbf{r}_1$ and $|\mathbf{r}_1 \times \mathbf{r}_2|$.

Example 2. For the cone with vertex at the origin and semi-vertical angle α, show that the tangent plane is the same at all points on the generating line.

The position vector of any point on the cone with semi-vertical angle α and the axis of the cone as z-axis is

$$\mathbf{r} = (u \cos v, u \sin v, u \cot \alpha)$$

Now let us find the fundamental coefficients.

$$\mathbf{r}_1 = (\cos v, \sin v, \cot \alpha), \mathbf{r}_2 = (-u \sin v, u \cos v, 0)$$

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + \cot^2\alpha = \operatorname{cosec}^2\alpha, F = 0$$

$$G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2(\sin^2 v + \cos^2 v) = u^2.$$

$$H^2 = EG - F^2 = u^2 \operatorname{cosec}^2\alpha \text{ so that } H = u \operatorname{cosec} \alpha$$

Now $\quad \mathbf{r}_1 \times \mathbf{r}_2 = (-u \cos v \cot\alpha, -u \sin v \cot \alpha, u)$

Hence $\quad \mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = (-\cos v \cos \alpha, -\sin v \cos \alpha, \sin \alpha)$

Thus the surface normal $\mathbf{N}$ is independent of u. This implies that the tangent plane is the same at all points along a generating line.

Example 3. For a right helicoid given by $(u \cos v, u \sin v, av)$, determine $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{N})$ at a point on the surface and the direction of the parametric curves. Find the direction making angle $\frac{\pi}{2}$ at a point on the surface with the parametric curve v = constant.

Now any point on the right helicoid is $\mathbf{r} = (u \cos v, u \sin v, av)$

Hence $\quad \mathbf{r}_1 = (\cos v, \sin v, 0), \mathbf{r}_2 = (-u \sin v, u \cos v, a)$

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1, F = 0, G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2 + a^2.$$

$$H^2 = EG - F^2 = u^2 + a^2 \text{ so that } H = \sqrt{u^2 + a^2}$$

Again $\mathbf{r}_1 \times \mathbf{r}_2 = (a \sin v, -a \cos v, u)$

Now $$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \left(\frac{a}{\sqrt{u^2 + a^2}} \sin v, -\frac{a}{\sqrt{u^2 + a^2}} \cos v, \frac{u}{\sqrt{u^2 + a^2}}\right)$$

Let the components of **N** be (N_1, N_2, N_3).

Using (ii), the direction coefficients of the parametric curves are

$$\left(\frac{1}{\sqrt{E}}, 0\right) = (1, 0) \text{ and } \left(0, \frac{1}{\sqrt{G}}\right) = \left(0, \frac{1}{\sqrt{u^2 + a^2}}\right)$$

If γ is the angle made by **N** with the z-axis, then $\cos \gamma = N_3 = \dfrac{u}{\sqrt{u^2 + a^2}}$

If (l', m') is the direction coefficient orthogonal to the parametric direction v = constant, then by Theoem 2 we have $l' = -\dfrac{1}{H}(Fl + Gm)$, $m' = \dfrac{1}{H}(El + Fm)$.

Substituting for l, m, E, F, G and H in the above step, we have $l' = 0$ and $m' = \dfrac{1}{\sqrt{u^2 + a^2}}$ which is the direction of the parametric system u = constant. This is what we expect, since the parametric curves are orthogonal.

2.11 FAMILIES OF CURVES

So far, we were concerned with a single curve lying on a surface and associated tangential direction. Now we shall introduce families of curves on a surface and study some basic properties of such families.

Definition 1. Let $\phi(u, v)$ be a single valued function of u, v possessing continuous partial derivatives ϕ_1, ϕ_2 which do not vanish together. Then the implicit equation $\phi(u, v) = c$ where c is a real parameter gives a family of curves on the surface $\mathbf{r} = \mathbf{r}(u, v)$.

For different values of c, we get different curves of the family lying on the surfaces. From the very definition, we note the following properties.

(i) Through every point (u, v) of the surface, there passes one and only one member of the family.

Let $\phi(u_0, v_0) = c_1$ where (u_0, v_0) is any point on the surface. Then $\phi(u, v) = c_1$ is a member of the family passing through (u_0, v_0). Hence through every point (u_0, v_0) on the surface, there passes one and only one member of the family.

(ii) As noted in (vi) of 2.10, the direction ratios of the tangent to the curve of the family at (u, v) is $(-\phi_2, \phi_1)$.