Let us denote by $\mu = \dfrac{x^3 y^3 z^3}{ABC} \lambda^3 \kappa$.

Operating with $\lambda \dfrac{d}{ds} = \Delta$ on both sides of (5), we have

$$\lambda \frac{d}{ds}[\mu \mathbf{b}] = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z} \right) \left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{c}(c' - c) \right)$$

So $\lambda \mu' \mathbf{b} + \lambda \mu (-\tau \mathbf{n}) = (3x(a' - a), 3y(b' - b), cz(c' - c))$ ...(6)

From equation (7) of the theorem, we have

$$\lambda^2 \kappa \mathbf{n} + \lambda \lambda' \mathbf{t} = \Delta \mathbf{h} = -\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3} \right) \qquad ...(7)$$

Taking scalar product of (6) and (7), we obtain

$$\lambda^3 \kappa \tau \mu = 3 \sum \frac{A^2}{x^2}(a' - a)$$

Substituting the value of μ and simplifying, we get.

$$\lambda^6 \kappa^2 \tau = \frac{3\,ABC}{x^3 y^3 z^3} \sum \frac{A^2}{x^2}(a' - a)$$

Substituting the values of λ and κ, we get

$$\tau = \frac{3x^3 y^3 z^3}{ABC} \frac{\sum \frac{A^2}{x^2}(a' - a)}{\sum \frac{x^6}{A^2}(a' - a)^2}$$

1.10. CONTACT BETWEEN CURVES AND SURFACES

Let γ be a curve $\mathbf{r}(u) = \{f(u), g(u), h(u)\}$ and let S be a surface $F(x, y, z) = 0$. Let us assume that the curve γ and the surface S are of high class in the sense that $\mathbf{r}(u)$ and $F(x, y, z)$ have continuous derivatives of sufficiently high order. From the equation of the curve, we take $x = f(u), y = g(u), z = h(u)$. If this point lies on the surface, we have $F(f(u), g(u), h(u)) = 0$ which is an equation in u giving the points of intersection of the curve and the surface. Depending upon the nature of the roots of the equation, we shall define the contact between curves and surfaces as follows.

Let u_0 be one such zero of $F(u) = 0$. Since $F(u)$ possesses the derivatives of sufficiently high order, $F(u)$ has the following power series representation in the neighbourhood of $u = u_0$.

$$F(u) = F(u_0) + \frac{(u-u_0)}{1!}F'(u_0) + \frac{(u-u_0)^2}{2!}F''(u_0) + \ldots + \frac{(u-u_0)^n}{n!}F^n(u_0) + O(u-u_0)^{n+1} \quad \ldots(1)$$

$$F(u) = hF'(u_0) + \frac{h^2}{2!}F''(u_0) + \frac{h^3}{3!}F'''(u_0) + \ldots + \frac{h^n}{n!}F^{(n)}(u_0) + O(h^{n+1})$$

Definition 1. If $F'(u_0) \neq 0$, then u_0 is a simple zero of $F(u) = 0$. Then the curve γ and the surface S is said to have simple intersection at $\mathbf{r}(u_0)$.

Definition 2. If $F'(u_0) = 0$ and $F''(u_0) \neq 0$, u_0 is a double zero of $F(u)$ and $F(u)$ is of second order of h. Then the curve γ and S are said to have two point contact.

Definition 3. If $F'(u_0) = F''(u_0) = 0$ and $F'''(u_0) \neq 0$, γ and S are said to have three point contact at $u = u_0$ under these conditions u_0 is a triple zero of $F(u)$.

In general if $F'(u_0) = F''(u_0) = \ldots = F^{(n-1)}(u_0) = 0$ and $F^{(n)}(u_0) \neq 0$, the curve γ and the surface S are said to have n point contact at $u = u_0$.

Theorem 1. The conditions of a surface having n point contact with the curve γ are invariant over a change of parameter.

Proof. Let $u = \phi(t)$ be the given parametric transformation. Since it is regular, we have $\phi^{(K)}(u) \neq 0$ for $k \geq 1$. Corresponding to the point $u = u_0$, we have $u_0 = \phi(t_0)$ at $t = t_0$.

Now $F(u) = F(\phi(t)) = f(t)$ where f is a function of t only.

$$\dot{f}(t) = \frac{d}{dt}F(u) = \frac{d}{du}F(u)\cdot\frac{du}{dt} = F'(u)\,\dot{\phi}(t) \quad \ldots(1)$$

$$\ddot{f}(t) = \frac{d}{dt}[F'(u)\dot{\phi}(\mathbf{t})] = F''(u)[\dot{\phi}(\mathbf{t})]^2 + F'(u)\ddot{\phi}(t) \quad \ldots(2)$$

If $\quad F'(u) = 0$, then $\dot{f}(t) = 0$ as $\dot{\phi}(t) \neq 0$

If $F'(u) = 0$ and $F''(u) \neq 0$, then from (1) and (2) we get $\dot{f}(t) = 0$ and $\ddot{f}(t) \neq 0$, since $\dot{\phi}(t), \ddot{\phi}(t) \neq 0$.

Thus if the surface S given by $F(u)$ has two point contact with the curve γ at $\mathbf{r}(u_0)$, then the surface S given by $f(t)$ has two point contact with γ at $\mathbf{r}(\phi(t_0))$

Differentiating (2) again we get

$$\dddot{f}(t) = F'''(u)\,[\dot{\phi}(\mathbf{t})]^3 + 3F''(u)\,\dot{\phi}(t)\ddot{\phi}(t) + F'(u)\,\dddot{\phi}(t) \quad \ldots(3)$$

If $F'(u) = 0$, $F''(u) = 0$ and $F'''(u) \neq 0$, then from (3) $\dot{f}(t) = 0$, $\ddot{f}(t) = 0$ and $\dddot{f}(t) \neq 0$ as $\phi(t)$ is regular. Hence the surface S given by $f(t)$ has three point contact with the curve γ at $\mathbf{r}[\phi(t_0)]$. Proceeding like this, if $F'(u) = F''(u) = \ldots = F^{(n-1)}(u) = 0$ and $F^{(n)}(u) \neq 0$ at $u = u_0$, then $\dot{f}(t) = \ddot{f}(t) = \ldots = f^{(n-1)}(t) = 0$ and $f^{(n)}_{(t)} \neq 0$ at $\mathbf{r}[\phi(t_0)]$. Thus the surface S given by $f(t)$ has n point contact with γ at

$\mathbf{r}[\phi(t_0)]$, provided the surface S given by $F(u)$ has n point contact with γ at $\mathbf{r}(u_0)$. Thus a surface having n point contact with the curve γ is invariant over a change of parameter. Hence we conclude that the property of the curve having n-point contact with S is a property of γ in the sense that any path which represents γ will have this property.

Theorem 2. The osculating plane at any point P has three point contact with the curve at P

Proof. Let P be any point on the curve and let the arc length be measured from P so that $s = 0$ at P and let the equation of the curve be $\mathbf{r} = \mathbf{r}(s)$. We know that the osculating plane at P is

$$[\mathbf{r}(s) - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0)] = 0$$

and let $F(s) = [\mathbf{r}(s) - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0)]$ ...(1)

We shall show that $F'(s) = F''(s) = 0$ and $F'''(s) \neq 0$ at P where $s = 0$ and this proves that the osculating plane has three point contact with the curve.

Expanding $\mathbf{r}(s)$ by Taylor's theorem in the neighbourhood of P,

$$\mathbf{r}(s) = \mathbf{r}(0) + \frac{\mathbf{r}'(0)}{1!}s + \frac{\mathbf{r}''(0)}{2!}s^2 + \frac{1}{3!}\mathbf{r}'''(0)s^3 + 0(s^4) \quad ...(2)$$

Neglecting powers of s greater than 3 in (2) and substituting it in (1), we obtain

$$F(s) = \left[\frac{s\mathbf{r}'(0)}{1!} + \frac{\mathbf{r}''(0)}{2!}s^2 + \frac{\mathbf{r}'''(0)}{3!}s^3, \mathbf{r}'(0), \mathbf{r}''(0)\right] \quad ...(3)$$

$$= \left[\frac{\mathbf{r}'(0)}{1!}, \mathbf{r}'(0), \mathbf{r}''(0)\right]s + [\mathbf{r}''(0), \mathbf{r}'(0), \mathbf{r}''(0)]\frac{s^2}{2!}$$

$$+ [\mathbf{r}'''(0), \mathbf{r}'(0), \mathbf{r}''(0)]\frac{s^3}{3!} \quad ...(3)$$

The first two terms of (3) vanish. Using $\kappa^2\tau = [\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ in (3), we obtain

$$F(s) = -\frac{\kappa^2\tau}{6}s^3.$$

Hence $F'(0) = 0$, $F''(0) = 0$ and $F'''(0) = -\kappa^2\tau \neq 0$, provided κ and τ do not vanish at P. This proves that in general the curve and the osculating plane have three point contact at P.

Note. In case if $\kappa = 0$, or $\tau = 0$, $F'''(0) = 0$ so that the plane must have at least four point contact with the curve.

1.11 OSCULATING CIRCLE AND OSCULATING SPHERE

With the help of notion of contact of curves and surfaces, we obtain for space curves the analogues of circle of curvature and radius of curvature known for plane curves. Since we are concerned with the contact of curves and surfaces, we may think of a sphere having four point contact with the given curve. This leads to the introduction of osculating sphere and the radius of spherical curvature.

Definition 1. Let γ be the given space curve and P be any point on it. The circle having three point contact with the given space curve at P is called the osculating circle at P.

Definition 2. The radius of the osculating circle is called the radius of curvature of the curve at P. It is denoted by ρ. The centre of the osculating circle is called the centre of curvature at P.

Using the above definitions, we note the following properties of the osculating circle.

1. Since the osculating plane has also three point contact with the curve at P, the osculating circle lies on the osculating plane. It is evident even otherwise if we define the osculating circle as the curve passing through three consecutive points on the curve as we have defined the osculating plane as the plane passing through three consecutive points on the curve.
2. Since the circle of curvature and the curve have the same tangent at P lying in the osculating plane, the centre of the circle lies on the principal normal at P.

Theorem 1. The radius of the osculating circle at P is the reciprocal of curvature of the curve at P and the position vector of its centre of the osculating circle is $\mathbf{c} = \mathbf{r} + \rho\mathbf{n}$ where $\rho = \dfrac{1}{\kappa}$.

Proof. Choosing arc-length s as parameter, let $\mathbf{c}$ be the position vector of the centre of the osculating circle. The centre $\mathbf{c}$ is at a distance ρ from P along the principal normal at P. Hence we have $\mathbf{c} - \mathbf{r} = \rho\mathbf{n}$. Hence its equation is $(\mathbf{c} - \mathbf{r}) \cdot \mathbf{n} = \rho$. We prove that $\rho = \dfrac{1}{\kappa}$.

Since any point $\mathbf{r} = \mathbf{r}(s)$ on the osculating circle satifies the equation of the sphere $(\mathbf{c} - \mathbf{R})^2 = \rho^2$ and lies in the osculating plane, the osculating circle is the intersection of the osculating plane and the sphere $(\mathbf{c} - \mathbf{R})^2 = \rho^2$ where $\mathbf{R}$ is the position vector of any point on the sphere. If $\mathbf{r}(s)$ is the point of intersection of this sphere and the curve, the sphere has three point contact with the curve at $\mathbf{r} = \mathbf{r}(s)$.

Let the point of intersection be $F(s) = (\mathbf{c} - \mathbf{r})^2 - \rho^2$.

The conditions for three point contact are

$$F(s) = 0,\ F'(s) = 0,\ F''(s) = 0 \qquad \text{...(1)}$$

$$F'(s) = 0 \text{ gives } (\mathbf{c} - \mathbf{r}) \cdot \mathbf{t} = 0 \qquad \text{...(2)}$$

Differentiating (2), $(\mathbf{c} - \mathbf{r}) \cdot \mathbf{t}' - \mathbf{t} \cdot \mathbf{t} = 0$

Since

$$\mathbf{t}' = \kappa\mathbf{n} \text{ and } \mathbf{t} \cdot \mathbf{t} = 1, \text{ we have } (\mathbf{c} - \mathbf{r}) \cdot \kappa\mathbf{n} = 1 \qquad \text{...(3)}$$

Comparing (3) with the equation of the osculating circle $(\mathbf{c} - \mathbf{r}) \cdot \mathbf{n} = \rho$, we get $\rho = \dfrac{1}{\kappa}$.

Thus we have proved that centre of the circle of curvature is $\mathbf{c} = \mathbf{r} + \rho\mathbf{n}$ and the radius of the circle of curvature is the reciprocal of the curvature of the curve at P.

Definition 3. A sphere having four point contact with the curve at a point P is called the osculating sphere at P on the curve.

Definition 4. The centre of the osculating sphere is called the centre of spherical curvature and its radius is called the radius of spherical curvature.

Theorem 2. If $\mathbf{r} = \mathbf{r}(s)$ is the given curve γ, then the centre C and radius R of spherical curvature at a point P on γ are given by

$$C = \mathbf{r} + \rho\mathbf{n} + \sigma\rho'\mathbf{b},\ R = \sqrt{\rho^2 + \sigma^2\rho'^2}$$

Proof. If C is the centre and R is the radius of the osculating sphere, then its equation is $(C - \mathbf{R})^2 = R^2$ where $\mathbf{R}$ is the position in vector of any point on the sphere. The points of intersection of the curve and the sphere are given by $F(s) = (C - \mathbf{r})^2 - R^2 = 0$. Since the sphere has four point contact with γ at P, the conditions of four point contact are $F(s) = F'(s) = F''(s) = F'''(s) = 0$ which give rise to the following equations.

$$F'(s) = 0 \text{ gives } (C - \mathbf{r})\cdot\mathbf{t} = 0 \qquad \ldots(1)$$

$$F''(s) = 0 \text{ gives } (C - \mathbf{r})\cdot\kappa\mathbf{n} - 1 = 0 \qquad \ldots(2)$$

$$F'''(s) = 0 \text{ gives } (C - \mathbf{r})\cdot[\kappa'\mathbf{n} + \kappa(\tau\mathbf{b} - \kappa\mathbf{t})] = 0 \text{ giving}$$

$$\kappa'(C - \mathbf{r})\cdot\mathbf{n} - \kappa^2(C - \mathbf{r})\cdot\mathbf{t} + \kappa\tau(C - \mathbf{r})\cdot\mathbf{b} = 0 \qquad \ldots(3)$$

Using (1) and (2) in (3), we get

$$\frac{\kappa'}{\kappa} + \kappa\tau(C - \mathbf{r})\cdot\mathbf{b} = 0$$

Let $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$. Then $\rho' = -\frac{\kappa'}{\kappa^2}$. Using these in the above equation, we have $(C - \mathbf{r})\cdot\mathbf{b} = \rho'\sigma$ ...(4)

From (1), (2) and (4) we have

$$(C - \mathbf{r})\cdot\mathbf{t} = 0,\ (C - \mathbf{r})\cdot\mathbf{n} = \rho,\ (C - \mathbf{r})\cdot\mathbf{b} = \rho'\sigma.$$

The above equations show that $(C - \mathbf{r})$ lies in the normal plane and its components along the normal and binormal are ρ and $\rho'\sigma$ respectively. So we can write $(C - \mathbf{r})$ as $(C - \mathbf{r}) = \rho\mathbf{n} + \rho'\sigma\mathbf{b}$. Hence the centre of the osculating sphere is $C = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b}$.

The radius R of the osculating sphere is given by

$$R^2 = (C - \mathbf{r})^2 = (\rho\mathbf{n} + \rho'\sigma\mathbf{b})\cdot(\rho\mathbf{n} + \rho'\sigma\mathbf{b}) = \rho^2 + \rho'^2\sigma^2$$

Hence $R = \sqrt{\rho^2 + \rho'^2\sigma^2}$.

As shown in Fig. 2, the centre of osculating shpere lies in the normal plane on a line parallel to the binormal called the polar axis. The intersection of the sphere with the osculating plane is the osculating circle.

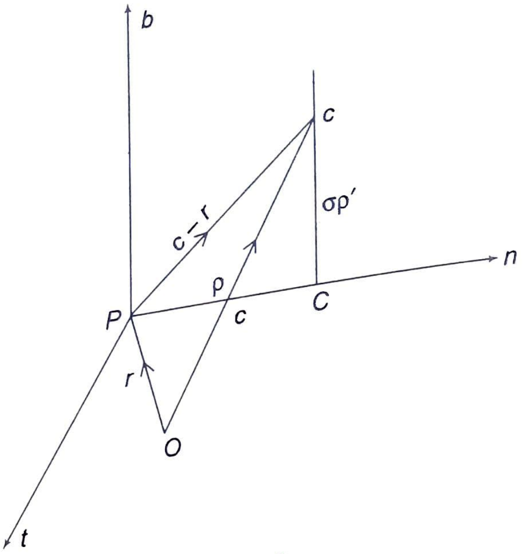

Fig. 2

Note. When κ is constant, $\rho = \dfrac{1}{\kappa}$ is also a constant and $\rho' = 0$ so that $R = \rho$. Thus for a curve of constant curvature, the centre of curvature and the centre of spherical curvature coincide and it is $c = \mathbf{r} + \rho\mathbf{n}$.

Example 1. Show that the radius of spherical curvature of a circular helix is equal to the radius of curvature.

Since ρ is a constant for a circular helix, the result follows from the note above.

Example 2. Find the centre of spherical curvature of the curve given by

$\mathbf{r} = (a \cos u, a \sin u, a \cos 2u)$ ...(1)

The osculating sphere $(\mathbf{r} - C)^2 = R^2$ with centre C and radius R has four point contact with the space curve. The conditions for the sphere to have four point contact with the space curve are

$$(\mathbf{r} - C) \cdot \dot{\mathbf{r}} = 0 \quad \text{...(2)}$$

$$(\mathbf{r} - C) \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}}^2 = 0 \quad \text{...(3)}$$

$$(\mathbf{r} - C) \cdot \dddot{\mathbf{r}} + 3\dot{\mathbf{r}}\,\ddot{\mathbf{r}} = 0 \quad \text{...(4)}$$

Let us take the centre $C = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ ...(5)

Using (1) and (5) in (2), (3) and (4), we get three equations in three unknowns. Solving these equations, we get the centre of spherical curvature.

Using the condition (2), we get

$$[(a \cos u - \alpha)\boldsymbol{i} + (a \sin u - \beta)\boldsymbol{j} + (a \cos 2u - \gamma)\boldsymbol{k}].$$

$$[(-a \sin u)\boldsymbol{i} + a \cos u\boldsymbol{j} - 2a \sin 2u\boldsymbol{k}] = 0 \text{ which gives}$$

$$\alpha \sin u - \beta \cos u + 2\gamma \sin 2u = 2a \cos 2u \sin 2u \quad \text{...(6)}$$

Using the condition (3), we get

$$[(a \cos u - \alpha)\boldsymbol{i} + (a \sin u - \beta)\boldsymbol{j} + (a \cos 2u - \gamma)\boldsymbol{k}].$$

$$[-a \cos u\boldsymbol{i} - a \sin u\boldsymbol{j} - 4a \cos 2u\boldsymbol{k}] + a^2(1 + 4 \sin^2 2u) = 0$$

Simplifying the above equation using dot product,

$$\alpha \cos u + \beta \sin u + 4\gamma \cos 2u = 4a(\cos^2 2u - \sin^2 2u) \qquad ...(7)$$

Using the condition (4), we have

$$[(a \cos u - \alpha)\boldsymbol{i} + (a \sin u - \beta)\boldsymbol{j} + (a \cos 2u - \gamma)\boldsymbol{k}] \cdot$$
$$[a \sin u\boldsymbol{i} - a \cos u\boldsymbol{j} + 8a \sin 2u\boldsymbol{k}]$$
$$+ 3[- a \sin u\boldsymbol{i} + a \cos u\boldsymbol{j} - 2a \sin 2u\boldsymbol{k}] \cdot [- \cos u\boldsymbol{i} - a \sin u\boldsymbol{j} - 4a \cos 2u\boldsymbol{k}] = 0$$

Simplifying the above equation, we get

$$\alpha \sin u - \beta \cos u + 8\gamma \sin 2u = 32a \sin 2u \cos 2u \qquad ...(8)$$

Now (8) – (6) gives $6\gamma \sin 2u = 30a \sin 2u \cos 2u$

Since $u \neq 0, \dfrac{\pi}{2}$ or π, $\gamma = 5a \cos 2u$.

Substituting this value of γ in (7), and (6), we get

$$\alpha \cos u + \beta \sin u = - 12a \cos^2 u - 4a \qquad ...(9)$$

and

$$\alpha \sin u - \beta \cos u = - 8a \sin 2u \cos 2u \qquad ...(10)$$

(9) cos u + (10) sin u gives

$$\begin{aligned}\alpha &= - 12\, a \cos^2 2u \cos u - 4a \cos u - 8a \sin 2u \cos 2u \sin u \\ &= - 8a \cos 2u\, (\cos 2u \cos u + \sin 2u \sin u) - 4a \cos u(\cos^2 2u + 1) \\ &= - 8a \cos 2u \cos u - 4a \cos u\, (\cos^2 2u + 1) \\ &= - 4a \cos u\, (1 + \cos 2u)^2 = - 16a \cos^5 u.\end{aligned}$$

Using this value of α and γ, we find β from the equation (6)

$$\begin{aligned}\beta \cos u &= - 16a \cos^5 u \sin u + 16a \sin u \cos u \cos 2u \\ &= 16a \sin u \cos u\, [\cos 2u - \cos^4 u] \text{ which gives} \\ \beta \cos u &= 16a \sin u \cos u[1 - 2 \sin^2 u - (1 - \sin^2 u)^2] \\ &= - 16a \sin^5 u \cos u \text{ so that } \beta = - 16a \sin^5 u\end{aligned}$$

Hence the coordinates of the centre of spherical curvature are

$$\mathbf{r} = (- 16a \cos^5 u, - 16a \sin^5 u, 5a \cos 2u).$$

Example 3. If the radius of spherical curvature is constant, prove that the curve either lies on a sphere or has a constant curvature.

The radius of spherical curvature is given by

$$R^2 = \rho^2 + (\sigma\rho')^2 \qquad ...(1)$$

Since R is constant, differentiating (1), we get

$$2\rho'\left[\rho + \sigma \frac{d}{ds}(\sigma\rho')\right] = 0 \qquad ...(2)$$

(2) shows that either $\rho' = 0$ or $\rho + \sigma \dfrac{d}{ds}(\sigma\rho') = 0$

If $\rho' = 0$, then ρ is constant. That is the curve has constant curvature.

Alternatively if $\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$, we shall prove that the curve lies on a sphere.

If a curve lies on a sphere, the osculating sphere at every point of the curve is the given sphere. So it is enough if we show that the osculating sphere is the same at every point of the curve. Since the radius of spherical curvature is constant, the osculating sphere has same radius at every point of the curve. So to complete the proof, we shall show that the centre of the osculating sphere is a fixed point given by a constant position vector. The position vector of the centre of spherical curvature is $\mathbf{C} = \mathbf{r} + \rho\mathbf{n} + \sigma\rho'\mathbf{b}$.

Differentiating with respect to s, we have

$$\frac{d\mathbf{C}}{ds} = \frac{d\mathbf{r}}{ds} + \rho'\mathbf{n} + \rho\frac{d\mathbf{n}}{ds} + \mathbf{b}\frac{d}{ds}(\sigma\rho') + \sigma\rho'\frac{d\mathbf{b}}{ds}$$

Using $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$, we have

$$\frac{d\mathbf{C}}{ds} = \mathbf{t} + \rho'\mathbf{n} + \rho\,(\tau\mathbf{b} - \kappa\mathbf{t}) + \mathbf{b}\frac{d}{ds}(\sigma\rho') + \sigma\rho'(-\,\tau\mathbf{n})$$

$$= \left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]\mathbf{b} = 0 \text{ by (2)}$$

Hence $\mathbf{C}$ is constant showing that the centre of the osculating sphere is independent of positions of points on the curve. Thus the osculating sphere at every point of the curve is the same sphere so that the curve lies on a sphere.

Example 4. Show that a necessary and sufficient condition that a curve lies on a sphere is

$$\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$$

at every point of the curve.

To prove the necessity of the condition, let us assume that the curve lies on a sphere. Then the sphere is the osculating sphere at every point of the curve so that the radius of the osculating sphere is constant.

The radius of the osculating sphere is

$$R^2 = \rho^2 + (\sigma\rho')^2 \qquad \text{...(1)}$$

Since R is a constant differentiating (1), we get

$$2\rho\rho' + 2(\sigma\rho')\,\frac{d}{ds}(\sigma\rho') = 0 \qquad \text{...(2)}$$

Since $\rho' \neq 0$, we get $\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$ proving the necessity of the condition.

Conversely let us assume that the condition is satisfied at every point of the curve. Multiplying the given condition throughout by ρ' and integrating with respect to s, we obtain $\rho^2 + (\sigma\rho')^2$ = constant which shows that under the given condition the radius of the osculating sphere is constant at every point of the curve. The centre **C** of the osculating sphere is

$$\mathbf{C} = \boldsymbol{\rho} + \rho\mathbf{n} + \sigma\rho'\mathbf{b}$$

Differentiating **C** and simplifying as in the previous example, we have $\frac{d\mathbf{C}}{ds} = \left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]\mathbf{b}$ which is zero by the given condition. Hence **C** is a constant vector which means that the centre of the osculating sphere is a fixed point Therefore the given curve must lie on a sphere. Hence the condition is sufficient.

1.12 LOCUS OF CENTRES OF SPHERICAL CURVATURE

Unless the curve lies on a sphere, the centres of spherical curvature change from point to point as the point moves on the curve. Hence it is but natural to study the locus of the centres of spherical curvature of the given curve. Let C be the given curve and C_1 be the locus of centres of spherical curvature. After finding the relation between moving triad (**t**, **n**, **b**) on C and the moving triad $(\mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1)$ on C_1, we express the curvature and torsion of C_1 in terms of those of C. We shall use the suffix 1 for the quantities pertaining to C_1 to distinguish them from the corresponding quantities of C.

Theorem 1. Let C be the given curve and C_1 be the locus of its centres of spherical curvature.

Then (i) $\mathbf{t}_1 = e\mathbf{b}$, $\mathbf{n}_1 = e_1\mathbf{n}$, $\mathbf{b}_1 = -ee_1\mathbf{t}$ where $e = e_1 = \pm 1$ and

(ii) The product of the torsions at the corresponding points is equal to the product of curvatures.

Proof. The position vector $\mathbf{r}_1$ of the centre of spherical curvature is given by

$$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \sigma\rho'\mathbf{b} \qquad ...(1)$$

Choosing the arc-length as the parameter and differentiating (1), we obtain

$$\frac{d\mathbf{r}_1}{ds} = \frac{d\mathbf{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \frac{d\mathbf{r}}{ds} + \rho'\mathbf{n} + \rho\mathbf{n}' + (\sigma'\rho'\mathbf{b} + \sigma\rho''\mathbf{b}) + \sigma\rho'\mathbf{b}'$$

So $$\frac{d\mathbf{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \mathbf{t} + \rho'\mathbf{n} + \rho(\tau\mathbf{b} - \kappa\mathbf{t}) + (\rho'\sigma' + \sigma\rho'')\mathbf{b} + \sigma\rho'(-\tau\mathbf{n})$$

Hence $\mathbf{t}_1 s'_1 = \left(\frac{\rho}{\sigma} + \rho'\sigma' + \sigma\rho''\right)\mathbf{b}$ so that $\mathbf{t}_1$ is parallel to **b**.

Now C_1 is parametrised by s and so s_1 is an increasing functions of s so that s'_1 is non-negative.

So if we take $\mathbf{t}_1 = e\mathbf{b}$ where $e = \pm 1$, ...(2)

then we have $s'_1 = e\left(\frac{\rho}{\sigma} + \rho'\sigma' + \sigma\rho''\right)$

Having found out $\mathbf{t}_1$, let us find out $\mathbf{n}_1$ and $\mathbf{b}_1$.

Differentiating (2) with respect to s,

$$\frac{d\mathbf{t}_1}{ds} = \frac{d\mathbf{t}_1}{ds_1}\frac{ds_1}{ds} = e\frac{d\mathbf{b}}{ds} = -e\tau\mathbf{n}$$

Since $\frac{d\mathbf{t}_1}{ds_1} = \kappa_1\mathbf{n}_1$, we have $\kappa_1\mathbf{n}_1 s'_1 = -e\tau\mathbf{n}$ so that $\mathbf{n}_1$ is parallel to $\mathbf{n}$.

So we can take $\mathbf{n}_1 = e_1\mathbf{n}$ ...(3)

where $e_1 = \pm 1$. Using this $\mathbf{n}_1$, $e_1\kappa_1 s'_1 = -e\tau$ ...(4)

Further $\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = e\mathbf{b} \times e_1\mathbf{n} = -ee_1\mathbf{t}$ ...(5)

Hence (2), (3) and (5) prove (i)

To prove (ii), let us find τ'_1, and κ'_1

$$\frac{d\mathbf{b}_1}{ds} = \frac{d\mathbf{b}_1}{ds_1}\cdot\frac{ds_1}{ds} = -ee_1\frac{d\mathbf{t}}{ds} = -ee_1\kappa\mathbf{n} \text{ so that } \tau_1\mathbf{n}_1 s'_1 = ee_1\kappa\mathbf{n}$$

Since $\mathbf{n}_1 = e_1\mathbf{n}$, we get $\tau_1 s'_1 = e\kappa$ ...(6)

From (4) and (6) $e\tau\tau_1 s'_1 = e\tau e\kappa = -e\kappa e_1\kappa_1 s_1'$ which gives $\tau\tau_1 = -e_1\kappa\kappa_1$.

Thus if $e_1 = -1$, we have $\tau\tau_1 = \kappa\kappa_1$ which proves (ii).

Note. If C is of constant curvature, then $\rho' = 0$. As we have already noted the centres of spherical curvature and curvature conicide. Using $\rho' = 0$,

We obtain $\frac{ds_1}{ds} = e\frac{\rho}{\sigma} = e\frac{\tau}{\kappa}$, $e = \pm 1$

From (4) of the theorem, $e_1\kappa_1 s'_1 = -e\tau$

Substituting for s'_1, we get $e_1\kappa_1 = -\kappa$ if $e = -1$

Choosing $e_1 = -1$, we find $\kappa_1 = \kappa$

Also from (6) of the Theorem $\tau_1 s'_1 = e\kappa$

Substituting for s'_1, we find $\tau_1 = \frac{\kappa^2}{\tau}$.

Note. If we measure the arc-length s_1 of C_1 in that direction which makes its unit tangent $\mathbf{t}_1$ have the same direction as $\mathbf{b}$, then $\mathbf{t}_1 = \mathbf{b}$. We may chose the direction of $\mathbf{n}_1$ opposite to $\mathbf{n}$ so that $\mathbf{n}_1 = -\mathbf{n}$. With this choice, we have $\mathbf{b}_1 = \mathbf{t}$. These are the particular cases of the above Theorem.

Theorem 2. The radius of curvature ρ_1 of the locus of the centres of curvature is

$$\rho_1 = \left[\left\{\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{R}\right\}^2 + \frac{\rho'^2\sigma^4}{\rho^2 R^4}\right]^{-\frac{1}{2}}$$

where R is the radius of spherical curvature.

Proof. We shall use suffix unity for the quantities related to the locus of the centres of curvature C_1.

By Theorem 1 of 1.11, the position vector $\mathbf{r}_1$ of the centre of curvature is

$$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} \qquad \text{...(1)}$$

Using the arc-length s as parameter, let us differentiate (1) and find s'_1

$$\frac{d\mathbf{r}_1}{ds} = \frac{d\mathbf{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \frac{d\mathbf{r}}{ds} + \rho'\mathbf{n} + \rho\frac{d\mathbf{n}}{ds}$$

$$\mathbf{t}_1 s'_1 = \mathbf{t} + \rho'\mathbf{n} + \rho(\tau\mathbf{b} - \kappa\mathbf{t}) \text{ which gives}$$

$$\frac{\sigma}{\rho}\mathbf{t}_1 s'_1 = \frac{\sigma}{\rho}\rho'\mathbf{n} + \mathbf{b} \qquad \text{...(2)}$$

Taking dot product of (2) with itself, we get

$$\frac{\sigma^2}{\rho^2}s'^2_1 = \frac{\sigma^2}{\rho^2}\rho'^2 + 1 = \frac{\sigma^2\rho'^2 + \rho^2}{\rho^2} = \frac{R^2}{\rho^2}$$

where R is the radius of spherical curvature.

Thus we have from the above step, $s'_1 = \dfrac{R}{\sigma}$ ...(3)

Differentiating (2) with respect to s again, we get

$$\frac{\sigma}{\rho}s'_1\frac{d\mathbf{t}_1}{ds_1}\cdot\frac{ds_1}{ds} + \mathbf{t}_1\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right) = \frac{\sigma}{\rho}\rho'\frac{d\mathbf{n}}{ds} + \frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\mathbf{n} + \frac{d\mathbf{b}}{ds}$$

$$\frac{\sigma}{\rho}s'^2_1\kappa_1\mathbf{n}_1 + \mathbf{t}_1\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right) = \frac{\sigma}{\rho}\rho'[\tau\mathbf{b} - \kappa\mathbf{t}] + \frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\mathbf{n} - \tau\mathbf{n}$$

Thus we obtain,

$$\frac{\sigma}{\rho}s'^2_1\kappa_1\mathbf{n}_1 + \mathbf{t}_1\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right) = -\frac{\sigma}{\rho^2}\rho'\mathbf{t} + \left[\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right) - \frac{1}{\sigma}\right]\mathbf{n} + \frac{\rho'}{\rho}\mathbf{b} \qquad \text{...(4)}$$

Taking the cross product of (2) and (4), the left side of the resulting equation is

$$\frac{\sigma^2}{\rho^2}s'^3_1\kappa_1\mathbf{t}_1 \times \mathbf{n}_1 = \kappa_1\frac{\sigma^2}{\rho^2}s'^3_1\mathbf{b}_1$$

and the right hand side is

$$\left[\frac{\rho'^2\sigma^2+\rho^2}{\rho^2\sigma}-\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\right]\mathbf{t}-\frac{\sigma}{\rho^2}\rho'\mathbf{n}+\frac{\sigma^2\rho'^2}{\rho^3}\mathbf{b}$$

Hence we get

$$\kappa_1\frac{\sigma^2}{\rho^2}s_1'^3\mathbf{b}_1=\left[\frac{\rho'^2\sigma^2+\rho^2}{\rho^2\sigma}-\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\right]\mathbf{t}-\frac{\sigma}{\rho^2}\rho'\mathbf{n}+\frac{\sigma^2\rho'^2}{\rho^3}\mathbf{b}.$$

Taking dot product on both sides with itself,

$$\kappa_1^2\frac{\sigma^4}{\rho^4}s_1'^6=\Bigg[\frac{\rho'^2\sigma^2+\rho^2}{\rho^2\sigma}-\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\Bigg]^2+\frac{\sigma^2}{\rho^4}\rho'^2+\frac{\sigma^4\rho'^4}{\rho^6}\Bigg]$$

Substituting for s'_1 from (3), we get

$$\kappa_1^2\frac{\sigma^4}{\rho^4}\cdot\frac{R^6}{\sigma^6}=\Bigg[\Bigg[\Bigg\{\frac{R^2}{\rho^2\sigma}-\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\Bigg\}^2+\frac{\rho'^2\sigma^2}{\rho^6}\cdot R^2\Bigg] \text{ giving}$$

$$\kappa_1^2=\frac{\rho^4\sigma^2}{R^6}\Bigg[\Bigg\{\frac{R^2}{\rho^2\sigma}-\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\Bigg\}^2+\frac{\rho'^2\sigma^2}{\rho^6}R^2\Bigg]$$

Thus

$$\kappa_1=\Bigg[\Bigg[\Bigg\{\frac{1}{R}-\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma}{\rho}\rho'\right)\Bigg\}^2+\frac{\sigma^4\rho'^2}{\rho^2R^4}\Bigg]^{\frac{1}{2}}$$

Since $\rho_1=\dfrac{1}{\kappa_1}$, we get the formula for κ_1.

Using some of the steps in the theorem, we shall find torsion τ_1, in the following corollary.

Corollary. $$\tau_1=\frac{\rho\sigma}{R^2}\left[\frac{1}{\rho^2}-\left\{\frac{d}{ds}\left(\frac{\mathrm{R}}{\rho}\right)\right\}^2\right]^{1/2}$$

We have from (2) of the theorem

$$\frac{\sigma}{\rho}\mathbf{t}_1s'_1=\frac{\sigma}{\rho}\rho'\mathbf{n}+\mathbf{b}$$

Taking cross product with $\mathbf{n}_1$ on both sides of the above step,

$$\frac{\sigma}{\rho}\mathbf{t}_1\times\mathbf{n}_1s'_1=\frac{\sigma}{\rho}\rho'\mathbf{n}\times\mathbf{n}_1+\mathbf{b}\times\mathbf{n}_1$$

But we know that $\mathbf{n}_1 = -\mathbf{n}$ so that we have $\frac{\sigma}{\rho}\mathbf{b}_1 s'_1 = \mathbf{t}$

Differentiating the above relation with respect to s,

$$\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right)\mathbf{b}_1 + \frac{\sigma}{\rho}s'_1\frac{d\mathbf{b}_1}{ds_1}\cdot\frac{ds_1}{ds} = \frac{d\mathbf{t}}{ds}$$

That is
$$\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right)\mathbf{b}_1 - \frac{\sigma}{\rho}s'^{2}_1\tau_1\mathbf{n}_1 = \kappa\mathbf{n}$$

Taking the dot product of the above equation with itself on both sides, we obtain

$$\left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right)\right\}^2 + \left(\frac{\sigma}{\rho}\right)^2 s'^{4}_1\tau_1^2 = \kappa^2$$

Let us use $s'_1 = \frac{R}{\sigma}$ and find τ_1.

$$\tau_1^2 = \frac{\rho^2}{\sigma^2}\frac{1}{s'^{4}_1}\left[\frac{1}{\rho^2} - \left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}s'_1\right)\right\}^2\right]$$

$$= \frac{\sigma^2\rho^2}{R^4}\left[\frac{1}{\rho^2} - \left\{\frac{d}{ds}\cdot\frac{R}{\rho}\right\}^2\right]$$

Thus we get
$$\tau_1 = \frac{\rho\sigma}{R^2}\left[\frac{1}{\rho^2} - \left\{\frac{d}{ds}\left(\frac{R}{\rho}\right)\right\}^2\right]^{\frac{1}{2}}$$

Example 1. If R is the radius of spherical curvature, show that $R = \frac{|\mathbf{t}\times\mathbf{t}''|}{\kappa^2\tau}$

We shall find $\mathbf{t}\times\mathbf{t}''$. Since $\mathbf{t}' = \kappa\mathbf{n}_1 = \frac{\mathbf{n}}{\rho}$, differentiating this relation, we get

$$\mathbf{t}'' = \frac{1}{\rho}\mathbf{n}' - \frac{\rho'}{\rho^2}\mathbf{n} = \frac{1}{\rho}(\tau\mathbf{b} - \kappa\mathbf{t}) - \frac{\rho'}{\rho^2}\mathbf{n}$$

$$= -\frac{1}{\rho^2}\mathbf{t} - \frac{\rho'}{\rho^2}\mathbf{n} + \frac{1}{\rho\sigma}\mathbf{b}$$

Hence
$$\mathbf{t}\times\mathbf{t}'' = -\frac{\rho'}{\rho^2}\mathbf{b} - \frac{1}{\rho\sigma}\mathbf{n}$$

Hence [illegible]

Since the radius of spherical curvature [illegible] we [illegible]

[illegible] which gives [illegible]

Example 2 Show that the tangent to the locus of the centres of the osculating spheres passes through the centre of the osculating circle.

Let C_1 be the locus of the centres of spherical curvature. Then the position vector $\mathbf{r}_1$ of any point P on C_1 is

$$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + (\sigma\rho')\mathbf{b}$$

Hence the unit tangent vector at P on C_1 is

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d\mathbf{r}_1}{ds}\frac{ds}{ds_1} = [\mathbf{t} + \rho'\mathbf{n} + \rho(\tau\mathbf{b} - \kappa\mathbf{t}) + \frac{d}{ds}(\sigma\rho')\mathbf{b} + (\sigma\rho')(-\tau\mathbf{n})]\frac{ds}{ds_1}$$

Simplifying, we have $\mathbf{t}_1 = \frac{ds}{ds_1}\left(\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right)\mathbf{b}$

The equation of the tangent at P to C_1 is given by

$$\mathbf{R} - \mathbf{r}_1 = k\mathbf{r}'_1 = k\mathbf{t}_1$$

where k is a scalar. Using (1) and (2) in (3), we get

$$\mathbf{R} - \mathbf{r} - \rho\mathbf{n} - (\sigma\rho')\mathbf{b} = k\frac{ds}{ds_1}\left(\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right)\mathbf{b}$$

Since k is an arbitrary constant, we can choose

$$k = -\frac{\sigma\rho'}{\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')}\frac{ds_1}{ds}$$

with this choice of k, (4) becomes $\mathbf{R} - \mathbf{r} = \rho\mathbf{n}$ which is the position vector of the centre of the osculating circle. Thus the tangent to the locus of the centre of spherical curvature passes through the centre of the osculating circle.

1.13 TANGENT SURFACES, INVOLUTES AND EVOLUTES

With the help of contact of curves with surfaces, we introduced osculating and osculating sphere. As the point moves on the curve, the centres of osculat circle and osculating sphere trace out curves namely the locus of cen curvature and the locus of the centre of spherical curvature. The two loc results of point property of the curves. Now instead of taking points on the cur let us consider tangents at different points of the curve. These tangents will generate a surface and we consider curves on this surface. These notions lead to the definitions of involutes and consequently evolutes of a given curve.

To start with, it is worthwhile to point out the basic differences between the evolutes of a plane curve and those of a space curve.

(i) The evolute of a plane curve is unique but the space curve has infinitely many evolutes.

(ii) Evolute of a plane curve is defined as the locus of the centres of curvature but we will show that the evolute of a space curve is neither the locus of the centres of curvature nor the locus of the centres of spherical curvature.

However the concept of involute of a plane curve has natural generalisation to the space curves. Once we obtain the involute $\tilde{C}$ of a curve C, we define C to be the evolute of $\tilde{C}$.

Definition 1. The surface generated by the tangent lines to the given curve C is called the tangent surface to C.

Using this definition, let us find the position vector of a point P on the tangent surface.

Let A be any point on the curve at an arcual distance s from a fixed point O on C. Since P is a point on the tangent surface, AP is tangent to C. If u is the distance of P from A, then the position vector of P on the tangent surface is $\mathbf{R} = \mathbf{r}(s) + u\mathbf{t}(s)$.

Since $\mathbf{R}$ is a function of two parameters u and s, we denote the position vector by $\mathbf{R}(s, u)$ so that we can write $\mathbf{R}(s, u) = \mathbf{r}(s) + u\mathbf{t}(s)$. ...(1)

Since (1) is a function of two parameter, it represents a surface.

If we assume any relation of the type $u = \lambda(s)$ as the point moves on the curve, (1) represents a single parameter family so that it represents a curve on the tangent surface of C. Hence any curve on the tangnet surface has the positions vector

$$\mathbf{R} = \mathbf{r}(s) + \lambda(s)\,\mathbf{t}(s) \qquad ...(2)$$

We take the class of the curve (2) to be the smaller of the class of C or λ.

Definition 2. A curve which lies on the tangent surface of C and intersects the generators of the tangent surface orthogonally is called the involute of C denoted by $\tilde{C}$.

From the definition it follows that the tangents of C are normal to $\tilde{C}$. This means that the tangent to C at a point P is orthogonal to the tangent at the corresponding point of $\tilde{C}$. The following theorem gives the equation of an involute. We use the suffix 1 for the quantities pertaining to $\tilde{C}$.

Theorem 1. If $\mathbf{r}_1$ is the position vector of a point P_1 on the involute $\tilde{C}$ of C, then $\mathbf{r}_1 = \mathbf{r} + (c - s)\mathbf{t}$ where c is an arbitrary constant and $\mathbf{r}$ is the position vector of P on C.

Proof. Since the involute lies on the tangent surface, the position vector $\mathbf{r}_1$ of a point P_1 on the involute (Fig. 3) is $\mathbf{r}_1 = \mathbf{r} + \lambda(s)\mathbf{t}$ ...(1)

Using the definition of the involute, we shall find $\lambda(s)$ in the following manner.

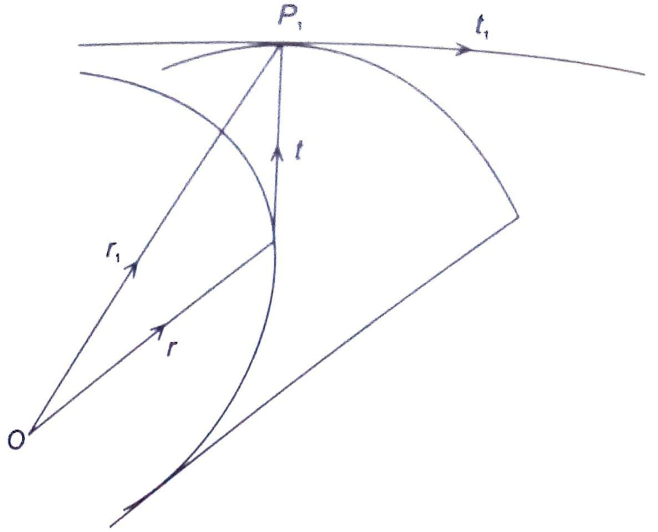

Fig. 3

Diffrentiating (1) with respect to s, we have

$$\frac{d\mathbf{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \frac{d\mathbf{r}}{ds} + \lambda'(s)\mathbf{t} + \lambda(s)\mathbf{t}'$$

That is $$\mathbf{t}_1\frac{ds_1}{ds} = \mathbf{t} + \lambda'(s)\mathbf{t} + \lambda(s)\kappa\mathbf{n} \qquad \text{...(2)}$$

Since the tangent to the involute cuts the generators orthogonally $\mathbf{t}\cdot\mathbf{t}_1 = 0$. Using this and taking dot product with $\mathbf{t}$ on both sides of (2) we get

$$1 + \lambda'(s) = 0 \quad \text{or} \quad \frac{d\lambda}{ds} = -1 \qquad \text{...(4)}$$

Integrating (4) with respect to s, we find $\lambda = (c - s)$

Hence the equation of the involute is $\mathbf{r}_1 = \mathbf{t} + (c - s)\,\mathbf{t}$ ...(5)

Since c is an arbitrary constant, the equation (5) shows that for a given curve C, there is an infinite system of involutes of C. For different choices of c, we get different involutes of the system.

Corollary 1. The distance between corresponding points of two involutes is constant.

Let P be a fixed point on C. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of the corresponding points on two involutes determined by $c = c_1$ and $c = c_2$.

Then

Then

$$\mathbf{r}_1 = \mathbf{r} + (c_1 - s)\,\mathbf{t},\ \mathbf{r}_2 = \mathbf{r} + (c_2 - s)\mathbf{t}$$

$$\mathbf{r}_1 - \mathbf{r}_2 = (c_1 - c_2)\,\mathbf{t} \text{ so that } |\mathbf{r}_1 - \mathbf{r}_2| = |c_1 - c_2| \text{ which is constant.}$$

Thus the length between two such corresponding points is constant.

Corollary 2. Since $\mathbf{t}$ is the same for different involutes, the tangents at the corresponding points of the involutes are parallel.

1.17 FUNDAMENTAL EXISTENCE THEOREM FOR SPACE CURVES

As illustrated in the example of previous section, given the curve, we can determine the intrinsic equations by finding curvature and torsion as functions of s. Therefore the question naturally arises whether the converse is true. That is, given curvature and torsion as functions of arc-length s, can we determine the curve under suitable conditions on $\kappa(s)$ and $\tau(s)$? To answer this question, we impose some conditions on $\kappa(s)$ and $\tau(s)$ and then we determine not only position of the curve in space but also show that all the curves congruent to it will have the same intrinsic equation. The following theorem known as the fundamental existence theorem for space curves asserts the existence and uniqueness of space curves interms of intrinsic equations, when the scalar function $\kappa(s)$ and $\tau(s)$ defining curvature and torsion are continuous functions.

Theorem. If $\kappa(s)$ and $\tau(s)$ are continuous functions for all non-negative real values of s, then there exists a unique space curve determined but for position in space for which κ the is curvature and τ is the torsion and s is the arc length measured from suitable base point.

Proof. The method of proof is to construct the position vector $\mathbf{r}(s)$ at any point on the curve and the moving triad $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$ with the help of the given intrinsic equations.

Now consider the following three simultaneous differential equations of first order in α, β and γ.

$$\frac{d\alpha}{ds} = \kappa\beta, \quad \frac{d\beta}{ds} = \tau\gamma - \kappa\alpha, \quad \frac{d\gamma}{ds} = -\tau\beta \qquad \ldots(1)$$

where α, β, γ are unknown functions of s and κ, τ are given functions $\kappa(s)$ and $\tau(s)$. Since $\kappa(s)$ and $\tau(s)$ are continuous functions, the above set of differential equations has a unique set of continuous solutions (α, β, γ) with prescribed initial conditions at $s = 0$ as guaranteed by the theorem on existence and uniqueness of solutions of a set of differential equations of first order.

The whole technique of proof is to use the three solutions $(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, 3$ with three different initial conditions and identify the vectors $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$ constructed on the basis of solutions with $(\mathbf{t}, \mathbf{n}, \mathbf{b})$.

The set of equations (1) admits a unique set of solutions which assume the prescribed value $(\alpha_0, \beta_0, \gamma_0)$ at $s = 0$.

Let $(\alpha_1, \beta_1, \gamma_1)$ be one such solution taking the prescribed value

$$\alpha_1(0) = 1, \beta_1(0) = 0, \text{ and } \gamma_1(0) = 0 \qquad \ldots(2)$$

In a similar manner, we can find two more solutions $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ having the prescribed conditions.

$$\alpha_2(0) = 0, \beta_2(0) = 1, \gamma_2(0) = 0 \qquad \ldots(3)$$

$$\alpha_3(0) = 0, \beta_3(0) = 0 \; \gamma_3(0) = 1 \quad \text{at } s = 0.$$

The following four steps establish the theorem.

Step 1. We shall establish the following properties of the above three solutions.

$$\begin{aligned} \alpha_1^2+\beta_1^2+\gamma_1^2 &= 1 & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 &= 0 \\ \alpha_2^2+\beta_2^2+\gamma_2^2 &= 1 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 &= 0 \\ \alpha_3^2+\beta_3^2+\gamma_3^2 &= 1 & \alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1 &= 0 \end{aligned} \qquad \ldots(4)$$

for all values of s. These properties will enable us to introduce an orthogonal matrix for finding $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ to define $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$.

$$\frac{d}{ds}(\alpha_1^2+\beta_1^2+\gamma_1^2) = 2\alpha_1\frac{d\alpha_1}{ds} + 2\beta_1\frac{d\beta_1}{ds} + 2\gamma_1\frac{d\gamma_1}{ds}$$

Since $(\alpha_1, \beta_1, \gamma_1)$ are solutions of (1), we get

$$\frac{d}{ds}(\alpha_1^2+\beta_1^2+\gamma_1^2) = 2\alpha_1\kappa\beta_1 + 2\beta_1(\tau\gamma_1 - \kappa\alpha_1) + 2\gamma_1(-\tau\beta_1) = 0$$

Hence $\alpha_1^2+\beta_1^2+\gamma_1^2 =$ constant c (say).

Using the initial conditions (2), we get $c = 1$ and so we have $\alpha_1^2+\beta_1^2+\gamma_1^2 = 1$.

In a similar manner, we can prove $\alpha_2^2+\beta_2^2+\gamma_2^2 = 1$, $\alpha_3^2+\beta_3^2+\gamma_3^2 = 1$.

To prove the other relations in (4), let us consider

$$\frac{d}{ds}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)$$

$$= \alpha_1\frac{d\alpha_2}{ds} + \alpha_2\frac{d\alpha_1}{ds} + \frac{d\beta_1}{ds}\beta_2 + \beta_1\frac{d\beta_2}{ds} + \frac{d\gamma_1}{ds}\gamma_2 + \gamma_1\frac{d\gamma_2}{ds}$$

Since $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are solutions of (1) substituting for the derivatives from (1)

$$= \alpha_1(\kappa\beta_2) + \alpha_2(\kappa\beta_1) + \beta_1[\tau\gamma_2 - \kappa\alpha_2] + \beta_2[\tau\gamma_1 - \kappa\alpha_1] + \gamma_2[-\tau\beta_1] + \gamma_1[-\tau\beta_2] = 0$$

This proves that $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 =$ constant d (say). Using the initial condition (3), $d = 0$ so that

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 = 0$$

In a similar manner, we get

$$\alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 = 0$$

$$\alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1 = 0$$

Step 2. We prove that $\mathbf{t} = (\alpha_1, \alpha_2, \alpha_3)$, $\mathbf{n} = (\beta_1, \beta_2, \beta_3)$ and $\mathbf{b} = (\gamma_1, \gamma_2, \gamma_3)$ are three mutually orthogonal unit vectors.

Now consider the matrix

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

The six relations proved in the first step show that the matrix A is an orthogonal matrix. This implies that if A' is the transpose of A, then $AA' = I$ where I is the identity matrix. Now the equation $AA' = I$ implies $A' = A^{-1}$ so that we have $A'A = I$. ...(5)

Now the matrix equation (5) is

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 & \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 & \beta_1^2 + \beta_2^2 + \beta_3^2 & \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 \\ \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 & \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 & \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which gives

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \qquad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$$
$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1, \qquad \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \qquad \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0$$

The above six relations show that these are three mutually orthogonal unit vectors

$\mathbf{t} = (\alpha_1, \alpha_2, \alpha_3)$, $\mathbf{n} = (\beta_1, \beta_2, \beta_3)$ and $\mathbf{b} = (\gamma_1, \gamma_2, \gamma_3)$ for each value of s.

Step 3. We find the position vector of a point on the curve.

Let us define the curve $\mathbf{r}(s) = \int_o^s \alpha(s)\, ds$ ...(6)

Differentiating with respect to s, $\frac{d\mathbf{r}}{ds} = \mathbf{t} = \alpha(s)$.

This shows that the arc-length s and the unit tangent vector to the curve (6) are s and $\mathbf{t} = \alpha(s)$.

Further $\frac{d\mathbf{t}}{ds} = \frac{d\alpha}{ds} = \kappa(s)\beta$ from (1)

Since $\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$, the unit normal vector $\mathbf{n}$ is parallel to the unit vector β and $\kappa = \kappa(s)$. If we take the sense of $\mathbf{n}$ as that of β, we get $\mathbf{n} = \beta$.

Since $\mathbf{b} = \mathbf{t} \times \mathbf{n} = \alpha \times \beta = \gamma$, we have $\mathbf{b} = \gamma$. So $\dfrac{d\mathbf{b}}{ds} = \dfrac{d\gamma}{ds} = -\tau(s)\,\beta$ from (1).

Since $\dfrac{d\mathbf{b}}{ds} = -\tau\mathbf{n}$, we have $\tau = \tau(s)$.

Hence if $\mathbf{r} = \int_{s}^{s} \mathbf{t}\, ds$, then $\mathbf{r}(s)$ is the position vector of a point on the curve with the arc length s as parameter and having $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ as the moving unit triad and $\kappa = \kappa(s)$ and $\tau = \tau(s)$ as curvature and torsion.

Step 4. We shall establish the uniqueness of the curve. To this end, we shall show that if the curves have same intrinsic equations, then they are congruent.

Let C and C_1 be two curves defined in terms of their respective arc-lengths having equal curvature and equal torsion for the same value of s.

Let A and A_1 be two points of C and C_1 corresponding to $s = 0$. Let C_1 be moved so that the points A and A_1 coincide. If $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and $(\mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1)$ refer to the curve C and C_1, then C_1 be suitably oriented so that $(\mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1)$ at A_1 coincide with $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ at A when $s = 0$. Since different points of the curves are determined by the same values of s and since both the curves have the same curvature and torsion, we get

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}_1) = \mathbf{t}\cdot\frac{d\mathbf{t}_1}{ds} + \frac{d\mathbf{t}}{ds}\cdot\mathbf{t}_1 = \mathbf{t}\cdot(\kappa\mathbf{n}_1) + \kappa\mathbf{n}\cdot\mathbf{t}_1$$

$$\frac{d}{ds}(\mathbf{n}\cdot\mathbf{n}_1) = \mathbf{n}\cdot\frac{d\mathbf{n}_1}{ds} + \frac{d\mathbf{n}}{ds}\cdot\mathbf{n}_1 = \mathbf{n}\cdot[\tau\mathbf{b}_1 - \kappa\mathbf{t}_1] + [\tau\mathbf{b} - \kappa\mathbf{t}]\cdot\mathbf{n}_1$$

$$\frac{d}{ds}(\mathbf{b}\cdot\mathbf{b}_1) = \mathbf{b}\cdot\frac{d\mathbf{b}_1}{ds} + \frac{d\mathbf{b}}{ds}\cdot\mathbf{b}_1 = \mathbf{b}\cdot(-\tau\mathbf{n}_1) + (-\tau\mathbf{n})\cdot\mathbf{b}_1$$

Since the sum of the terms of the right hand side of the above equations is zero.

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}_1 + \mathbf{n}\cdot\mathbf{n}_1 + \mathbf{b}\cdot\mathbf{b}_1) = 0$$

Integrating the above equation, we obtain

$\mathbf{t}\cdot\mathbf{t}_1 + \mathbf{n}\cdot\mathbf{n}_1 + \mathbf{b}\cdot\mathbf{b}_1 = c$ where c is a constant.

As $s = 0$, $\mathbf{t} = \mathbf{t}_1$, $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{b} = \mathbf{b}_1$ so that $c = 3$

Thus we obtain $\mathbf{t}\cdot\mathbf{t}_1 + \mathbf{n}\cdot\mathbf{n}_1 + \mathbf{b}\cdot\mathbf{b}_1 = 3$

Since the dot product of two unit vectors gives cosine, the above equation gives the sum of three cosines. But the sum of the three cosines is equal to 3 only when each angle is zero. This implies that at all pairs of corresponding points $\mathbf{t} = \mathbf{t}$ $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{b} = \mathbf{b}_1$.

Further $\mathbf{t} = \mathbf{t}_1$ gives $\dfrac{d}{ds}(\mathbf{r} - \mathbf{r}_1) = 0$ which gives $\mathbf{r} - \mathbf{r}_1 =$ a constant d (say). A $s = 0$, $\mathbf{r} = \mathbf{r}_1$ so that $d = 0$. Hence $\mathbf{r} = \mathbf{r}_1$ identically. Thus the two curves C and C coincide so that the curve is uniquely determined except as to its position in spa This completes the proof of the theorem.

Hence $(\alpha_1, \beta_1, \gamma_1) = (\cos\sigma(s), -\sin\sigma(s), 0)$

(ii) When $\alpha(0) = 0, \beta(0) = 1, \gamma(0) = 0$, we get

$$A_2 = 0, B_2 = 1, C_2 = 0$$

Hence $(\alpha_2, \beta_2, \gamma_2) = (\sin\sigma(s), \cos\sigma(s), 0)$

(iii) When $\alpha(0) = 0, \beta(0) = 0, \gamma(0) = 1$,

$$A_3 = 0, B_3 = 0, C_3 = 1$$

Hence $(\alpha_3, \beta_3, \gamma_3) = (0, 0, 1)$

Using the above solutions $(\alpha_i, \beta_i, \gamma_i), i = 1, 2, 3$ we can define $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ at any point of the curve.

$$\mathbf{t} = (\alpha_1, \alpha_2, \alpha_3) = (\cos\sigma(s), \sin\sigma(s), 0)$$

$$\mathbf{n} = (\beta_1, \beta_2, \beta_3) = (-\sin\sigma(s), \cos\sigma(s), 0)$$

$$\mathbf{b} = (\gamma_1, \gamma_2, \gamma_3) = (0, 0, 1) \text{ where } \sigma(s) = \sqrt{\frac{2}{a}}\, s^{\frac{1}{2}}$$

and

$$\mathbf{r}(s) = \int_o^s \mathbf{t}\, d\sigma(s) = \int_o^s \left(\cos\sqrt{\frac{2s}{a}}, \sin\sqrt{\frac{2s}{a}}, 0\right)\frac{1}{\sqrt{2as}}\, ds$$

We can easily check the orthogonality conditions of the theorem.

Note. The solutions of the equations involving α, β, γ become difficult in most of the cases, in particular for the curves of class ≥ 3. If $\kappa(s)$ and $\tau(s)$ are of class ≥ 3, eliminating β, γ, we obtain a third order equation in α with variable coefficients. In some of the simple cases we obtain solution of the equation by a change of variable as in Example 2. But such a third order equation can be reduced to a first order Reccati equation whose solutions are well-studied. Solving this Reccati equation of first order, we obtain its solution.

1.18 HELICES

We conclude this chapter with a brief discussion of the properties of a wide class of space curves known as helices which we used as examples in the previous sections.

Definition 1. A space curve lying on a cylinder and cutting the generators of the cylinder at a constant angle is called a cylindrical helix.

The above definition implies that the tangent to the curve makes a constant angle α with a fixed line known as the axis of the helix.

We can obtain more general helices than cylindrical helices if the cylinder is replaced by other surfaces like cone. But we consider only cylindrical helices in our study. The following theorem characterises the cylindrical helices.

Theorem 1. A necessary and sufficient condition for a curve to be helix is that the ratio of the curvature to torsion is constant at all points.

Proof. To prove the necessity of the condition, let **a** be the unit vector in the direction of the axis. Since the helix cuts the generators at a constant angle, let angle between the generator and the tangent at any point P on the helix be α. So from the definition of the helix, we have $\mathbf{t}\cdot\mathbf{a} = \cos\alpha$ ...(1)

Differentiating (1) with respect to s, we get

$$\mathbf{t}'\cdot\mathbf{a} + \mathbf{t}\cdot\mathbf{a}' = 0 \qquad ...(2)$$

Since $\mathbf{a}$ is a constant vector and $\mathbf{t}' = \kappa\mathbf{n}$, we get from (2)

$$\kappa\mathbf{n}\cdot\mathbf{a} = 0 \qquad ...(3)$$

If $\kappa = 0$, the curve is a straight line and the conclusion of the theorem is obvious. As we have excluded the case when $\kappa = 0$, (3) gives $\mathbf{n}\cdot\mathbf{a} = 0$ showing that $\mathbf{a}$ is perpendicular to the normal at P. Since $\mathbf{a}$ passes through P making a constant angle α with the tangent $\mathbf{t}$ at P and perpendicular to the normal at P, it lies in the rectifying plane at P. Hence ($\cos\alpha$, $\sin\alpha$) are the components of $\mathbf{a}$ in the rectifying plane so that we can take $\mathbf{a} = \mathbf{t}\cos\alpha + \mathbf{b}\sin\alpha$ ...(4)

Differentiating (4) with respect to s and using $\mathbf{a}' = 0$,

$$(\mathbf{t}'\cos\alpha + \mathbf{b}'\sin\alpha) = (\kappa\cos\alpha - \tau\sin\alpha)\mathbf{n} = 0$$

Since $\mathbf{n} \neq 0$, we have $\kappa\cos\alpha - \tau\sin\alpha = 0$ giving $\dfrac{\kappa}{\tau} = \tan\alpha$ which is constant, proving the necessity of the condition.

To prove the converse, let us assume $\dfrac{\kappa}{\tau} =$ constant λ (say) and prove that the curve is a helix.

Given any constant λ, we can always find the smallest angle α such that $\tan\alpha = \lambda$. So we can take $\dfrac{\kappa}{\tau} = \tan\alpha$ giving $(\kappa\cos\alpha - \tau\sin\alpha) = 0$ ...(5)

Since $\mathbf{n} \neq 0$, (5) implies $\mathbf{n}(\kappa\cos\alpha - \tau\sin\alpha) = 0$ ...(6)

(6) can be rewritten as $\dfrac{d}{ds}(\mathbf{t}\cos\alpha + \mathbf{b}\sin\alpha) = 0$

This proves that $\mathbf{t}\cos\alpha + \mathbf{b}\sin\alpha$ is a constant vector $\mathbf{a}$ (say). Then $\mathbf{a}\cdot\mathbf{t} = (\mathbf{t}\cos\alpha + \mathbf{b}\sin\alpha)\cdot\mathbf{t} = \cos\alpha$ which proves that the curve is a helix.

In the above definition of the cylindrical helix, we have not specified the base curve which is the cross section of the cylinder by a horizontal plane. However if we take the circle to be the base curve, we get a helix on a circular cylinder. Such helices are called circular helices. With proper choice of the coordinate axes, we shall find the equations of a circular helix.

Theorem 2. If the z-axis is the axis of the cylinder as well as that of the helix, the parametric equation of the helix is of the form

$$x = a\cos u,\ y = a\sin u,\ z = bu$$

where the base circle is $x^2 + y^2 = a^2$, $z = 0$ and b is a suitably chosen constant.

Proof. Let P be any point on the helix with the position vector $\mathbf{r}$ and P_1 be its projection on the XOY plane with the position vector $\mathbf{r}_1$. Let $\mathbf{a}$ be the unit vector in the direction of the axis of the helix. By our choice of the axis of the helix, $\mathbf{a}$ is

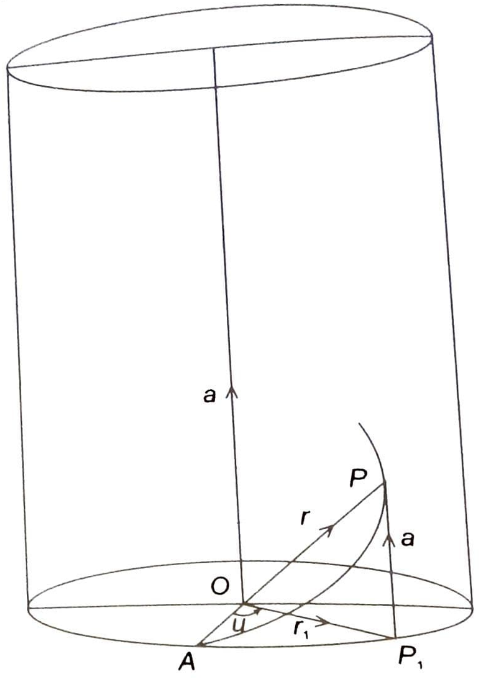

Fig. 6

parallel to the z-axis. So P_1P is parallel to **a** and hence $P_1P = \mathbf{r}\cdot\mathbf{a}$. Hence we tak $\overrightarrow{P_1P} = (\mathbf{r}\cdot\mathbf{a})\mathbf{a}$. Using this we write the position vector of P_1 as

$$\mathbf{r}_1 = \mathbf{r} - (\mathbf{a}\cdot\mathbf{r})\mathbf{a}$$

Differentiating (1) with respect to s, we get

$$\frac{d\mathbf{r}_1}{ds} = \mathbf{t} - (\mathbf{a}\cdot\mathbf{t})\mathbf{a} = \mathbf{t} - \mathbf{a}\cos\alpha$$

Hence taking dot product of (2) with itself,

$$\frac{d\mathbf{r}_1}{ds}\cdot\frac{d\mathbf{r}_1}{ds} = (\mathbf{t} - \mathbf{a}\cos\alpha)\cdot(\mathbf{t} - \mathbf{a}\cos\alpha) = \sin^2\alpha$$

If s_1 is the arc length of the projection of the helix on the XOY pla $d\mathbf{r}_1\cdot d\mathbf{r}_1 = d\mathbf{r}_1^2$. Using this in (3), we get $ds_1 = \sin\alpha\, ds$ which implies $s_1 = \sin\alpha\, s$.

Since the helix is a circular helix $\mathbf{t}\cdot\mathbf{k} = \cos\alpha$, so that

$$\mathbf{t}\cdot\mathbf{k} = \frac{d\mathbf{r}}{ds}\cdot\mathbf{k} = \frac{d}{ds}(\mathbf{r}\cdot\mathbf{k}) = \frac{dz}{ds} = \cos\alpha, \text{ since } \mathbf{r}\cdot\mathbf{k} = z$$

Hence from the above step, $z = s\cos\alpha$

Using (4) in (5), we obtain $z = s_1\cot\alpha$

From the Fig. 6, $AP_1 = s_1 = au$ so that $z = au\cot\alpha$.

If (x, y, z) are the coordinates of P, then we have from Fig. 6,

$$x = a\cos u,\ y = a\sin u,\ z = bu \text{ where } b = a\cot\alpha$$