Chapter 3

HILBERT-SCHMIDT THEORY : SYMMETRIC KERNELS

OUTLINE

- Square Integrable Function
- Hilbert Space
- Orthogonal System of Functions
- Gram-Schmidt Orthonormalization Process
- Symmetric Kernel Hilbert Schmidt Theorem

3.1 INTRODUCTION

The part of Fredholm theory which involves integral operators generated by real symmetric kernels is referred to as the Hilbert-Schmidt theory of integral equation. Owing to the richness of its result, the theory has attracted extensive attention of those interested in practical applications of integral equations as well as those interested in abstract theory, specially functional analysis. In this chapter, we are going to focus our attention to those aspects of the theory which constitute the interface between differential equation and integral equations.

GENERAL DEFINITIONS

Definition 1

A function $f(x)$ is said to be square integrable if

$$\int_a^b |f(x)|^2 \, dx < \infty$$

A square integrable function $f(x)$ is called an I_2-function, i.e., a function $f(x)$ is said to be I_2-function if the following conditions are satisfied :

(i) $$\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt < \infty \; \forall \; x \in [a,b], \forall \; t \in [a,b]$$

(ii) $$\int_a^b |k(x,t)|^2 \, dx < \infty \; ; \quad \forall \; x \in [a,b]$$

(iii) $\int_a^b |k(x,t)|^2 \, dx < \infty$; $\forall t \in [a, b]$

Definition 2

Let f and g be two complex l_2-functions of real variable x, then inner product or scalar product, denoted by (f, g) and is defined as

$(f, g) = \int_a^b f(x) \, \bar{g}(x) \, dx$, where, bar denotes the complex conjugate.

Definition 3

Two functions f and g are said to be orthogonal if

$(f, g) = 0$, i.e., $\int_a^b f(x) \, \bar{g}(x) \, dx = 0$

Definition 4

The norm of a function f(x) is given by

$$\| f(x) \| = \left[\int_a^b f(x) \, \bar{f}(x) \, dx \right]^{1/2} = \left[\int_a^b |f(x)|^2 \, dx \right]^{1/2}$$

Definition 5

A function f(x) is said to be normalized if $\| f(x) \| = 1$.

Remark.

1. A non-zero function, with non-zero norm can always be normalized by dividing it by its norm.

3.2 COMPLEX HILBERT SPACE

Let H be a complex Banach space. Then, H is called a Hilbert space if a complex number (x, y) called the inner product of x and y, is associated to each of the two vectors x and y in such a way that

(i) $\overline{(x, y)} = (y, x)$

(ii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

and (iii) $(x, x) = \| x \|^2$; for all $x, y, z \in H$ and for all scalars α, β.

Examples of Hilbert Spaces

(1) Consider the Banach space consisting of all n tuples of complex numbers with the norm of a vector $x = (x_1, x_2, \ldots\ldots, x_n)$ defined by

$$\| x \| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

We shall show that if the inner product of two vectors $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$ is defined by

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$$

Then, it is a Hilbert space.

For all arbitrary vectors $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ and for arbitrary scalars and , we have

(i) $$\overline{(x, y)} = \overline{\left(\sum_{i=1}^{n} x_i \bar{y}_i\right)} = \left(\overline{x_1\bar{y}_1 + x_2\bar{y}_2 + ... + x_n\bar{y}_n}\right)$$

$$= (\overline{x_1\bar{y}_1}) + ... + (\overline{x_n, \bar{y}_n})$$

$$= \bar{x}_1(\overline{\bar{y}_1}) + \bar{x}_2(\overline{\bar{y}_2}) + ... + \bar{x}_n(\overline{\bar{y}_n})$$

$$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + ... + \bar{x}_n y_n$$

$$= y_1\bar{x}_1 + y_2\bar{x}_2 + ... + y_n\bar{x}_n$$

$$= (y, x)$$

(ii) $$\alpha x + \beta y = \alpha(x_1, x_2, ..., x_n) + \beta(y_1, y_2, ..., y_n)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, ..., \alpha x_n + \beta y_n)$$

Therefore,

$$(\alpha x + \beta y, z) = (\alpha x_1 + \beta y_1)\bar{z}_1 + ... + (\alpha x_n + \beta y_n)\bar{z}_n$$

$$= \alpha(x_1\bar{z}_1 + x_2\bar{z}_2 + ... + x_n\bar{z}_n) + \beta(y_1\bar{z}_1 + y_2\bar{z}_2 + ..., + y_n\bar{z}_n)$$

$$= \alpha(x, z) + \beta(y, z)$$

(iii) $$(x, x) = \sum_{i=1}^{n} x_i\,\bar{x}_i = \sum_{i=1}^{n} |\,x_i\,|^2 = \|\,x\,\|^2$$

Hence, the space under consideration is a Hilbert space.

3.3 ORTHONORMAL SYSTEM OF FUNCTIONS

We know that a finite or infinite set $\{f_i(x)\}$ defined on an interval $a \le x \le b$ is said to be an orthogonal set if

$$(f_i, f_j) = 0$$

i.e., $$\int_a^b f_i(x)\, f_j(x)\, dx = 0 \qquad i \ne j \qquad ......(1)$$

If none of the elements of this set is a zero vector, then it is called a proper orthogonal set.

The set $\{f_i(x)\}$ is said to be orthonormal if

$$(f_i, f_j) = \int_a^b f_i(x)\, f_j(x)\, dx = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$$

Remark

- Any function f(x) for which $||\,f(x)\,|| = 1$ is known as normalized.

Hermitian or Self-Adjoint Operator

The integral operator

$$K = \int_a^b k(x, t)\, dt$$

is said to be Hermitian or self-adjoint if it satisfies the condition

$$(Kf, g) = (f, Kg)$$

Hilbert-Schmidt Kernel

A kernel $k(x, t)$ is said to be Hilbert-Schmidt kernel if it is Hermitian and square integrable.

3.4 GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

[MEERUT-2006]

Let $\{f_1, f_2, \ldots, f_k, \ldots\}$ be the set of given functions. We can construct an orthogonal set $\{g_1, g_2, \ldots, g_k, \ldots\}$ by Gram-Schmidt process as follows.

Let $\quad g_1 = \dfrac{f_1}{\| f_1 \|}$

To find g_2, we can define

$$w_2(x) = f_2(x) - (f_2, f_1)\, g_1$$

The function w_2 is orthogonal to g_1. Therefore, g_2 can be constructed by setting

$$g_2 = \frac{w_2}{\| w_2 \|}$$

Proceeding in the same way, we get

$$w_k(x) = f_k(x) - \sum_{i=1}^{k-1} (f_k, f_i)\, g_i, \qquad g_k = \frac{w_k}{\| w_k \|}$$

Also, if we are given set of orthonormal functions, we can convert it into an orthonormal set simply by dividing each function by its norm. Now, starting from an arbitrary orthonormal system, it is possible to construct the theory of Fourier series. Suppose we want to find the best approximation of an arbitrary function f(x) in terms of a linear combination of an orthonormal set $(g_1, g_2, \ldots, g_n)$.

Now, for any $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\left\| f - \sum_{i=1}^{n} \alpha_i g_i \right\|^2 = \| f \|^2 + \sum_{i=1}^{n} |(f, g_i) - \alpha_i|^2 - \sum_{i=1}^{n} |(f, g_i)|^2 \qquad \ldots\ldots(1)$$

Clearly, the minimum can be attained by setting $\alpha_i = (f, g_i) = a_i$ (say).

Here, the numbers a_i are known as Fourier coefficients of the function f(x) relative to the orthonormal system $\{g_i\}$. Then (1) can be written as

$$\left\| f - \sum_{i=1}^{n} \alpha_i g_i \right\|^2 = \| f \|^2 - \sum_{i=1}^{n} |a_i|^2 \qquad \ldots\ldots(2)$$

Since L.H.S of (2) is non-negative, we get

$$\sum_{i=1}^{n} |a_i|^2 \leq \| f \|^2$$

which gives the Bessel's inequality for the infinite set $[g_i]$, i.e.,

$$\sum_{i=1}^{\infty} |a_i|^2 \leq \| f \|^2 \qquad \ldots\ldots(3)$$

Suppose, we are given an infinite orthonormal system $[g_i(x)]$ in I_2 and a sequence of constants $<\alpha_i>$, then the convergence of the series $\sum_{k=1}^{\infty} |\alpha_k^2|$ is clearly a necessary condition for the existence of an I_2-function $f(x)$, whose fourier coefficients is with respect to the system f_i and α_i.

Schwarz Inequality

If $f(x)$ and $g(x)$ be any two I_2-functions in a Hilbert space H, then $|(f, g)| \leq \| f \| . \| g \|$ [MEERUT-92, 94, 98, 2000]

Proof : If g = 0, then $\| g \| = 0$ and $|(f, g)| = 0$. Therefore, in this case both sides vanish and result is trivially true.

Now, let $g \neq 0$. Now, for any scalar λ, we have

$$(f + \lambda g, f + \lambda g) \geq 0$$

$$\Rightarrow \quad (f, f + \lambda g) + \lambda(g, f + \lambda g) \geq 0$$

$$\Rightarrow \quad (f, f) + \bar{\lambda}(f, g) + \lambda(g, f) + \lambda\bar{\lambda}(g, g) \geq 0$$

$$\Rightarrow \quad \| f \|^2 + \bar{\lambda}(f, g) + \lambda(g, f) + |\lambda|^2 \| g \|^2 \geq 0 \qquad \ldots\ldots(1)$$

Since, $g \neq 0 \Rightarrow \| g \| \neq 0$, therefore putting $\lambda = -\frac{(f, g)}{\| g \|^2}$ in (1), we get

$$\| f \|^2 - \frac{\overline{(f, g)}}{\| g \|^2}(f, g) - \frac{(f, g)}{\| g \|^2}(g, f) + \frac{|(f, g)|^2}{(\| g \|)^4} \| g \|^2 \geq 0$$

$$\Rightarrow \quad \| f \|^2 - \frac{|(f, g)|^2}{\| g \|^2} - \frac{(f, g)\overline{(f, g)}}{\| g \|^2} + \frac{|(f, g)|^2}{\| g \|^2} \geq 0$$

$$\Rightarrow \quad \| f \|^2 - \frac{|(f, g)|^2}{\| g \|^2} - \frac{|(f, g)|^2}{\| g \|^2} + \frac{|(f, g)|^2}{\| g \|^2} \geq 0$$

$$\Rightarrow \quad \| f \|^2 - \frac{|(f, g)|^2}{\| g \|^2} \geq 0$$

$$\Rightarrow \quad \| f \|^2 . \| g \|^2 - |(f, g)|^2 \geq 0$$

$$\Rightarrow \quad |(f, g)| \leq \| f \| . \| g \|$$

Minkowski Inequality

If f(x) and g(x) be any two L_2-functions in a Hilbert space H, then $|(f, g)| \le \| f \| + \| g \|$.

Bessel's Inequality

Statement : *For every square integrable function* $f(x)$

$$\sum_{i=1}^{\infty} |C_i|^2 = \sum_{i=1}^{\infty} |(f, \phi_i)|^2 \le ||f||^2,$$

where $f(x)$ *is real and continuous and* $\phi_i(x) : i = 1, 2, \ldots\ldots$ *is real and continuous and consisting normalized orthogonal set.* **[MEERUT-1997, 99]**

Proof : Consider

$$\int_a^b |f(x) - \sum C_i\phi_i(x)|^2\, dx = \int_a^b |f(x)|^2\, dx + \sum_{i=1}^{n} \int_a^b |C_i|^2 |\phi_i(x)|^2\, dx$$

$$-\sum_{i=1}^{n} \int_a^b \overline{f(x)}\, C_i\phi_i(x)\, dx - \sum_{i=1}^{n} \int_a^b f(x)\, \overline{C}_i\phi_i(x)\, dx \qquad \ldots\ldots(1)$$

Now, since $\int_a^b |\phi_i(x)|^2\, dx = 1$, $\int_a^b f(x)\, \overline{\phi}_i(x)\, dx = C_i$, $\int_a^b \overline{f(x)}\, \phi_i(x)\, dx = \overline{C_i}$

implies that $\int_a^b \left| f(x) - \sum_{i=1}^{n} C_i\phi_i(x) \right|^2 dx = \int_a^b |f(x)|^2\, dx - \sum_{i=1}^{n} |C_i|^2 \qquad \ldots\ldots(2)$

The integral in (2) has non-negative value, therefore, for every n, we have

$$\sum_{i=1}^{n} |C_i|^2 \le \int_a^b |f(x)|^2\, dx = ||f||^2 \qquad \ldots\ldots(3)$$

It follows that the series $\sum_{i=1}^{n} |C_i|^2$ is always convergent and its sum satisfies the inequality

$$\sum_{i=1}^{\infty} |C_i|^2 \le \int_a^b |f(x)|^2\, dx = \| f \|^2$$

Remark

1. When $n \to \infty$, the LHS of (2) tends to zero, then

   $\sum_{i=1}^{\infty} |C_i|^2 = \| f \|^2$ is known as Parseval's equation.

OTHER USEFUL DEFINITIONS

Definition 1

Given a sequence of functions $< f_n(x) >$ and a function $f(x)$ in L_2-space defined on an interval I, then the sequence $< f_n >$ converges uniformly on I if

$$\sup_{x \in I} | f_m(x) - f_n(x) | \to 0 \text{ as } m, n \to \infty$$

Definition 2

The sequence $< f_n(x) >$ converges uniformly to $f(x)$ if

$$\sup_{x \in I} | f(x) - f_n(x) | \to 0 \text{ as } n \to \infty$$

Definition 3

The sequence $< f_n(x) >$ converges in the mean on $[a, b]$ if

$$\int_a^b | f_m(x) - f_n(x) |^2 \, dx \to 0 \text{ as } m, n \to \infty$$

Also, it converges in mean to $f(x)$ if

$$\lim_{n \to \infty} \int_a^b | f(x) - f_n(x) |^2 \, dx = 0$$

Definition 4

A series of functions $S = \sum_{i=1}^{\infty} f_i(x)$, $\forall \, x \in I$

converges uniformly (or in mean) to $F(x)$, if the sequence of partial sums

$$s_n(x) = \sum_{i=n}^{n+p} f_i(x)$$

converges uniformly (or in the mean) to F(x).

Definition 5

The series S, defined above is said to be absolutely convergent if the series

$\sum_{i=1}^{\infty} | f_i(x) |$ is pointwise convergent.

Remark

- On a finite closed domain , uniform convergence implies convergence in the mean. The converse is not true. For example, as the open interval $] \, 0, 1 \, [$, the sequence $< e^{-nx} >$ converges in the mean but not uniformly.

3.5 RIESZ-FISCHER THEOREM [MEERUT-2004, 05, 05BP]

Statement : *Let $\{\phi_i(x)\}$ is an orthonormal system of I_2-functions defined and integrable together with the squares of their moduli in the domain of (a, b) and $< \alpha_i >$ is a sequence of complex numbers, then the series*

$$\sum_{i=1}^{\infty} | \alpha_i |^2$$

converges. There exists a unique function $f(x)$, integrable together with the square of its modulus for which the numbers α_i are the fourier coefficients with regard to an orthonormal system $\{\phi_i(x)\}$ to which the fourier series converges in the mean.

Proof : Consider the sequence of partial sums

$$s_n(x) = \sum_{i=1}^{n} \alpha_i\, \phi_i(x) \qquad \text{......(1)}$$

Since, we know that

$$\int_a^b | s_{n+m}(x) - s_n(x) |^2\, dx = | \alpha_{n+1}^2 | + | \alpha_{n+2}^2 | + \ldots\ldots + | \alpha_{n+m}^2 | \qquad \text{......(2)}$$

Since the sequence $\sum_{i=1}^{\infty} | \alpha_i |^2$ is convergent for every $\varepsilon > 0$, there exists an arbitrarily chosen positive small number δ such that

$$| \, | s_{n+m}(x) - s_n(x) | \, | < \epsilon, \qquad \forall\, n > \delta,\ m \qquad \text{......(3)}$$

Thus, there exists a unique function $f(x)$, integrable in the domain $\Omega(a, b)$ together with the square of its modulus to which the sequences (1) converges in the mean and therefore

$$\left\| f - \sum_{i=1}^{n} \alpha_i\, \phi_i \right\| \to 0 \ \text{ as } \ n \to \infty \qquad \text{......(4)}$$

Since the numbers α_i are the fourier coefficients of the function $f(x)$ with regard to the system $\{\phi_i(x)\}$, then

$$\left\| f - \sum_{i=1}^{n} \alpha_i\, \phi_i \right\|^2 = \| f \|^2 + \sum_{i=1}^{n} | C_i |^2 + \sum_{i=1}^{n} | \alpha_i - C_i |^2 \to 0 \ \text{ as } \ n \to \infty \qquad \text{......(5)}$$

Using Bessel's inequality, we have

$$\alpha_i = C_i = \int_a^b f(x)\, \bar{\phi}_i(x)\, dx \qquad \text{......(6)}$$

The fourier series $\sum_{i=1}^{\infty} C_i\, \phi_i(x)$ of the function $f(x)$ with regard to the system $\{\phi_i(x)\}$ is convergent in the mean to that function, that is

$$\left\| f - \sum_{i=1}^{n} C_i\, \phi_i \right\| \to 0 \ \text{ as } \ n \to \infty \qquad \text{......(7)}$$

Also, from (5), we observed that the fourier coefficients C_i of the function f(x) with respect to the system $\{\phi_i(x)\}$ satisfy Parseval's equation

$$\sum_{i=1}^{\infty} | C_i |^2 = | \, | f | \, |^2 \qquad \text{......(8)}$$

3.6 SYMMETRIC KERNEL

[MEERUT-2006(BP)]

A kernel $k(x, t)$ is said to be symmetric (or complex symmetric or Hermitian) if

$$k(x, t) = \bar{k}(t, x)$$

where the bar denotes the complex conjugate. In case of a real kernel, the symmetry reduces to the equality

$$k(x, t) = k(t, x)$$

GENERAL THEOREMS

THEOREM-1

All iterated kernels of a symmetric kernel are also symmetric.

[MEERUT-1997, 99, 2000,06(BP), GARHWAL-1999, 2002, 04]

Proof : By definition of iterated kernel, we have

$$k_1(x, t) = k(x, t)$$

$$k_2(x, t) = \int_a^b k_1(x, z)\, k(z, t)\, dz$$

$$\ldots \quad \ldots \quad \ldots$$

$$k_n(x, t) = \int_a^b k_{n-1}(x, z)\, k(z, t)\, dz$$

Also, given that kernel $k(x, t)$ is symmetric, therefore

$$k(x, t) = \bar{k}(t, x) \qquad \ldots\ldots(1)$$

Now, consider

$$k_2(x, t) = \int_a^b k(x, z)\, k_1(z, t)\, dz = \int_a^b k(x, z)\, k(z, t)\, dz \qquad \text{[Using 1]}$$

$$= \int_a^b \bar{k}(z, x)\, \bar{k}(t, z)\, dz = \int_a^b \bar{k}(t, z)\, \bar{k}(z, x)\, dz$$

$$= \int_a^b \bar{k}(t, z)\, \bar{k}_1(z, x)\, dz = \bar{k}_2(t, x)$$

$\Rightarrow$ Result is true for n = 2, i.e., $k_2(x, t)$ is symmetric.

Let us suppose $k_n(x, t)$ is symmetric for n = m, then we have

$$k_m(x, t) = \bar{k}_m(t, x) \qquad \ldots\ldots(2)$$

We shall prove that $k_n(x, t)$ is symmetric for n = m + 1.

Consider

$$k_{m+1}(x, t) = \int_a^b k(x, z)\, k_m(z, t)\, dz = \int_a^b \bar{k}(z, x)\, \bar{k}_m(t, z)\, dz$$

$$= \int_a^b \bar{k}_m(t, z)\, \bar{k}(z, x)\, dz = \bar{k}_{m+1}(t, x)$$

Hence, by principle of mathematical induction, we can say that the iterated kernel $k_n(x, t)$ is symmetric for all n.

Remark

- The above relation implies $k_n(x, x) = \bar{k}_n(x, x)$ and therefore, the function $k_n(x, x)$ is real.

THEOREM-2

The eigen function of a symmetric kernel corresponding to different eigen values are orthogonal. [MEERUT-1995, 1999, 2001, 03, 05(BP), GARHWAL-2002]

Proof : Let $\phi_m(x)$ and $\phi_n(x)$ are eigen functions of a symmetric kernel $k(x, t)$ for eigen values λ_m and λ_n respectively of the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_a^b k(x, t)\, \phi(t)\, dt \qquad \ldots\ldots(1)$$

Also, since kernel $k(x, t)$ is symmetric, therefore

$$k(x, t) = k(t, x) \qquad \ldots\ldots(2)$$

Here, we observed that $\lambda = 0$ can not be an eigen value since it gives the trivial solution $\phi(x) = 0$.

Now, ϕ_m and ϕ_n both satisfy the integral equation (1). Therefore, we have

$$\phi_m(x) = \lambda_m \int_a^b k(x, t)\, \phi_m(t)\, dt \qquad \ldots\ldots(3)$$

and

$$\phi_n(x) = \lambda_n \int_a^b k(x, t)\, \phi_n(t)\, dt \qquad \ldots\ldots(4)$$

Multiplying (3) by $\phi_n(x)$ and integrating w.r.t. x over an interval (a, b), we have

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \lambda_m \int_a^b \phi_n(x) \left\{ \int_a^b k(x, t)\, \phi_m(t)\, dt \right\} dx$$

$$= \lambda_m \int_a^b \phi_m(t) \left\{ \int_a^b k(x, t) \phi_n(x)\, dx \right\} dt$$

(By changing the order of integration) ......(5)

Since, kernel $k(x, t)$ is symmetric, therefore using (2), (5) can be written as

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \lambda_m \int_a^b \phi_m(t) \left\{ \int_a^b k(t, x)\, \phi_n(x)\, dx \right\} dt \qquad \ldots\ldots(6)$$

By interchanging the variables x and t in (4), we get

$$\phi_n(t) = \lambda_n \int_a^b k(t, x)\, \phi_n(x)\, dx \qquad \ldots\ldots(7)$$

Using relation (7), the equation (6) reduces to

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \frac{\lambda_m}{\lambda_n} \int_a^b \phi_m(t)\, \phi_n(t)\, dt = \frac{\lambda_m}{\lambda_n} \int_a^b \phi_m(x)\, \phi_n(x)\, dx$$

$$\Rightarrow \quad \left(1 - \frac{\lambda_m}{\lambda_n}\right) \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0 \qquad \ldots\ldots(8)$$

Since, $\lambda_m \neq \lambda_n$, therefore (8) gives

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0$$

Therefore, we can say if $\phi_m(x)$ and $\phi_n(x)$ are eigen function of (1) corresponding to distinct eigen values, then ϕ_m and ϕ_n are orthogonal over the interval (a, b). Hence, the eigen functions of a symmetric kernel, corresponding to different eigen values are orthogonal.

THEOREM-3

The eigen values of a symmetric kernel are real.

[MEERUT-1998, 2005, GARHWAL-1999, 2003]

Proof : Let $\phi(x)$ be an eigen function corresponding to an eigen value λ. Then, by definition of eigen value, we have

$$\phi(x) - \lambda\, k\, \phi(x) = 0 \qquad \text{......(1)}$$

Multiplying (1) by $\bar{\phi}(x)$ and integrating with respect to x from a to b, we get

$$\| \phi(x) \|^2 - \lambda(k\phi, \bar{\phi}) = 0$$

$$\Rightarrow \quad \lambda = \| \phi(x) \|^2 / (k\phi, \bar{\phi}) \qquad \text{......(2)}$$

Since both numerator and denominator of RHS of (2) are real. Hence, λ is real.

THEOREM-4

If $k(x, t)$ is real and symmetric, continuous and identically not equal to zero, then all the characteristics constants are real. **[MEERUT-1996, 98, 2001, 03, 04, 05, 07]**

Proof : Let, if possible, the characteristic constant λ_o is not real, then we can write

$$\lambda_o = \mu_o + i\nu_o \qquad [\nu_o \neq 0]$$

We know that the homogeneous integral equation

$$\phi(x) = \lambda_o \int_a^b k(x, t)\, \phi(t)\, dt \qquad \text{......(1)}$$

has at least one continuous solution $u(x) \neq 0$.

Therefore,

$$u(x) = \lambda_o \int_a^b k(x, t)\, u(t)\, dt$$

$$\Rightarrow \quad u(x) = (\mu_o + i\nu_o) \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(2)}$$

Separating into real and imaginary parts [considering u(x) as real]

$$u(x) = \mu_o \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(3)}$$

and $$0 = \nu_o \int_a^b k(x, t)\, u(t)\, dt \qquad \text{......(4)}$$

Using (4), we get

$$\int_a^b k(x, t)\, u(t)\, dt = 0 \qquad [\because \nu_o \neq 0]$$

$$\Rightarrow \quad u(x) = 0$$

which is contrary to our assumption $u(x) \neq 0$. Hence, λ_o must be real.

THEOREM-5

The multiplicity of any non-zero eigen value is finite for every symmetric kernel for which $\int_a^b \int_a^b | k(x, t) |^2\, dx\, dt$ is finite. **[GARHWAL-2001]**

Proof : Let $\phi_{1\lambda}(x), \phi_{2\lambda}(x), \ldots\ldots, \phi_{n\lambda}(x)$ be the linearly independent eigen functions, corresponding to a non-zero eigen value λ. Then using Gram-Schmidt procedure, we can find linear combinations of these functions which form an orthonormal system $\{u_{k\lambda}(x)\}$. Then $\{\bar{u}_{k\lambda}(x)\}$ also form an orthonormal system.

Now, let

$$k(x, t) \sim \sum_i a_i \bar{u}_{i\lambda}(t)$$

where $a_i = \int_a^b k(x, t)\, u_{i\lambda}(t) = \lambda^{-1} u_{i\lambda}(x)$

be the series associated with kernel $k(x, t)$ for a fixed x. Using Bessel's inequality, we get

$$\int_a^b |k(x, t)|^2\, dt \geq \sum_i \frac{1}{\lambda^2} |u_{i\lambda}(x)|^2$$

Integrating w.r.t. x, we get

$$\int_a^b \int_a^b |k(x, t)|^2\, dx\, dt \geq \sum_i \frac{1}{\lambda^2}$$

$$\Rightarrow \quad \int_a^b \int_a^b |k(x, t)|^2\, dx\, dt \geq \frac{m}{\lambda^2} \qquad \text{......(1)}$$

where, m is the multiplicity of λ. Since, LHS of (1) is finite. Hence, m is finite.

THEOREM-6

The eigen values of a symmetric kernel form a finite or infinite sequence $<\lambda_n>$ with no finite limit point.

Proof : Using above theorem, by including each eigen value in the sequence a number of times equal to its multiplicity, we get

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \leq \int_a^b \int_a^b |k(x, t)|^2\, dx\, dt \qquad \text{......(1)}$$

Let $<u_\gamma(x)>$ be the orthonormal eigen function corresponding to different non-zero eigen values λ_i, then using Bessel's inequality, we have

$$\sum_i \frac{1}{\lambda_i^2} \leq \int_a^b |k(x, t)|^2\, dx\, dt < \infty$$

$\Rightarrow$ if there exists an enumerable infinity of λ_i, then we must have

$$\sum_{n=-1}^{\infty} \frac{1}{\lambda_n^2} < \infty$$

showing that $\lim\left(\frac{1}{\lambda_i}\right) \to 0$ and ∞ is the only limit point of the eigen value.

THEOREM-7

The set of eigen values of the second iterated kernel coincide with the set of squares of the eigen values of the given kernel.

Proof : Let λ be an eigen value of the kernel $k(x, t)$ corresponding to the eigen function $\phi(x)$. Then, we have

$$\phi = \lambda k \phi$$

$\Rightarrow \quad (I - \lambda k)\,\phi = 0$ ......(1)

Operating both sides of (1) with the operator $(I + \lambda k)$, we get

$$(I - \lambda^2 k^2)\,\phi = 0$$

or $\quad \phi(x) = \lambda^2 \int_a^b k_2(x, t)\,\phi(t)\,dt$

$\Rightarrow \quad \lambda^2$ is an eigen value of the kernel $k_2(x, t)$.

Conversely, let $\mu = \lambda^2$ be an eigen value of the kernel $k_2(x, t)$ corresponding to the eigen function $\phi(x)$, then we can write

$$(I - \lambda^2 k^2)\,\phi = 0$$

$\Rightarrow \quad (I - \lambda k)(I + \lambda k)\,\phi = 0$ ......(2)

If λ is an eigen value of kernel k, then required result is proved.

If not, let us assume that

$(I + \lambda k)\,\phi = \phi'(x)$ ......(3)

Since λ is not an eigen value of k by our assumption, therefore (3) shows that $\phi'(x) = 0$

$\Rightarrow \quad (I + \lambda k)\,\phi = 0$

$\Rightarrow$ $-\lambda$ is an eigen value of the kernel k and hence the result.

3.7 EXPANSION OF SYMMETRIC KERNEL IN EIGEN FUNCTION

[GARHWAL-2004]

Let $k(x, t)$ be a symmetric kernel which contains an infinite number of eigen values. Let these eigen values

$\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ ......(1)

are real and non-zero in such a way that eigen value is repeated as many times as the ordinal number of its rank, or the number of linearly independent eigen functions.

Let us assume

$$0 < |\lambda_1| \le |\lambda_2| \le \ldots\ldots \le |\lambda_p| \le |\lambda_{p+1}| \le \ldots\ldots$$

Clearly $|\lambda_p| \to \infty$ as $p \to \infty$

Now consider

$\phi_1(x), \phi_2(x) \ldots\ldots, \phi_p(x)$ ......(2)

be a sequence of eigen functions corresponding to eigen values (1). We know that a symmetric kernel is identical to the kernel of its associated equation and its eigen values are real, then two function of sequence (2) corresponding to two distinct eigen values are orthogonal. The sequence of eigen function (2) of a given symmetric kernel k(x, t) form an orthonormal system, i.e.,

$$\int_a^m \phi_m(x)\,\bar{\phi}_n(x)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \qquad \text{......(3)}$$

Now consider the fourier series $\sum_{p=1}^{\infty} C_p\,\phi_p(x)$ of the kernel k(x, t) with regard to the orthogonal system $\{\phi_p(x)\}$ of its eigen function, treating t as a constant. The coefficient of the series becomes

$$C_p(t) = \int_a^b k(x,t)\,\overline{\phi_p(x)}\,dx \qquad \text{......(4)}$$

Since $\phi_p(x)$ is an eigen function, therefore, it satisfy the homogeneous Fredholm integral equation

$$\phi_p(t) = \lambda_p \int_a^b k(t,x)\,\phi_p(x)\,dx \qquad \text{......(5)}$$

Therefore,

$$\bar{\phi}_p(t) = \lambda_p \int_a^b \overline{k(t,x)}\,\bar{\phi}_p(x)\,dx = \lambda_p \int_a^b k(x,t)\,\bar{\phi}_p(x)\,dx \qquad \text{......(6)}$$

$$\Rightarrow \qquad C_p(t) = \frac{\bar{\phi}_p(t)}{\lambda_p} \qquad \text{......(7)}$$

The fourier series of the kernel $k(x,t)$ with regard to the orthogonal system $\{\phi_p(x)\}$ of its eigen value has the form

$$\sum_{p=1}^{\infty} \frac{\phi_p(x)\,\bar{\phi}_p(t)}{\lambda_p} \qquad \text{......(8)}$$

THEOREM-1

If the fourier series of a symmetric kernel k(x, t) converges uniformly with regard to each of the variables than its sum is almost everywhere equal to the kernel

$$k(x,t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\overline{\phi_n(t)}}{\lambda_n}$$

Proof : Let us assume

$$P(x,t) = k(x,t) - \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\overline{\phi_n(t)}}{\lambda_n} \qquad \text{......(1)}$$

is not almost everywhere equal to zero, i.e.,

$$\int_a^b \int_a^b |P(x,t)|^2\,dx\,dt > 0 \qquad \text{......(2)}$$

The kernel P(x, t) of the integral equation

$$\psi(x) = \lambda \int_a^b P(x,t)\,\psi(t)\,dt \qquad \text{......(3)}$$

is symmetric and satisfy the inequality (2). There exists a number λ such that the equation (3) has a solution $\psi(x)$ not almost everywhere equal to zero. We shall prove the solution $\psi(x)$ is orthogonal to all the eigen functions $\phi_p(x)$ of the kernel $k(x, t)$.

$$\psi(x) = \lambda \int_a^b \left\{ k(x, t) - \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\bar{\phi}_n(t)}{\lambda_n} \right\} \psi(t)\, dt$$

Multiplying by $\bar{\phi}_p(x)$ both the sides, and integrating with respect to x over an interval (a, b), we have

$$\int_a^b \psi(x)\,\bar{\phi}_p(x)dx = \lambda \int_a^b \int_a^b k(x, t)\,\bar{\phi}_p(x)\,\psi(t)\,dx\,dt$$

$$-\sum_{n=1}^{\infty} \lambda \int_a^b \int_a^b \frac{\bar{\phi}_p(x)\,\phi_n(x)\,\bar{\phi}_n(t)}{\lambda_n} .\psi(t)\,dx\,dt \qquad \text{......(4)}$$

$\Rightarrow$ $$\int_a^b \psi(x)\,\bar{\phi}_p(x)dx = \lambda \int_a^b \int_a^b k(x, t)\,\bar{\phi}_p(x)\,\psi(t)\,dx\,dt$$

$$-\sum_{n=1}^{\infty} \lambda \int_a^b \left\{ \int_a^b \bar{\phi}_p(x)\,\phi_n(x)\,dx \right\} \frac{\bar{\phi}_n(t)}{\lambda_n} \psi(t)\,dt$$

or $$\int_a^b \psi(x)\,\bar{\phi}_p(x)dx = \lambda \int_a^b \int_a^b k(x, t)\,\bar{\phi}_p(x)\,\psi(t)\,dx\,dt - \lambda \int_a^b \frac{\bar{\phi}_p(t)}{\lambda_n} \psi(t)\,dt$$

$\Rightarrow$ $$\int_a^b \psi(x)\,\bar{\phi}_p(x)dx = \lambda \int_a^b \left\{ \int_a^b k(x, t)\,\bar{\phi}_p(x)\,dx - \frac{\bar{\phi}_p(t)}{\lambda_n} \right\} \psi(t)\,dt \qquad \text{......(5)}$$

Using (1) and (3), we get

$$\psi(x) = \lambda \int_a^b \left\{ k(x, t) - \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\bar{\phi}_n(t)}{\lambda_n} \right\} \psi(t)\,dt$$

$$= \lambda \int_a^b k(x, t)\,\psi(t)\,dt - \sum_{n=1}^{\infty} \lambda\,\phi_n(x) \int_a^b \frac{\bar{\phi}_n(t)\,\psi(t)}{\lambda_n}\,dt \qquad \text{[Using (5)]}$$

$\Rightarrow$ $$\psi(x) = \lambda \int_a^b k(x, t)\,\psi(t)\,dt \qquad \text{......(6)}$$

The function $\psi(x)$ is also an eigen function of the kernel $k(x, t)$, therefore, it may be represented as a linear combination of certain eigen functions of the sequence $\{\phi_n(x)\}$ corresponding to the eigen value λ.

$$\psi(x) = C_1\phi_{n1}(x) + C_2\phi_{n2}(x) + \ldots\ldots + C_m\phi_{nm}(x) \qquad \text{......(7)}$$

where, m is the rank of the eigen value λ.

Multiplying the relation (7) by $\phi_{nr}(x)$, both the sides and integrating with respect to x over the interval (a, b), we have

$$\int_a^b \psi(x)\,\phi_{nr}(x)\,dx = C_r = 0\ (r = 1, 2, \ldots\ldots, m)$$

Therefore, $\psi(x) = 0$, which is a contradiction. Therefore, (2) is not justified and the relation

$$k(x, t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\overline{\phi_n(t)}}{\lambda_n} \qquad \text{......(8)}$$

holds almost everywhere when the kernel $k(x, t)$ is continuous.

Hence, every symmetric kernel with a finite number of eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ can be expressed in the form

$$k(x, t) = \sum_{m=1}^{n} \frac{\phi_m(x)\,\overline{\phi_m(t)}}{\lambda_m}$$

where $\phi_m(x)$ is an eigen function corresponding to λ_m, neglecting the term with norm equal to zero.

THEOREM-2

Let $\phi_n(x)$ is an orthonormal system of all eigen functions of the symmetric kernel $k(x, t)$ corresponding to eigen values λ_n, then the symmetric kernel

$$P(x, t) = k(x, t) - \sum_{n=1}^{m} \frac{\phi_n(x)\,\overline{\phi_n(t)}}{\lambda_n}$$

has the eigen values $\lambda_{m+1}, \lambda_{m+2}, \ldots\ldots$, where m is any positive integer.

Proof : Since, we know that

$$\phi(x) - \lambda \int_a^b P(x, t)\,\phi(t)\,dt = 0 \qquad \text{......(1)}$$

is equivalent to

$$\phi(x) - \lambda \int_a^b k(x, t)\,\phi(t)\,dt + \lambda \sum_{n=1}^{m} \frac{\phi_n(x)}{\lambda_n}(\phi, \phi_n)\,dt = 0 \qquad \text{......(2)}$$

Let us write $\lambda = \lambda_j$ and $\phi(x) = \phi_j(x),\ j \geq m+1$ on LHS of (2) and using the orthogonality condition

$$\phi_j(x) - \lambda_j \int_a^b k(x, t)\,\phi_j(t)\,dt = 0 \qquad \text{......(3)}$$

showing that $\phi_j(x)$ and λ_j for $j \geq n+1$ are eigen functions and eigen values of the kernel $P(x, t)$.

Now, let λ and $\phi(x)$ be an eigen value and eigen function of the kernel $P(x, t)$ so that

$$\phi(x) - \lambda k\,\phi(x) + \lambda \sum_{n=1}^{m} \frac{\phi_n(x)}{\lambda_n}(\phi, \phi_n) = 0 \qquad \text{......(4)}$$

Taking the scalar product of (4) with $\phi_j(x)$, $j \leq m$ and using the orthonormality of $\phi_j(x)$, we have

$$(\phi, \phi_j) - \lambda(k\phi, \phi_j) + \frac{\lambda}{\lambda_j}(\phi, \phi_j) = 0 \qquad \text{......(5)}$$

Now, $(k\phi, \phi_j) = (\phi, k\phi_j) = \frac{1}{\lambda_j}(\phi, \phi_j)$ ......(6)

Using (5) and (6), we get

$$(\phi, \phi_j) + \frac{\lambda}{\lambda_j}\{(\phi, \phi_j) - (\phi, \phi_j)\} = (\phi, \phi_j) = 0 \quad ......(7)$$

In view of (7), we find that the last term in LHS of (4) vanishes and hence (4) gives

$$\phi(x) - \lambda\int_a^b k(x, t)\,\phi(t)\,dt = 0$$

which shows that λ and $\phi(x)$ are eigen value and eigen function of the kernel $k(x, t)$ and that $\phi \neq \phi_j,\ j \leq m$. Also, ϕ is orthonormal to all $\phi_j, j \leq m$ and $\phi(x)$ and λ are surely contained in the sequence $\{\phi_k(x)\}$ and $[\lambda_k], k \geq n+1$ respectively.

THEOREM-3 (MERCER'S THEOREM)

If $k(x, t)$ is a positive definite continuous kernel with orthonormal eigen functions $[\phi_k]$, then the series

$$\sum_{k=1}^{\infty}\frac{\phi_k(x)\,\phi_k(t)}{\lambda_k} \text{ converges to } k(x, t) \text{ absolutely and uniformly.}$$

Proof : We can write

$$P(x, t) = k(x, t) - \sum_{k=1}^{n}\frac{\phi_k(x)\,\phi_k(t)}{\lambda_k}$$

Then clearly $P(x, t)$ is also a positive definite kernel.

Therefore, $P(x, x) = k(x, x) - \sum_{k=1}^{n}\frac{\phi_k^2(x)}{\lambda_k} \geq 0$

i.e., $\sum_{k=1}^{n}\frac{\phi_k^2(x)}{\lambda_k} \leq k(x, x)$

Now, since $k(x, t)$ is continuous in the closed domain $a \leq x, t \leq b$, therefore, k(x, t) is bounded. Thus,

$$\sum_{k=1}^{n}\frac{\phi_k^2(x)}{\lambda_k} \text{ is bounded for all n.}$$

Hence, the series of positive terms

$$\sum_{k=1}^{\infty}\frac{\phi_k^2(x)}{\lambda_k} \text{ is convergent for all x.}$$

Also, by Schwarz inequality

$$\left[\sum_{k=m}^{m+p}\left|\frac{\phi_k(x)\,\phi_k(t)}{\lambda_k}\right|\right]^2 \leq \sum_{k=m}^{m+p}\frac{\phi_k^2(x)}{\lambda_k}\cdot\sum_{k=m}^{m+p}\frac{\phi_k^2(t)}{\lambda_k} \quad ......(1)$$

Therefore, the given series converges absolutely. Hence, the series converges uniformly in x for fixed k and in t for fixed t. Now, each ϕ_i being an integral is a continuous function of its argument. Hence, the sum function defined by the series $\sum_{n=1}^{\infty} \frac{\phi_n(x)\,\phi_n(t)}{\lambda_n}$ is a continuous function of x for fixed t and conversely. Since uniform convergence implies mean convergence, the series converges to $k(x, t)$. Uniform convergence of the series in both x and t together follows from the inequality (1).

3.8 HILBERT-SCHMIDT THEOREM

Statement : *Any function f(x), which can be expressed in the form*

$$f(x) = \int_a^b k(x, t)\, h(t)\, dt$$

is almost everywhere the sum of its Fourier series with regard to the orthogonal system $\phi_n(x)$ of eigen functions of the symmetric kernel $k(x, t)$. The kernel $k(x, t)$ is integrable together with the square of its modulus with regard to both of its variables i.e. x and t. The integral $\int |k(x, t)|^2\, dx$ is bounded and $h \in I_2[a, b]$.

[MEERUT-1996, 2000, 01,02,03,06(BP),07,07(BP), GARHWAL-2002]

Proof : We have to prove that

$$f(x) = \sum_{n=1}^{\infty} f_n\, \phi_n(x) \qquad \text{......(1)}$$

where the coefficients f_n are the fourier coefficients of the function f(x) with respect to the system $\{\phi_n(x)\}$, i.e.,

$$f_n = \int_a^b f(x)\, \overline{\phi_n(x)}\, dx \qquad \text{......(2)}$$

We observe that

(i) the fourier series $\sum_{n=1}^{\infty} f_n\, \phi_n(x)$ is convergent in domain (a, b)

(ii) its sum is the function f(x), given

Consider the homogeneous integral equation

$$f(x) = \int_a^b k(x, t)\, h(t)\, dt \qquad \text{......(3)}$$

Put this value in (2), we get

$$f_n = \int_a^b \int_a^b k(x, t)\, \overline{\phi_n}(x)\, h(t)\, dx\, dt$$

Now since $\int_a^b k(x, t)\, \overline{\phi}_n(x)\, dx = \int_a^b \overline{k}(t, x)\, \overline{\phi}_n(x)\, dx = \frac{1}{\lambda_n}\, \overline{\phi}_n(t)$

Therefore,

$$f_n = \int_a^b \frac{1}{\lambda_n}\, \overline{\phi}_n(t)\, h(t)\, dt = \frac{h_n}{\lambda_n} \qquad \text{......(4)}$$

where h_n are the fourier coefficients of the given function h with regard to the system $\{\phi_n(x)\}$.

$$\therefore \quad h_n = \int_a^b h(x)\,\bar{\phi}_n(x)\,dx \tag{5}$$

i.e.,
$$\sum_{n=1}^{\infty} f_n\,\phi_n(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n}\,\phi_n(x) \tag{6}$$

The series (6) is convergent if

$$\left[\sum_{k=n+1}^{n+p} \left|\frac{h_k}{\lambda_k}\,\phi_k(x)\right|\right]^2 \le \sum_{k=n+1}^{n+p} |h_k|^2 \sum_{k=n+1}^{n+p} \left|\frac{\phi_k(x)}{\lambda_k}\right|^2 \tag{7}$$

Now, $\sum_{k=n+1}^{n+p} |h_k|^2 < \epsilon$, for any arbitrary chosen positive small number ϵ. The summation $\sum_{k=n+1}^{n+p} \left|\frac{\phi_k(x)}{\lambda_k}\right|^2$ is bounded since $\int_a^b |k(x,t)|^2\,dx$ is bounded. Therefore, the fourier series (6) of the function f(x) with respect to the system $\{\phi_n(x)\}$ is absolutely and uniformly convergent on (a, b).

Let $s(x)$ denotes the sum of the series (6), therefore

$$s(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n}\,\phi_n(x) \tag{8}$$

is equal to f(x).

Now, consider $P(x) = f(x) - s(x)$ (9)

Here, the function f(x) and s(x) have the same fourier coefficients with regard to the system $\{\phi_n(x)\}$.

Multiplying (9) by $\bar{\phi}_n(x)$ and integrating w.r.t. x over the interval (a, b), we get

$$\int_a^b P(x)\,\bar{\phi}_n(x)\,dx = \int_a^b f(x)\,\bar{\phi}_n(x)\,dx - \int_a^b s(x)\,\bar{\phi}_n(x)\,dx$$

i.e.,
$$\int_a^b P(x)\,\bar{\phi}_n(x)\,dx = 0 \tag{10}$$

which implies that P(x) is orthogonal to all the eigen function $\phi_n(x)$ of the kernel k(x, t). But we know that the function P(x) is orthogonal to the kernel k(x, t), which implies

$$\int_a^b k(t,x)\,\bar{P}(t)\,dt = 0 \tag{11}$$

Now, multiplying both the sides by h(x) and integrating with regard to x over the interval (a, b), we have

$$\int_a^b \int_a^b k(t,x)\,\bar{P}(t)\,h(x)\,dx\,dt = 0 \quad \Rightarrow \quad \int_a^b \bar{P}(t)\,f(t)\,dt = 0$$

which implies that the function f is orthogonal to P.

Also, $\int_a^b |P(x)|^2\,dx = \int_a^b \bar{P}(x)\left[f(x) - s(x)\right]dx$

$$\Rightarrow \quad \int_a^b h^2(x)\, dx = \int_a^b h(x)\, g(x)\, dx - \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n} \int_a^b \phi_n(x)\, h(x)\, dx = 0$$

$\Rightarrow \quad h(x) = 0$ almost everywhere. Hence, the function $f(x)$ is almost everywhere equal to the sum of its absolutely and uniformly convergent Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n} \phi_n(x)$$

HILBERT'S THEOREM

Statement. Every symmetric kernel with a norm not equal to zero has at least one eigen value. **[MEERUT 2000, 03, 05(BP), 06]**

Proof. Consider the non-homogeneous Fredholm integral equation of second kind

$$g(x) = f(x) + \lambda \int_a^b K(x, t)\, g(t)\, dt \tag{1}$$

The solution of the integral equation in terms of a power series in λ is given by

$$g(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x, t)\, f(t)\, dt \tag{2}$$

Multiplying (2) by $\bar{f}(x)$ and integrating in an interval (a, b), we have

$$\int_a^b g(x)\, \bar{f}(x)\, dx = \sum_{n=0}^{\infty} A_n\, \lambda^n \tag{3}$$

where $\quad A_n = \int_a^b \int_a^b K_n(x, t)\, \bar{f}(x)\, f(t)\, dx\, dt \tag{4}$

with $K_0 = 1,\ K_1(x, t) = K(x, t)$

The recurrence relation for iterated kernels are defined by

$$K_{m+n}(x, t) = \int_a^b K_m(x, z)\, K_n(z, t)\, dz \tag{5}$$

$$\Rightarrow \quad K_{2n}(x, t) = \int_a^b K_n(x, z)\, K_n(z, t)\, dz$$

Now since $K_n(x, t) = \bar{K}_n(t, x)$, therefore, we have

$$A_{2n} = \int_a^b \left[\int_a^b K_n(x, z)\, \bar{f}(x)\, dx \right] \left[\int_a^b K_n(z, t)\, f(t)\, dt \right] dz$$

$$= \int_a^b \left| \int_a^b K_n(x, z)\, \bar{f}(x)\, dx \right|^2 dz \tag{6}$$

Therefore, we can say that all the coefficients of series (3) with even subscripts are non-negative real numbers.

i.e., $\quad A_{2n} \geq 0$

Also, from (4) and (5), we get

$$A_{2n} = \int_a^b \left[\int_a^b K_{n-1}(x, z)\, \bar{f}(x)\, dx \right] \left[\int_a^b K_{n+1}(z, t)\, f(t)\, dt \right] dz$$

Now, using Schwarz's inequality, we get

$$A_{2n}^2 \le \int_a^b |k_{n-1}(x,t)\,\bar{F}(x)\,dx|^2 ds \,.\, \int_a^b \left| \int_a^b k_{n+1}(x,s)\,\bar{F}(x)\,dx \right|^2 ds$$

Using (6), we have

$$A_{2n}^2 \le A_{2n-2} \cdot A_{2n+2} \text{ hold for every } n \ge 2. \qquad \text{......(7)}$$

Particularly

$$A_2 = \int_a^b \int_a^b k_2(x,t)\,\bar{F}(x)\,F(t)\,dx\,dt$$

$$= \int_a^b \left| \int_a^b k(x,s)\,\bar{F}(x)\,dx \right|^2 ds \qquad \text{......(8)}$$

Now, $A_4 = \int_a^b \int_a^b k_4(x,t)\,\bar{F}(x)\,F(t)\,dx\,dt$

$$\therefore \quad A_4 = \int_a^b \left| \int_a^b k_2(x,t)\,\bar{F}(x)\,dx \right|^2 dt \qquad \text{......(9)}$$

Since the kernel k(x, s) has a non-zero norm (by our assumption), therefore, there exists a function $F \in I_2$ such that $A_2 > 0$.

Now, we shall prove that $A_4 > 0$. If $A_4 = 0$, then (9) gives

$$\int_a^b k_2(x,t)\,\bar{F}(x)\,dx = 0$$

almost everywhere with regard to t in the domain (a, b) and then from (9), we have $A_2 = 0$, which is a contradiction.

Therefore, all the coefficients with even indices are positive and satisfy the inequality

$$\frac{A_{2n+2}}{A_{2n}} \ge \frac{A_{2n}}{A_{2n-2}} \qquad \text{......(10)}$$

The above inequality forms a non-decreasing sequence. The series $\Sigma A_n \lambda^n$ can not be convergent for every value of λ, unless $A_4 = 0$. If $A_4 > 0$, then from (7), we observe that the same will be true for $A_6, A_8, \ldots$ etc. The ratio A_{2n} / A_{2n-2} will be increasing. The series considered by taking the terms of even order in the series $\Sigma A_n \lambda^n$ can not, therefore, be convergent for every value of λ, if A_4 is not zero. If A_4 be zero then it is necessary and sufficient that F(x) be orthogonal to the kernel $k_2(x, t)$.

Now, we shall find the interval in which at least one eigen value λ_0 of the kernel k(x, t) is contained. The terms of the series (10) satisfy the inequality

$$\frac{A_{2n+2}}{A_{2n}} \cdot \frac{\lambda^{2n+2}}{\lambda^{2n}} \ge \frac{A_4}{A_2} \lambda^2$$

Therefore, the series diverges if

$$(A_4 / A_2)\,|\lambda|^2 > 1 \Rightarrow |\lambda| > \left[(A_2 / A_4)^{1/2} \right]$$

Hence, one eigen value of the kernel k(x, t) is contained in the interval

$$\left\{ -\sqrt{(A_2 / A_4)},\ +\sqrt{(A_2 / A_4)} \right\}$$

which is real.

ADDITIONAL THEOREM

THEOREM-1

If $f_m(x)$ are a complete normalized orthogonal system of fundamental functions for $k(x, t)$ to the characteristic constants λ_m, then $f_m(x)$ are a complete normalized orthogonal system of fundamental functions for $k_n(x, t)$ to the characteristic constants λ_m^n.

Proof : We have

$$f_m(x) = \lambda_m \int_a^b k(x, t)\, \psi_m(t)\, dt \qquad \text{......(1)}$$

Multiplying (1) by $k(z, x)$, both the sides and integrating w.r.t. x over (a, b), we get

$$\int_a^b k(z, x)\, f_m(x)\, dx = \lambda_m \int_a^b \int_a^b k(z, x)\, k(x, t)\, \psi_m(t)\, dt\, dx$$

$$\Rightarrow \quad \frac{f_m(z)}{\lambda_m} = \lambda_m \int_a^b k_2(z, t)\, f_m(t)\, dt$$

Replacing z by x, we get

$$f_m(x) = \lambda_m^2 \int_a^b k_2(x, t)\, f_m(t)\, dt \qquad \text{......(2)}$$

Similarly

$$f_m(x) = \lambda_m^3 \int_a^b k_3(x, t)\, f_m(t)\, dt$$

$$f_m(x) = \lambda_m^4 \int_a^b k_4(x, t)\, f_m(t)\, dt$$

$$\vdots$$

In general

$$f_m(x) = \lambda_m^n \int_a^b k_n(x, t)\, f_m(t)\, dt \qquad \text{......(3)}$$

Therefore, λ_m^n is the characteristic constant to $k_n(x, t)$ and $f_m(x)$ is the eigen function of the iterated kernel $k_n(x, t)$ belonging to λ_m^n.

Now, it remains to prove that the iterated kernel $k_n(x, t)$ has no other characteristic constants than λ_m^n.

We know that every fundamental function $\phi(x)$ of $k_n(x, t)$ may be expressed as a linear combination such as

$$\phi(x) = C_{m1}\, f_{m1}(x) + C_{m2} f_{m2}(x) + \ldots\ldots + C_{mr}\, f_{mr}(x) \qquad \ldots\ldots(4)$$

Let μ be any other characteristic constant of the iterated kernel $k_n(x, t)$ and $\phi(x)$ is an eigen function belonging to μ.

Then,

$$\phi(x) = \mu \int_a^b k_n(x, t)\, \phi(t)\, dt \qquad \ldots\ldots(5)$$

Let $\mu_1, \mu_2, \ldots\ldots, \mu_n$ be the nth roots of μ, then, we can write $\quad \mu = \mu_m^n \quad [m \in N]$

Now, we can write

$$n \cdot g_i(x) = \phi(x) + \mu_i \int_a^b k(x, t)\, \phi(t)\, dt + \mu_i^2 \int_a^b k_2(x, t)\, \phi(t)\, dt$$

$$+ \mu_i^{n-1} \int_a^b k_{n-1}(x, t)\, \phi(t)\, dt, \qquad [i \in N] \qquad \ldots\ldots(6)$$

On adding these equations, we get

$$g_1(x) + g_2(x) + \ldots\ldots + g_n(x) = \phi(x) \qquad \ldots\ldots(7)$$

Since $\quad \mu_1^r + \mu_2^r + \ldots\ldots + \mu_n^r = 0 \qquad [r = 1, 2, \ldots\ldots, n-1]$

Multiplying (6) by $k(z, x)$ and integrating with respect to x over (a, b), we get

$$n \int_a^b k(z, x)\, g_i(x)\, dx = \int_a^b k(x, z)\, \phi(x)\, dx + \sum_{r=1}^{n-1} \mu_i^r \int_a^b \int_a^b k(z, x)\, k_r(x, t)\, \phi(t)\, dt\, dx$$

$$\ldots\ldots(8)$$

Multiplying (8) by μ_i and using definition of iterated kernel and (5) for x = z, t = x, we get

$$n\mu_i \int_a^b k(z, x)\, g_i(x)\, dx = \mu_i \int_a^b k(z, t)\, \phi(t)\, dt + \mu_i^2 \int_a^b k_2(z, t)\, \phi(t)\, dt + \ldots$$

$$+ \mu_i^{n-1} \int_a^b k_{n-1}(z, t)\, \phi(t)\, dt + \phi(z) \qquad \ldots\ldots(9)$$

From (6) and (9), we have

$$\mu_i \int_a^b k(z, x)\, \phi_i(x)\, dx = \phi_i(z) \qquad \ldots\ldots(10)$$

If $\phi_i(x) \neq 0$, then μ_i is a characteristic constant of $k(x, t)$ and $\phi_i(x)$ is a fundamental function of $k(x, t)$ belonging to μ_i. Therefore, there exists a value ν of m, such that $\mu_i = \lambda_\nu$ and $\mu = \lambda_\nu^n$. Also, $\phi_i(x)$ being a fundamental function of $k(x, t)$ can be expressed as

$$\phi_i(x) = C_{i1}\, f_{i1} + C_{i2}\, f_{i2} + \ldots\ldots + C_{ik}\, f_{ik} \qquad \ldots\ldots(11)$$

But, $\phi_i(x) \neq 0$, otherwise from (7), we get $\phi(x) = 0$, which is a contradiction.

Therefore $\qquad \mu = \lambda_v^n \qquad \ldots\ldots(12)$

Using (11) and (12), we have

$$\phi(x) = C_{m1} f_{m1}(x) + C_{m2} f_{m2}(x) + \ldots\ldots + C_{mr}\, f_{mr}(x)$$

Hence, the iterated kernel $k_n(x, t)$ has no other characteristic constants than λ_m^n.

3.9 SOLUTION OF THE FREDHOLM INTEGRAL EQUATION OF FIRST KIND

[MEERUT-2005]

Let us consider the equation

$$F(x) = \int_a^b k(x, t)\, \phi(t)\, dt \qquad \ldots\ldots(1)$$

where $k(x, t)$ is continuous, real and symmetric.

The equation (1) has no continuous solution unless $F(x)$ is expressed as a linear combination of the characteristic function corresponding to the associated homogeneous equation of second kind.

$$\phi(x) = \lambda \int_a^b k(x, t)\, \phi(t)\, dt \qquad \ldots\ldots(2)$$

Let us assume that (1) has a continuous solution, then $F(x)$ can be expanded in a series as follows

$$F(x) = \sum f_n\, g_n\, (x) \quad : \; a \le x \le b \qquad \ldots\ldots(3)$$

where $f_n = \int_a^b F(x)\, g_n(x)\, dx \qquad \ldots\ldots(4)$

and g_n is the n^{th} characteristic function of (2).

Consider the homogeneous integral equation is

$$g_n(x) = \lambda_n \int_a^b k(x, t)\, g_n(t)\, dt \qquad \ldots\ldots(5)$$

From (1), (3) and (5), we have

$$\int_a^b k(x, t)\, \phi(t)\, dt = \sum \lambda_n\, f_n \int_a^b k(x, t)\, g_n(t)\, dt$$

or $$\int_a^b k(x, t)\, \phi(t)\, dt = \sum_n \lambda_n\, f_n\, g_n(t)\, dt = 0 \qquad \ldots\ldots(6)$$

or $$\int_a^b k(x, t)\, \psi(t)\, dt = 0 \qquad \ldots\ldots(7)$$

where $\phi(x) = \psi(x) = \sum \lambda_n\, f_n\, g_n(x) \qquad \ldots\ldots(8)$

which is the solution of (7).

Now, since the equation (1) contains a continuous solution, which must be of the form (8). From (7), we can say that either it is satisfied by the trivial function $\psi(x) = 0$ or it possesses infinitely many solutions.

Multiplying (7) by $g_n(x)$ and integrating w.r.t. x over (a, b), we have

$$\int_a^b g_n(x) \left\{ \int_a^b k(x, t)\, \psi(t)\, dt \right\} dx = 0$$

$$\Rightarrow \quad \int_a^b \psi(t) \left\{ \int_a^b k(t, x)\, g_n(x)\, dx \right\} dt = 0$$

$$\Rightarrow \quad \frac{1}{\lambda_n} \int_a^b \psi(t)\, g_n(t)\, dt = 0 \qquad \ldots(9)$$

Therefore, if the equation (7) contains a non-trivial solution, then that solution must be orthogonal to all the characteristic function g_n. If the set of functions are finite then there exists infinitely many linearly independent functions and if the functions g_n possess an infinite complete set, then no continuous non-trivial function can be simultaneously orthogonal to all functions of the set. Hence, in this case the function $\phi(x) = 0$ in (8).

3.10 SCHMIDT'S SOLUTION OF THE NON-HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF SECOND KIND

[MEERUT-1997,99,2000,01,02,04,05(BP),06]

Consider a non-homogeneous Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b k(x, t)\, y(t)\, dt \qquad \text{......(1)}$$

where $k(x, t)$ is continuous, real and symmetric and λ is not an eigen value.

Now, equation (1) can be written as

$$y(x) - f(x) = \lambda \int_a^b k(x, t)\, y(t)\, dt \qquad \text{......(2)}$$

Then, by Hilbert-Schmidt theorem, we have

$$y(x) - f(x) = \sum_{m=1} a_m\, \phi_m(x) \qquad a \le x \le b \qquad \text{......(3)}$$

where $\phi_m(x)\ [m = 1, 2, ...]$ are the normalized eigen functions of homogeneous integral equation

$$y(x) = \lambda \int_a^b k(x, t)\, y(t)\, dt \qquad \text{......(4)}$$

Let $\lambda_m (m \in \mathbf{N})$ be the corresponding eigen values of (4)

Also, let $\lambda \ne \lambda_m \quad \forall\ m \in \mathbf{N}$ ......(5)

Since, $\phi_m(x)$, $(m \in \mathbf{N})$ are normalized, then, we have

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \begin{cases} 0, & m \ne n \\ 1, & m = n \end{cases} \qquad \text{......(6)}$$

Multiplying both sides of (3) by $\phi_m(x)$ and then integrating w.r.t. x from a to b, we have

$$\int_a^b y(x)\, \phi_m(x)\, dx - \int_a^b f(x)\, \phi_m(x)\, dx = a_1 \int_a^b \phi_1(x)\, \phi_m(x)\, dx + ...$$
$$+ a_m \int_a^b \phi_m(x)\, \phi_m(x)\, dx + ... \qquad \text{......(7)}$$

Let $\quad C_m = \int_a^b y(x)\, \phi_m(x)\, dx$ ......(8)

and $\quad f_m = \int_a^b f(x)\, \phi_m(x)\, dx$ ......(9)

Using (6), (8) and (9) in (7), we get

$$C_m - f_m = a_m \qquad \text{......(10)}$$

Now, multiplying both sides of (1) by $\phi_m(x)$ and then integrating w.r.t. x, we get

$$\int_a^b y(x)\,\phi_m(x)\,dx = \int_a^b f(x)\,\phi_m(x)\,dx + \lambda\int_a^b\left\{\int_a^b k(x,t)\,y(t)\,dt\right\}\phi_m(x)\,dx$$

$$\Rightarrow \quad C_m = f_m + \lambda\int_a^b y(t)\left\{\int_a^b k(x,t)\,\phi_m(x)\,dx\right\}dt$$

$$= f_m + \lambda\int_a^b y(t)\left\{\int_a^b k(x,t)\,\phi_m(x)\,dx\right\}dt \qquad \text{......(11)}$$

Since $\phi_m(x)$ is eigen function corresponding to the eigen value λ_m of (4), therefore by definition, we have

$$\phi_m(x) = \lambda_m\int_a^b k(x,t)\,\phi_m(t)\,dt = \lambda_m\int_a^b k(x,z)\,\phi_m(z)\,dz$$

So, $\quad \phi_m(t) = \lambda_m\int_a^b k(t,z)\,\phi_m(z)\,dz = \lambda_m\int_a^b k(t,x)\,\phi_m(x)\,dx$

i.e., $$\int_a^b k(t,x)\,\phi_m(x)\,dx = \frac{\phi_m(t)}{\lambda_m} \qquad \text{......(12)}$$

Put this value in (11), we get

$$C_m = f_m + \lambda\int_a^b \frac{y(t)\,\phi_m(t)}{\lambda_m}\,dt = f_m + \frac{\lambda}{\lambda_m}\int_a^b y(x)\,\phi_m(x)\,dx$$

$$= f_m + \frac{\lambda\,C_m}{\lambda_m} \qquad \text{[Using (8)]} \qquad \text{......(13)}$$

From (10), we have

$$C_m = a_m + f_m \qquad \text{......(14)}$$

Using (13) and (14), we get

$$a_m + f_m = f_m + \frac{\lambda}{\lambda_m}(a_m + f_m) \quad \Rightarrow \quad a_m\left(1 - \frac{\lambda}{\lambda_m}\right) = \frac{\lambda}{\lambda_m}\cdot f_m$$

$$\Rightarrow \quad a_m(\lambda_m - \lambda) = \lambda\, f_m$$

i.e., $$a_m = \frac{\lambda}{\lambda_m - \lambda}\, f_m \qquad [\lambda_m \neq \lambda] \qquad \text{......(15)}$$

Put this value in (3), we get

$$y(x) - f(x) = \sum \frac{\lambda\, f_m}{\lambda_m - \lambda}\,\phi_m(x)$$

$$\Rightarrow \quad y(x) = f(x) + \lambda\sum_m \frac{f_m}{\lambda_m - \lambda}\,\phi_m(x) \qquad \text{......(16)}$$

From (9), we have

$$f_m = \int_a^b f(t)\,\phi_m(t)\,dt \qquad \text{......(17)}$$

Put this value of f_m in (16), we get

$$y(x) = f(x) + \lambda\sum_m \frac{\phi_m(x)}{\lambda_m - \lambda}\int_a^b f(t)\,\phi_m(t)\,dt$$

$$\Rightarrow \quad y(x) = f(x) + \lambda \int_a^b \left[\sum_m \frac{\phi_m(x)\,\phi_m(t)}{\lambda_m - \lambda} \right] f(t)\,dt \qquad \text{......(18)}$$

DISCUSSION

Case I : Unique Solution : If condition (5) is satisfied, then from (15), we can find the well defined value of a_m. Put this value of a_m in (3). Hence, solution given by (16) exists uniquely if and only if λ does not take on an eigen value.

Case II : No Solution : If λ_k be the k^{th} eigen value then, let $\lambda = \lambda_k$ and $f_k \neq 0$, i.e., $\int_a^b f(x)\,\phi_k(x)\,dx \neq 0$, i.e., $\phi_k(x)$ is not orthogonal to f(x).

Therefore, we find that no solution exists, since the term $\dfrac{f_k\,\phi_k(x)}{\lambda_k - \lambda}$ is not defined.

Case III : Infinitely Many Solutions Exist : Let $\lambda = \lambda_k$, where λ_k is the k^{th} eigen value and also let

$$f_k = 0, \text{ i.e., } \int_a^b f(x)\,\phi_k(x)\,dx = 0$$

i.e., $\phi_k(x)$ is orthogonal to $f(x)$.

Then for m = k, equation (13) gives

$$C_k = 0 + \frac{\lambda}{\lambda} C_k \qquad [\because \text{ given } \lambda = \lambda_k]$$

$\therefore \quad C_k = C_k$, which is a trivial identity.

Therefore, from (15), the coefficient a_k of $\phi_k(x)$ in (16), which formally assumes the form $\dfrac{0}{0}$ is truly arbitrary. Hence, in this case solution (16) can be written as follows.

$$y(x) = f(x) + A\,\phi_k(x) + \lambda \sum \frac{f_m}{\lambda_m - \lambda}\,\phi_m(x) \qquad [m \neq k]$$

where A is any constant.

Remark

❶ From expression (18), we can write

$$R(x, t; \lambda) = \sum \frac{\phi_m(x)\phi_m(t)}{\lambda_m - \lambda}$$

Here, $R(x, t\,; \lambda)$ is known as Resolvent kernel.

SOLVED EXAMPLES

GP 5m **EXAMPLE 1** *solve V.I. eqn $y(x) = 1+x - \int_0^x y(t)dt$*

$y(x) = 1+\int_0^x y(t)dt$

Solve the following homogeneous Fredholm integral equation using Schmidt solution.

$$f(x) = \lambda \int_0^1 e^x\, e^t\, f(t)\,dt \qquad \text{[KANPUR-2005]}$$

Solution : Here, the given integral equation is

$$f(x) = \lambda \int_0^1 e^x e^t f(t)\, dt \qquad \ldots\ldots(1)$$

Let $C = \int_0^1 e^t f(t)\, dt$

Then (1) gives

$$f(x) = \lambda C e^x \qquad \ldots\ldots(2)$$

Using (1) and (2), we get

$$\lambda C e^x = \lambda e^x \int_0^1 e^t(\lambda C e^t)\, dt \Rightarrow \lambda C e^x = \frac{1}{2}\lambda^2 .C.e^x (e^2 - 1)$$

i.e., $\lambda C\{2 - \lambda(e^2 - 1)\} = 0$

if $\lambda = 0$ or $C = 0$, then $f = 0$. Let us assume neither $C = 0$ nor $\lambda = 0$.
Then the required eigen value λ is given by

$$\lambda = \frac{2}{(e^2 - 1)}$$

Putting this value in (2), we get

$$f(x) = \frac{2Ce^x}{e^2 - 1}$$

Hence, the required eigen function is $f(x) = e^x$ to the corresponding eigen value $\frac{2}{e^2 - 1}$.

EXAMPLE 2

Solve the integral equation of the first kind

$$u(x) = \int_0^1 k(x, t)\, u(t)\, dt$$

[MEERUT-1995, 96,97,98,2006(BP)]

where $k(x, t) = \begin{cases} x(1 - t), & x < t \\ t(1 - x), & x > t \end{cases}$

Solution : By proceeding in the usual procedure, we can reduce the given problem to the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda u = 0 \qquad \ldots\ldots(1)$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$ ......(2)

The solution of (1) is given by $u(x) = C_1 \cos\sqrt{\lambda}\,.\,x + C_2 \sin\sqrt{\lambda}\,.\,x$.

using given boundary conditions, we get $C_1 = 0$ and $\lambda = n^2\pi^2$ $(C_2 \neq 0)$

Therefore, the required eigen values λ_n are given by $\lambda_n = n^2\pi^2, n = 1, 2, 3, \ldots\ldots$

Also, the corresponding eigen functions $u_n(x)$ are given by

$$u_n(x) = \sqrt{2}\sin n\pi x, \quad n = 1, 2, 3, \ldots\ldots \qquad C_2 = 1 \text{ (let)}$$

EXAMPLE 3

Find the eigen values and eigen functions of the homogeneous integral equation

$$u(x) = \lambda \int_0^{\pi} k(x,t)\, u(t)\, dt$$

where $k(x,t) = \begin{cases} \cos x \sin t, & 0 \le x \le t \\ \cos t \sin x, & t \le x \le \pi \end{cases}$ **[MEERUT-1995,96]**

Solution : The given equation can be written as

$$u(x) = \lambda \int_0^{x} k(x,t)\, u(t)\, dt + \lambda \int_x^{\pi} k(x,t)\, u(t)\, dt$$

$$= \lambda \sin x \int_0^{x} \cos t\, u(t)\, dt + \lambda \cos x \int_x^{\pi} \sin t\, u(t)\, dt \qquad \text{......(1)}$$

Differentiating both sides of (1), we have

$$u'(x) = \lambda \cos x \int_0^{x} \cos t\, u(t)\, dt + \lambda \sin x \cos x\, u(x) - \lambda \sin x \int_x^{\pi} \sin t\, u(t)\, dt - \lambda \sin x \cos x\, u(x)$$

$$= \lambda \cos x \int_0^{x} \cos t\, u(t)\, dt - \lambda \sin x \int_x^{\pi} \sin t\, u(t)\, dt$$

Again differentiating, we have

$$u''(x) = \lambda u(x) - \left[\lambda \sin x \int_0^{x} \cos t\, u(t)\, dt + \lambda \cos x \int_x^{\pi} \sin t\, u(t)\, dt \right]$$

$$\Rightarrow \quad u''(x) - (\lambda - 1)\, u(x) = 0$$

Therefore, the given integral equation reduces to the boundary value problem

$$u''(x) - (\lambda - 1)\, u(x) = 0 \qquad \text{......(2)}$$

with boundary conditions

$$u(\pi) = 0, \quad u'(0) = 0 \qquad \text{......(3)}$$

Now, there are following cases :

Case I : **When** $\lambda - 1 = 0$

Then equation (2) gives

$$u''(x) = 0 \quad \Rightarrow \quad u(x) = Ax + B$$

Using the given boundary conditions, we have

$$A = B = 0$$

$\therefore$ the given integral equation has only the trivial solution.

$$u(x) = 0, \quad \lambda = 1.$$

Case II : **When** $\lambda - 1 > 0$

Then, equation (3) gives

$$u(x) = A \cosh \sqrt{(\lambda - 1)}.x + B \sinh \sqrt{(\lambda - 1)}.x$$

Using boundary conditions, we get

$$A \cosh \pi\sqrt{(\lambda - 1)} + B \sinh \pi\sqrt{(\lambda - 1)} = 0, \quad B = 0$$

or $A = 0, B = 0, \lambda - 1 > 0$

Therefore, the given integral equation has no eigen values and hence no eigen function.

Case III : When $\lambda - 1 < 0$

Then solution of (2) is given by

$$u(x) = A\cos\sqrt{(1-\lambda)}.x + B\sin\sqrt{(1-\lambda)}\,x$$

Using boundary conditions, we get

$$A\cos\pi\sqrt{1-\lambda} + B\sin\pi\sqrt{1-\lambda} = 0 \quad \text{and} \quad B\sqrt{1-\lambda} = 0 \qquad \text{......(4)}$$

For non trivial solution of the system, we must have

$$D(\lambda) = \begin{vmatrix} \cos\pi\sqrt{(1-\lambda)} & \sin\pi\sqrt{(1-\lambda)} \\ 0 & \sqrt{(1-\lambda)} \end{vmatrix} = \sqrt{1-\lambda}\,\cos\pi\sqrt{1-\lambda} = 0$$

$\therefore$ the required eigen values are

$$\pi\sqrt{(1-\lambda)} = \frac{\pi}{2} + n\pi, \quad n \in \mathbf{Z}$$

Solve for λ, we get

$$\lambda = 1 - \left(n + \frac{1}{2}\right)^2$$

Therefore, from (4), we get A = 0 and B = 0.
which has an infinite number of non-zero solutions.
Hence, the given integral equation has an infinite number of solutions of the form

$$u(x) = \cos\left(n + \frac{1}{2}x\right)$$

EXAMPLE 4

Using Hilbert-Schmidt theorem, solve the following integral equation

$$u(x) = (x+1)^2 + \int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt$$

[GARHWAL-2004]

Solution : Here, the given equation is

$$u(x) = (x+1)^2 + \int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt \qquad \text{......(1)}$$

Comparing with standard equation, we get

$$f(x) = (x+1)^2 \quad and \quad \lambda = 1 \qquad \text{......(2)}$$

Firstly, we will find the eigen values and the corresponding normalized eigen functions of

$$u(x) = \lambda\int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt \qquad \text{......(3)}$$

Equation (3) can be rewritten as

$$u(x) = \lambda x \int_{-1}^{1} t\, u(t)\, dt + \lambda x^2 \int_{-1}^{1} t^2 . u(t)\, dt \qquad ......(4)$$

$$= \lambda C_1 x + \lambda C_2 x^2 \qquad ......(5)$$

where $C_1 = \int_{-1}^{1} t . u(t)\, dt$ ......(6)

and $C_2 = \int_{-1}^{1} t^2 . u(t)\, dt$ ......(7)

from (5), we can find

$$u(t) = \lambda C_1 . t + \lambda C_2 . t^2 \qquad ......(8)$$

Put this value in (6), we get

$$C_1 = \int_{-1}^{1} t(\lambda C_1 t + \lambda C_2 t^2)\, dt \Rightarrow C_1 = C_1\lambda \left[\frac{t^3}{3}\right]_{-1}^{1} + C_2\lambda\left[\frac{t^4}{4}\right]_{-1}^{1}$$

$$\Rightarrow C_1 = \frac{2C_2\lambda}{3} + 0 \Rightarrow C_1\left(1 - \frac{2\lambda}{3}\right) + 0.C_2 = 0 \qquad ......(9)$$

Similarly, from (7), we can find

$$C_2 = \int_{-1}^{1} t^2(\lambda C_1 t + \lambda C_2 t^2)\, dt = C_1\lambda\left[\frac{t^4}{4}\right]_{-1}^{1} + \lambda C_2\left[\frac{t^5}{5}\right]_{-1}^{1}$$

or $C_2 = 0 + \dfrac{2C_2\lambda}{5} \Rightarrow 0.C_1 + \left(1 - \dfrac{2\lambda}{5}\right)C_2 = 0$ ......(10)

For non-trivial solution of (9) and (10), we must have

$$D(\lambda) = \begin{vmatrix} 1 - \frac{2\lambda}{3} & 0 \\ 0 & 1 - \frac{2\lambda}{5} \end{vmatrix} = 0 \quad \text{or} \quad \left\{1 - \frac{2\lambda}{3}\right\}\left\{1 - \frac{2\lambda}{5}\right\} = 0$$

$$\Rightarrow \lambda = \frac{3}{2} \text{ or } \frac{5}{2}$$

Therefore, the required eigen values are

$$\lambda_1 = \frac{3}{2} \text{ and } \lambda_2 = \frac{5}{2} \qquad ......(11)$$

Now, we will proceed to find the required eigen functions

(i) **Corresponding to** $\lambda_1 = \frac{3}{2}$

Putting $\lambda = \lambda_1 = \frac{3}{2}$ in (9) and (10), we get

$$C_1.0 + 0.C_2 = 0 \text{ and } 0.C_1 + \left[1 - \frac{2}{5}.\frac{3}{2}\right]C_2 = 0$$

On solving, we get $C_2 = 0$ and C_1 is arbitrary.

Putting all these values in (5), the required eigen function $u_1(x)$ corresponding to

$\lambda_1 = \frac{3}{2}$ is given by

$$u_1(x) = \frac{3}{2} C_1 x$$

Let us set $\frac{3}{2} C_1 = 1$, then $u_1(x) = x$

Now, the corresponding normalized eigen function $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{u_1(x)}{\left[\int_{-1}^{1} \{u_1(x)\}^2 dx\right]^{1/2}} = \frac{x}{\left[\int_{-1}^{1} x^2 dx\right]^{1/2}} = \frac{x}{\left\{\left[\frac{x^3}{3}\right]_{-1}^{1}\right\}^{1/2}}$$

$$= \frac{x}{\sqrt{2/3}} = x \cdot \left(\frac{3}{2}\right)^{1/2} = \frac{x\sqrt{6}}{2} \qquad \text{......(12)}$$

(ii) Corresponding to $\lambda_2 = \frac{5}{2}$

Putting $\lambda = \lambda_2 = \frac{5}{2}$ in (9) and (10), we get

$$\left[1 - \frac{2}{3} \cdot \frac{5}{2}\right] C_1 + 0.C_2 = 0 \quad \text{and} \quad 0.C_1 - 0.C_2 = 0$$

On solving, we get $C_1 = 0$ and C_2 is arbitrary.
Putting these values in (5), we get

$$u_2(x) = \frac{5}{2} C_2 . x^2$$

Let $\frac{5}{2} C_2 = 1$, then $u_2(x) = x^2$

Now, the corresponding normalized eigen function $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{u_2(x)}{\left[\int_{-1}^{1} \{u_2(x)\}^2 dx\right]^{1/2}} = \frac{x^2}{\left[\int_{-1}^{1} x^4 dx\right]^{1/2}} = \frac{\sqrt{10}}{2} x^2 \qquad \text{......(13)}$$

Also, $f_1 = \int_{-1}^{1} f(x) \phi_1(x) dx$

$$= \int_{-1}^{1} (x+1)^2 \left(\frac{\sqrt{6}}{2} . x\right) dx = \frac{\sqrt{6}}{2} \int_{-1}^{1} (x^2 + 2x + 1) x \, dx$$

$$= \frac{\sqrt{6}}{2} \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} = \frac{2\sqrt{6}}{3} \qquad \text{......(14)}$$

and $f_2 = \int_{-1}^{1} f(x) \phi_2(x) dx = \int_{-1}^{1} (x+1)^2 \left(\frac{\sqrt{10}}{2} . x^2\right) dx$

$$= \frac{\sqrt{10}}{2} \left[\frac{x^5}{2} + \frac{2x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{15}\sqrt{10} \qquad \text{......(15)}$$

From (2), $\lambda = 1$, also $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{5}{2}$.

Hence, $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Then (1) will have a unique solution given by

$$u(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$= (x+1)^2 + \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x) = (x+1)^2 + \frac{f_1\phi_1(x)}{\lambda_1 - 1} + \frac{f_2\phi_2(x)}{\lambda_2 - 1}$$

$$\Rightarrow \quad u(x) = (x+1)^2 + \frac{\left\{\left(\frac{2}{3}\right)\sqrt{6}\right\}\{\sqrt{6}/2\,x\}}{\frac{3}{2} - 1} + \frac{\left\{\left(\frac{8}{15}\right)\sqrt{10}\right\}\{\sqrt{10}/2\,x^2\}}{\frac{5}{2} - 1}$$

$$\Rightarrow \quad u(x) = (x+1)^2 + 4x + \frac{16}{9}x^2 = x^2 + 2x + 1 + 4x + \frac{16}{9}x^2$$

Hence, $u(x) = \frac{25}{9}x^2 + 6x + 1$

EXAMPLE 5

Solve the following symmetric integral equation with the help of Hilbert-Schmidt theorem

$$u(x) = 1 + \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt$$

Solution : The given integral equation is

$$u(x) = 1 + \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt \qquad \text{......(1)}$$

Comparing (1) with the following given standard equation

$$u(x) = f(x) + \lambda \int_0^{\pi} k(x,t)\, u(t)\, dt \qquad \text{......(2)}$$

We get $f(x) = 1,\ \lambda = \lambda,\ k(x,t) = \cos(x+t)$ ......(3)

Now, we will proceed to find eigen values and corresponding normalized eigen function of

$$u(x) = \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt \qquad \text{......(4)}$$

Equation (4) can be written as

$$u(x) = \lambda \int_0^{\pi} (\cos x \cos t - \sin x \sin t)\, u(t)\, dt$$

$$= \lambda \cos x \int_0^{\pi} \cos t\, u(t)\, dt - \lambda \sin x \int_0^{\pi} \sin t\, u(t)\, dt \qquad \text{......(5)}$$

Let $C_1 = \int_0^{\pi} \cos t\, u(t)\, dt$ ......(6)

$C_2 = \int_0^{\pi} \sin t\, u(t)\, dt$ ......(7)

Using (6) and (7), (5) gives

$$u(x) = \lambda C_1 \cos x - \lambda C_2 \sin x \qquad \ldots\ldots(8)$$

$$\Rightarrow \quad u(t) = \lambda C_1 \cos t - \lambda C_2 \sin t \qquad \ldots\ldots(9)$$

Putting the value of u(t), from (9) in (6), we get

$$C_1 = \int_0^{\pi} \cos t(\lambda C_1 \cos t - \lambda C_2 \sin t)\, dt$$

$$= \frac{\lambda C_1}{2} \int_0^{\pi} (1 + \cos 2t)\, dt - \frac{\lambda C_2}{2} \int_0^{\pi} \sin 2t\, dt$$

$$\Rightarrow \quad \frac{\lambda C_1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi} + \frac{\lambda C_2}{4}[\cos 2t]_0^{\pi}$$

$$\Rightarrow \quad C_1 = \frac{\lambda C_1 \pi}{2} \quad \text{or} \quad C_1(2 - \lambda\pi) + 0.C_2 = 0 \qquad \ldots\ldots(10)$$

Similarly, from (7), we get

$$C_2 = \int_0^{\pi} \sin t\, (\lambda C_1 \cos t - \lambda C_2 \sin t)\, dt$$

$$= \frac{\lambda C_1}{2} \int_0^{\pi} \sin 2t\, dt - \frac{\lambda C_2}{2} \int_0^{\pi} (1 - \cos 2t)\, dt$$

$$= -\frac{\lambda C_1}{4}[\cos 2t]_0^{\pi} - \frac{\lambda C_2}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$\Rightarrow \quad C_2 = -\frac{\lambda C_2 \pi}{2} \quad \text{or} \quad 0.C_1 + (2 + \lambda\pi)\, C_2 = 0 \qquad \ldots\ldots(11)$$

For non-trivial solution of (10) and (11), we must have

$$D(\lambda) = \begin{vmatrix} 2 - \lambda\pi & 0 \\ 0 & 2 + \lambda\pi \end{vmatrix} = 0$$

$$\Rightarrow \quad (2 - \lambda\pi)\,(2 + \lambda\pi) = 0 \quad \Rightarrow \quad \lambda = \frac{2}{\pi} \quad \text{or} \quad -\frac{2}{\pi}$$

$\therefore$ the required eigen values are given by $\lambda_1 = \frac{2}{\pi}$, $\lambda_2 = -\frac{2}{\pi}$.

Now, we find the eigen function corresponding to given values

(i) Corresponding to $\lambda = \lambda_1 = \frac{2}{\pi}$

Putting $\lambda = \lambda_1 = \frac{2}{\pi}$ in (10) and (11), we get

$$0.C_1 + 0.C_2 = 0 \quad \text{and} \quad 0.C_1 + 4.C_2 = 0$$

On solving, we get

$$C_2 = 0 \quad \text{and} \quad C_1 \text{ is arbitrary.}$$

Putting these values in (8), the required eigen function $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{u_1(x)}{\left[\int_0^{\pi} \{u_1(x)\}^2\, dx\right]^{1/2}} = \frac{\cos x}{\left[\int_0^{\pi} \cos^2 x\, dx\right]^{1/2}} = \frac{\cos x}{\left[\int_0^{\pi} \frac{1 + \cos 2x}{2}\, dx\right]^{1/2}}$$

$$= \frac{\cos x}{\left\{\left[\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)\right]_0^{\pi}\right\}^{1/2}} = \frac{\cos x}{\sqrt{\pi/2}} = \left(\frac{2}{\pi}\right)^{1/2} \cos x \qquad \text{......(13)}$$

(ii) Corresponding to $\lambda_2 = -\frac{2}{\pi}$

Putting $\lambda = \lambda_2 = -\frac{2}{\pi}$ in (10) and (11), we get

$$4C_1 + 0.C_2 = 0 \quad \text{and} \quad 0.C_1 + 0.C_2 = 0$$

On solving, we get

$$C_1 = 0 \text{ and } C_2 \text{ is arbitrary.}$$

Now, from (8), the required eigen function $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{u_2(x)}{\left[\int_0^{\pi}\{u_2(x)\}^2\right]^{1/2}} = \frac{\sin x}{\left[\int_0^{\pi}\sin^2 x\,dx\right]^{1/2}} = \frac{\sin x}{\left[\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx\right]^{1/2}}$$

$$= \frac{\sin x}{\left\{\left[\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\right]_0^{\pi}\right\}^{1/2}} = \frac{\sin x}{\sqrt{\pi/2}} = \left(\frac{2}{\pi}\right)^{1/2} \sin x \qquad \text{......(14)}$$

$$f_1 = \int_0^{\pi} f(x)\,\phi_1(x)\,dx = \int_0^{\pi}\cos x\left(\frac{2}{\pi}\right)^{1/2} dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2}[\sin x]_0^{\pi} = 0 \qquad \text{......(15)}$$

and $$f_2 = \int_0^{\pi} f(x)\,\phi_2(x)\,dx = \int_0^{\pi}\sin x\left(\frac{2}{\pi}\right)^{1/2} dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2}[-\cos x]_0^{\pi} = 2\left[\frac{2}{\pi}\right]^{1/2} \qquad \text{......(16)}$$

Here, we discuss the following cases

Case-I : Let $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Then, solution of (1) is given by

$$u(x) = f(x) + \lambda\sum_{m=1}^{2}\frac{f_m}{\lambda_m - \lambda}\phi_m(x) = f(x) + \frac{\lambda}{\lambda_1 - \lambda} f_1\,\phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2\,\phi_2(x)$$

$$\Rightarrow \quad u(x) = 1 + \left[\frac{\lambda\,\phi_1(x)}{(2/\pi) - \lambda}\right].0 + \frac{\lambda}{-(2/\pi) - \lambda}.2\left[\frac{2}{\pi}\right]^{1/2}.\left[\frac{2}{\pi}\right]^{1/2}.\sin x$$

$$\Rightarrow \quad u(x) = 1 - \frac{4\lambda\sin x}{2 + \lambda\pi} \qquad \text{......(17)}$$

Case II : Let $\lambda = \lambda_2 = -\frac{2}{\pi}$. Since $f_2 \neq 0$, therefore (1) has no solution.

Case III : Let $\lambda = \lambda_1 = \frac{2}{\pi}$. Since $f_1 = 0$, there exist infinitely many solutions given by

$$y(x) = f(x) + A\,\phi_1(x) + \lambda \sum_{m=1}^{2}{}' \frac{f_m}{\lambda_m - \lambda}\phi_m(x) \quad (\lambda_m \neq m)$$

$$\Rightarrow \quad y(x) = f(x) + A\phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2\,\phi_2(x)$$

or
$$u(x) = 1 + A\left(\frac{2}{\pi}\right)^{1/2} \cos x + \frac{(2/\pi)}{\frac{2}{\pi} - \frac{2}{\pi}} \cdot 2 \cdot \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2} \sin x$$

$$\Rightarrow \quad u(x) = 1 + C\cos x - \frac{2\sin x}{\pi}, \text{ where } C\left(= \frac{A\sqrt{2}}{\sqrt{\pi}}\right) \text{ is an arbitrary constant.}$$

EXAMPLE 6

Find the solution of the symmetric integral equation

$$u(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt$$

using Hilbert-Schmidt theorem. **[MEERUT-2000, 01, GARHWAL-2000]**

Solution : Here, the given equation is

$$u(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt \qquad \text{......(1)}$$

On comparing with the standard equation, we get

$$f(x) = x^2 + 1 \quad \text{and} \quad \lambda = \frac{3}{2} \qquad \text{......(2)}$$

Now, we will proceed to find the eigen values and corresponding normalized eigen functions of

$$u(x) = \lambda\int_{-1}^{1}(xt + x^2t^2)\,u(t)\,dt \qquad \text{......(3)}$$

Now, proceed same as example (5), we get, the required eigen values as

$$\lambda_1 = \frac{3}{2} \quad \text{and} \quad \lambda_2 = \frac{5}{2}.$$

and the corresponding eigen functions are given by

$$\phi_1(x) = \frac{(x\sqrt{6})}{2} \quad \text{and} \quad \phi_2(x) = \frac{(x^2\sqrt{10})}{2} \qquad \text{......(4)}$$

Now, $$f_1 = \int_{-1}^{1} f(x)\,\phi_1(x)\,dx = \int_{-1}^{1}(x^2+1)\frac{x\sqrt{6}}{2}\,dx = \frac{\sqrt{6}}{2}\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{-1}^{1} = 0 \qquad \text{......(5)}$$

and $$f_2 = \int_{-1}^{1} f(x)\,\phi_2(x)\,dx$$

$$= \int_{-1}^{1}(x^2+1)\frac{x^2\sqrt{10}}{2}\,dx = \frac{\sqrt{10}}{2}\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{8\sqrt{10}}{15} \qquad \text{......(6)}$$

Since $\lambda = \lambda_1$ and $f_1 = 0$, therefore, there exist infinitely many solutions of (1), which are given by

$$u(x) = f(x) + A\phi_1(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \quad [m \neq 1] \qquad \text{......(7)}$$

Hence, (7) gives

$$u(x) = f(x) + A\phi_1(x) + \lambda \frac{f_2}{\lambda_2 - \lambda} \phi_2(x)$$

$$= x^2 + 1 + A\left(\frac{x\sqrt{6}}{2}\right) + \frac{3}{2} \cdot \frac{(8/15).\sqrt{10}}{(5/2)-(3/2)} \cdot \frac{x^2\sqrt{10}}{2}$$

$\Rightarrow \quad y(x) = x^2 + 1 + Cx + 4x^2 = 5x^2 + Cx + 1$, where $C = (A\sqrt{6})/2$.

EXAMPLE 7

Solve the following symmetric integral equation

$$u(x) = e^x + \lambda \int_0^1 k(x, t)\, u(t)\, dt$$

where, $k(x, t) = \begin{cases} \dfrac{\sinh x \sinh (t-1)}{\sinh 1}; & 0 \le x \le t \\ \dfrac{\sinh t \sinh (x-1)}{\sinh 1}; & t \le x \le 1 \end{cases}$

[MEERUT-1995, 2000, GARHWAL-1999, KANPUR-2005]

Solution : Here, the given equation is

$$u(x) = e^x + \lambda \int_0^1 k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

where, $k(x, t) = \begin{cases} \dfrac{\sinh x \sinh (t-1)}{\sinh 1}; & 0 \le x \le t \\ \dfrac{\sinh t \sinh (x-1)}{\sinh 1}; & t \le x \le 1 \end{cases}$ ......(2)

Comparing the given equation with

$$u(x) = f(x) + \lambda \int_0^1 k(x, t)\, u(t)\, dt \qquad \text{......(3)}$$

we get $f(x) = e^x$ ......(4)

Now, we will proceed to find the eigen values and the corresponding functions of the homogeneous integral equation.

$$u(x) = \lambda \int_0^1 k(x, t)\, u(t)\, dt \qquad \text{......(5)}$$

Now, (5) can be written as

$$u(x) = \lambda \left[\int_0^x k(x, t)\, u(t)\, dt + \int_x^1 k(x, t)\, u(t)\, dt \right]$$

$$= \int_0^x \frac{\lambda \sinh t \sinh(x-1).u(t)}{\sinh 1}\, dt + \int_x^1 \frac{\lambda \sinh x \sinh(t-1).u(t)}{\sinh 1}\, dt \qquad \text{......(6)}$$

Using Leibnitz's rule, differentiating both sides of (6) w.r.t. x, we get

$$u'(x) = \int_0^x \frac{\lambda \sinh t \cosh(x-1).u(t)}{\sinh 1}\,dt + \frac{\lambda \sinh x \sinh(x-1).u(x)}{\sinh 1}$$

$$+ \int_x^1 \frac{\lambda \cosh x \sinh(t-1).u(t)}{\sinh 1}\,dt - \frac{\lambda \sinh x \sinh(x-1).u(x)}{\sinh 1}$$

$$= \int_0^x \frac{\lambda \sinh t \cosh(x-1)}{\sinh 1}\,u(t)\,dt + \int_x^1 \frac{\lambda \cosh x \sinh(t-1).u(t)}{\sinh 1}\,dt \qquad \text{......(7)}$$

Again differentiating in the same way, we get

$$u''(x) = \int_0^x \frac{\lambda \sinh t \sinh(x-1).u(t)}{\sinh 1}\,dt + \frac{\lambda \sinh x \cosh(x-1)}{\sinh 1}\,u(x)$$

$$+ \int_x^1 \frac{\lambda \sinh x \sinh(t-1)}{\sinh 1}\,u(t)\,dt - \frac{\lambda \cosh x \sinh(x-1).u(x)}{\sinh 1}$$

$$= u(x) + \frac{\lambda\, u(x)}{\sinh 1}\left[\sinh x \cosh(x-1) - \cosh x \sinh(x-1)\right]$$

$\Rightarrow \qquad u''(x) = u(x) + \dfrac{\lambda\, u(x)}{\sinh 1}\sinh\{x-(x-1)\}$

$\Rightarrow \qquad u''(x) - (1+\lambda)\,u(x) = 0$ ......(8)

Putting x = 0 in (6), we get u(0) = 0 ......(9)

Putting x = 1 in (6), we get u(1) = 0 ......(10)

Now, we have the following cases :

Case I : Let $1 + \lambda = 0$, i.e., $\lambda = -1$

Putting this value in (8), we get

$u''(x) = 0$

$\Rightarrow \qquad u(x) = Ax + B$ ......(11)

Using boundary conditions (9) and (10) in (11), we get

$B = 0$ ......(12)

and $\qquad A + B = 0$ ......(13)

On solving, we get A = B = 0.

Therefore, (11) gives $u(x) = 0$, which is not an eigen function and therefore $\lambda = -1$ is not an eigen value.

Case II : Let $\lambda + 1 = k^2, \quad k \neq 0$

Putting this value in (8), we get $u'' - k^2\, u(x) = 0$.

whose general solution is given by

$u(x) = Ae^{kx} + Be^{-kx}$ ......(14)

Using (9) and (10), we get

$0 = A + B$ ......(15)

and $\qquad 0 = Ae^{k} + Be^{-k}$ ......(16)

On solving, we get A = B = 0. Therefore, from (14), we get $u(x) = 0$, which is not an eigen function.

Case III : Let $\lambda + 1 = -k^2 \quad (k \neq 0)$

In this case equation (8) gives

$$u''(x) + k^2 u(x) = 0$$

with general solution

$$u(x) = A\cos kx + B\sin kx \qquad \text{......(17)}$$

Using (9) and (10) in (17), we get

$$0 = A \qquad \text{......(18)}$$

and $$0 = A\cos k + B\sin k \qquad \text{......(19)}$$

On solving, we get

$$B\sin k = 0 \qquad \text{......(20)}$$

Take $B \neq 0$ (otherwise A = 0, B = 0), then (20) gives $\sin k = 0$, which implies

$$k = n\pi, \qquad n \in \mathbf{Z}$$

Therefore,

$$1 + \lambda = -k^2 = -n^2\pi^2 \text{ or } \lambda = -(1 + n^2\pi^2)$$

Hence, the required eigen values are given by

$$\lambda_n = -(1 + n^2\pi^2), \; n \in \mathbf{N}$$

Putting A = 0 and k = $n\pi$ in (17), we get $u(x) = B\sin n\pi x$.

Setting B = 1, we get

$$u_n(x) = \sin n\pi x, \qquad n \in \mathbf{N}$$

Now, we shall find the eigen function $\phi_n(x)$ such that

$$\phi_n(x) = \frac{u_n(x)}{\left[\int_0^1 \{u_n(x)\}^2 dx\right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x \, dx\right]^{1/2}}$$

$$= \frac{\sin n\pi x}{\left[\int_0^1 \frac{1 - \cos 2n\pi x}{2} dx\right]^{1/2}} = \frac{\sin n\pi x}{\left\{\left[\frac{1}{2}\left(x - \frac{\sin 2n\pi x}{2n\pi}\right)\right]_0^1\right\}^{1/2}}$$

$$= \frac{\sin n\pi x}{1/\sqrt{2}} = \sqrt{2}.\sin n\pi x$$

Also, $$f_n = \int_0^1 f(x)\,\phi_n(x)\,dx$$

$$= \int_0^1 e^x.(\sqrt{2}.\sin n\pi x)\,dx$$

$$= \sqrt{2}\left[\frac{e^x}{1 + n^2\pi^2}(\sin n\pi x - n\pi\cos n\pi x)\right]_0^1$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{\sqrt{2}}{1+n^2\pi^2}\left[-en\pi \cos n\pi - (-n\pi)\right]$$

$$= \frac{n\pi\sqrt{2}\,\{1-e(-1)^n\}}{1+n^2\pi^2}, \quad n \in \mathrm{N} \qquad \text{......(21)}$$

Now, we discuss the following cases :

Case (a) : Let λ be not an eigen value, i.e.,

$\lambda \neq \lambda_n$, for n = 1, 2, 3,.........

Then, unique solution of (1) is given by

$$u(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(x)$$

$$= e^x + \lambda \sum_{n=1}^{\infty} \frac{n\pi\sqrt{2}\,\{1-e(-1)^n\}}{1+n^2\pi^2} \cdot \frac{\sqrt{2}.\sin n\pi x}{-n^2\pi^2 - 1 - \lambda}$$

$$= e^x - 2\lambda\pi \sum_{n=1}^{\infty} \frac{n\{1-e(-1)^n\}\sin n\pi x}{(1+n^2\pi^2)(1+n^2\pi^2+\lambda)}$$

Case (b) : Let $\lambda = \lambda_n = -1 - n^2\pi^2$, $n \in \mathrm{N}$. Then, since from (21), $f_n \neq 0$. for $n \in \mathrm{N}$. Hence (1) have no solution.

EXAMPLE 8

Solve the symmetric integral equation

$$u(x) = f(x) + \lambda \int_a^b k(x)\,k(t)\,u(t)\,dt$$ **[GARHWAL-2000]**

Solution : Here, the given equation is

$$u(x) = f(x) + \lambda \int_a^b k(x)\,k(t)\,u(t)\,dt \qquad \text{......(1)}$$

Now, we will proceed to find the eigen values and the corresponding normalized eigen functions of homogeneous equation.

$$u(x) = \lambda \int_a^b k(x)\,k(t)\,u(t)\,dt \qquad \text{......(2)}$$

Equation (1) can be written as

$$u(x) = \lambda\, k(x) \int_a^b k(t)\,u(t)\,dt \qquad \text{......(3)}$$

Let us assume

$$C = \int_a^b k(t)\,u(t)\,dt \qquad \text{......(4)}$$

Then, (3) becomes

$$u(x) = \lambda C k(x) \qquad \text{......(5)}$$

$$\Rightarrow \quad u(t) = \lambda C k(t) \qquad \text{......(6)}$$

Putting this value of u(t) in (4), we get

$$C = \int_a^b \left(k(t) . \lambda C k(t)\right) dt$$

$$= C\left[1 - \lambda \int_a^b \{k(t)\}^2 \, dt\right] = 0 \qquad \text{......(7)}$$

For non zero solution of (5), we must have $C \neq 0$.

Then (7) gives

$$1 - \lambda \int_a^b \{k(t)\}^2 \, dt = 0$$

i.e.,
$$\lambda_1 = \frac{1}{\int_a^b \{k(t)\}^2 \, dt} \qquad \text{......(8)}$$

which is the only eigen value of (2). Putting the value of λ in (5), we get

$$u_1(x) = \frac{Ck(x)}{\int_a^b \{k(t)\}^2 \, dt}$$

Setting, $\dfrac{C}{\left[\int_a^b \{k(t)\}^2 \, dt\right]} = 1$, we take

$$u_1(x) = k(x) \qquad \text{......(9)}$$

Hence, the corresponding normalized eigen function $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{u_1(x)}{\left[\int_a^b \{u_1(x)\}^2 \, dx\right]^{1/2}} = \frac{k(x)}{\left[\int_a^b \{k(x)\}^2 \, dx\right]^{1/2}} \qquad \text{......(10)}$$

and
$$f_1 = \int_a^b f(x) \, \phi_1(x) \, dx = \int_a^b \left[\{f(x) \, k(x)\} \Big/ \left[\int_a^b \{k(x)\}^2 \, dx\right]^{1/2}\right] dx$$

$$= \left[\int_a^b f(x) \, k(x) \, dx\right] \Big/ \left[\int_a^b \{k(x)\}^2 \, dx\right]^{1/2} \qquad \text{......(11)}$$

Now, we discuss the following cases :

Case I : Let $\lambda \neq \lambda_1$

Then, equation (1) has the unique solution given by

$$u(x) = f(x) + \lambda \sum_{m=1}^{1} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$= f(x) + \frac{\lambda}{\lambda_1 - \lambda} f_1 \, \phi_1(x)$$

$$= f(x) + \frac{\lambda}{\left\{\left[\int_a^b \{k(x)\}^2 \, dx\right]^{-1} - \lambda\right\}} \cdot \frac{\int_a^b f(x) \, k(x) \, dx}{\left[\int_a^b \{k(x)\}^2 \, dx\right]^{1/2}} \cdot \frac{k(x)}{\left[\int_a^b \{k(x)\}^2 \, dx\right]^{1/2}}$$

$$\Rightarrow \quad u(x) = f(x) + \frac{\lambda k(x) \int_a^b f(x) \, k(x) \, dx}{1 - \lambda \int_a^b \{k(x)\}^2 \, dx}$$

Case II : Let $\lambda = \lambda_1$.

Suppose f(x) is not orthogonal to $\phi_1(x)$, i.e.,

$$f_1 = \int_a^b f(x)\,\phi_1(x)\,dx \neq 0$$

Then equation (1) has no solution.

Case III : Let $\lambda = \lambda_1$ Suppose f(x) is orthogonal to $\phi_1(x)$, i.e.,

$$f_1 = \int_a^b f(x)\,\phi_1(x)\,dx = 0$$

Then equation (1) have infinitely many solutions given by

$u(x) = f(x) + A\,\phi_1(x)$, where A is any arbitrary constant.

$$\Rightarrow \quad u(x) = f(x) + A.k(x).\left[\int_a^b \{k(x)\}^2\,dx\right]^{-1/2}$$

$$\Rightarrow \quad u(x) = f(x) + C.k(x), \text{ where } C = A\left[\int_a^b \{k(x)\}^2\,dx\right]^{-1/2}, \text{ a constant.}$$

EXAMPLE 9

Using Hilbert Schmidt theorem, solve $u(x) = x + \lambda \int_0^1 k(x, t)\,u(t)\,dt$

where $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$

[MEERUT-2001,02,05, 07(BP), GARHWAL-2004, KANPUR-2005]

Solution : Here, the given equation is

$$u(x) = x + \lambda \int_0^1 k(x, t)\,u(t)\,dt \quad \ldots\ldots(1)$$

with $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases} \quad \ldots\ldots(2)$

Comparing (1) with the equation

$$u(x) = f(x) + \lambda \int_0^1 k(x, t)\,u(t)\,dt$$

we get

$$f(x) = x \quad \ldots\ldots(3)$$

Now, we will proceed to find the eigen values and the corresponding eigen functions of the homogeneous equation

$$u(x) = \lambda \int_0^1 k(x, t)\,u(t)\,dt \quad \ldots\ldots(4)$$

Equation (4) can be written as

$$u(x) = \lambda\left[\int_0^x k(x, t)\,u(t)\,dt + \int_x^1 k(x, t)\,u(t)\,dt\right]$$

$$= \int_0^x \lambda t(x-1)\,u(t)\,dt + \int_x^1 \lambda x(t-1)\,u(t)\,dt \quad \ldots\ldots(5)$$

Differentiating w.r.t. x by using Leibnitz's rule, we get

$$u'(x) = \frac{d}{dx}\int_0^x \lambda t(x-1)\,u(t)\,dt + \frac{d}{dx}\int_x^1 \lambda x(t-1)\,u(t)\,dt$$

$$= \int_0^x \frac{\partial}{\partial x}\{\lambda t(x-1)\,u(t)\}\,dt + \lambda x(x-1)\,u(x)\frac{dx}{dx}$$

$$-\lambda(x-1)u(0).\frac{d0}{dx} + \int_x^1 \frac{\partial}{\partial x}\{\lambda x(t-1)\,u(t)\}\,dt$$

$$+\lambda x(1-1)\,u(1).\frac{d1}{dx} - \lambda x(x-1)\,u(x).\frac{dx}{dx}$$

$$= \int_0^x \lambda t\,u(t)\,dt + \lambda x(x-1)\,u(x) + \int_x^1 \lambda(t-1)\,u(t)\,dt - \lambda x(x-1)\,u(x)$$

$$= \int_0^x \lambda t\,u(t)\,dt + \int_x^1 \lambda(t-1)\,u(t)\,dt \qquad ......(6)$$

Again differentiating, we get

$$u''(x) = \frac{d}{dx}\int_0^x \lambda t\,u(t)\,dt + \frac{d}{dx}\int_x^1 \lambda(t-1)\,u(t)\,dt$$

$$= \int_0^x \frac{\partial}{\partial x}\{\lambda t\,u(t)\}\,dt + \lambda x\,u(x)\frac{dx}{dx} - \lambda.0.u(0).\frac{d0}{dx}$$

$$+ \int_x^1 \frac{\partial}{\partial x}\{\lambda(t-1)u(t)\}\,dt + \lambda(1-1)\,u(1).\frac{d1}{dx} - \lambda(x-1)\,u(x).\frac{dx}{dx}$$

$\Rightarrow \quad u''(x) = \lambda x\,u(x) - \lambda(x-1)\,u(x)$

$\Rightarrow \quad u''(x) = \lambda u(x)$

$\Rightarrow \quad u''(x) - \lambda\,u(x) = 0 \qquad ......(7)$

Also, from (5), we get

$u(0) = 0 \qquad ......(8)$

and $\quad u(1) = 0 \qquad ......(9)$

Now, to find the eigen values and corresponding eigen functions, we shall discuss the following cases :

Case I : Let $\lambda = 0$

Then (7) gives

$u''(x) = 0$

$\Rightarrow \quad u(x) = Ax + B \qquad ......(10)$

Using (8) and (9), we get

$0 = B$ and $0 = A + B$

On solving, we get A = B = 0.

Therefore, from (10), we get $u(x) = 0$, which is not an eigen function and so $\lambda = 0$ is not an eigen value.

Case II : Let $\lambda = k^2$ $(k \neq 0)$

Then (7) gives

Case II : Let $\lambda = \lambda_1$

Suppose f(x) is not orthogonal to $\phi_1(x)$, i.e.,

$$f_1 = \int_a^b f(x)\,\phi_1(x)\,dx \neq 0$$

Then equation (1) has no solution.

Case III : Let $\lambda = \lambda_1$ Suppose f(x) is orthogonal to $\phi_1(x)$, i.e.,

$$f_1 = \int_a^b f(x)\,\phi_1(x)\,dx = 0$$

Then equation (1) have infinitely many solutions given by

$u(x) = f(x) + A\,\phi_1(x)$, where A is any arbitrary constant.

$$\Rightarrow \quad u(x) = f(x) + A.k(x).\left[\int_a^b \{k(x)\}^2\,dx\right]^{-1/2}$$

$$\Rightarrow \quad u(x) = f(x) + C.k(x), \text{ where } C = A\left[\int_a^b \{k(x)\}^2\,dx\right]^{-1/2}, \text{ a constant.}$$

EXAMPLE 9

Using Hilbert Schmidt theorem, solve $u(x) = x + \lambda \int_0^1 k(x, t)\,u(t)\,dt$

where $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$

[MEERUT-2001,02,05, 07(BP), GARHWAL-2004, KANPUR-2005]

Solution : Here, the given equation is

$$u(x) = x + \lambda \int_0^1 k(x, t)\,u(t)\,dt \quad \text{......(1)}$$

with $k(x, t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$ ......(2)

Comparing (1) with the equation

$$u(x) = f(x) + \lambda \int_0^1 k(x, t)\,u(t)\,dt$$

we get

$$f(x) = x \quad \text{......(3)}$$

Now, we will proceed to find the eigen values and the corresponding eigen functions of the homogeneous equation

$$u(x) = \lambda \int_0^1 k(x, t)\,u(t)\,dt \quad \text{......(4)}$$

Equation (4) can be written as

$$u(x) = \lambda\left[\int_0^x k(x, t)\,u(t)\,dt + \int_x^1 k(x, t)\,u(t)\,dt\right]$$

$$= \int_0^x \lambda t(x-1)\,u(t)\,dt + \int_x^1 \lambda x(t-1)\,u(t)\,dt \quad \text{......(5)}$$

$$u''(x) - k^2 u(x) = 0$$

whose general solution is given by

$$u(x) = Ae^{kx} + Be^{-kx} \qquad \text{......(11)}$$

Using (8) and (9) in (13), we get

$$0 = A + B \qquad \text{......(12)}$$

$$0 = Ae^{k} + Be^{-k} \qquad \text{......(13)}$$

On solving, we get

$$A = B = 0$$

Then (11) gives u(x) = 0, which is again not an eigen function.

Case III : Let $\lambda = -k^2, (k \neq 0)$

Then, (7) gives

$$u''(x) + \lambda^2 u(x) = 0$$

whose general solution is given by

$$u(x) = A\cos kx + B\sin kx \qquad \text{......(14)}$$

Using (8) and (9) in (14), we get

$$0 = A \qquad \text{......(15)}$$

and $$0 = A\cos k + B\sin k \qquad \text{......(16)}$$

On solving, we get

$$B\sin k = 0 \qquad \text{......(17)}$$

Then, we must take $B \neq 0$.

Therefore, (17) gives

$$k = n\pi, \text{ where } n \in Z$$

$\therefore$ $$\lambda = -k^2 = -n^2\pi^2$$

Therefore, the required eigen values λ_n are given by

$$\lambda = -n^2\pi^2, \quad n = 1, 2, 3, \ldots\ldots \qquad \text{......(18)}$$

$\Rightarrow$ $$u(x) = B\sin n\pi x$$

Setting B = 1, we get $u(x) = \sin n\pi x$ ......(19)

Now, normalized eigen functions $\phi_n(x)$ can be obtained as

$$\phi_n(x) = \frac{u_n(x)}{\left[\int_0^1 \{u_n(x)\}^2\,dx\right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x\right]^{1/2}}$$

$$= \frac{\sin n\pi x}{\left[\int_0^1 \frac{1-\cos 2n\pi x}{2}\,dx\right]^{1/2}} = \frac{\sin n\pi x}{\left\{\left[\frac{1}{2}\left(x - \frac{\sin 2n\pi x}{2n\pi}\right)\right]_0^1\right\}^{1/2}}$$

$$= \frac{\sin n\pi x}{1/\sqrt{2}} = \sqrt{2}\sin n\pi x$$

Then, $f_n = \int_0^1 f(x)\,\phi_n(x)\,dx = \int_0^1 x(\sqrt{2}.\sin n\pi x)\,dx$

$$= \sqrt{2}\left\{\left[x\left(-\frac{\cos n\pi x}{n\pi}\right)\right]_0^1 - \int_0^1\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right\}$$

$$= \sqrt{2}\left\{-\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx\right\}$$

$$= \sqrt{2}\left\{-\frac{(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}[\sin n\pi x]_0^1\right\} = \frac{(-1)^{n+1}\sqrt{2}}{n\pi}$$

Case (a) : Let $\lambda \neq \lambda_n$, then (1) will have unique solution given by

$$u(x) = f(x) + \lambda\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n - \lambda}\,\phi_n(x)$$

$$= x + \lambda\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sqrt{2}}{n\pi}\cdot\frac{\sqrt{2}\sin n\pi x}{-n^2\pi^2 - \lambda}$$

$$= x + \frac{2\lambda}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n(n^2\pi^2 + \lambda)}$$

Case (b) : Let $\lambda = \lambda_n = -n^2\pi^2$, $n \in \mathbf{Z}$, then $f_n \neq 0$.

Hence, (1) have no solution.

EXAMPLE 10

Find the eigen values and the corresponding eigen functions of the equation

$$u(x) = f(x) + \lambda\int_0^{2\pi}\sin(x+t)\,u(t)\,dt$$

where $f(x) = x$. *Also, find the solution of this equation, where* λ *is not an eigen value.*

Solution : Here, the given equation is

$$u(x) = f(x) + \lambda\int_0^{2\pi}\sin(x+t)\,u(t)\,dt \qquad \text{......(1)}$$

with $\quad f(x) = x \qquad$ ......(2)

Now, we will proceed to find the eigen values and the corresponding normalized eigen functions of the homogeneous equation

$$u(x) = \lambda\int_0^{2\pi}\sin(x+t)\,u(t)\,dt \qquad \text{......(3)}$$

Equation (3) can also be rewritten as

$$u(x) = \lambda\int_0^{2\pi}(\sin x\cos t + \cos x\sin t)\,u(t)\,dt$$

$$= \lambda\sin x\int_0^{2\pi}\cos t\,u(t)\,dt + \lambda\cos x\int_0^{2\pi}\sin t\,u(t)\,dt \qquad \text{......(4)}$$

Let $\quad C_1 = \int_0^{2\pi}\cos t\,u(t)\,dt \qquad$ ......(5)

and $\quad C_2 = \int_0^{2\pi} \sin t \, u(t)\, dt$ ......(6)

Put these values of C_1 and C_2 in (4), we get

$$u(x) = \lambda C_1 \sin x + \lambda C_2 \cos x \quad \text{......(7)}$$

$$\Rightarrow \quad u(t) = \lambda C_1 \sin t + \lambda C_2 \cos t \quad \text{......(8)}$$

Putting this value in (5), we get

$$C_1 = \int_0^{2\pi} \cos t(\lambda C_1 \sin t + \lambda C_2 \cos t)\, dt$$

$$= \frac{\lambda C_1}{2}\int_0^{2\pi} \sin 2t\, dt + \frac{\lambda C_2}{2}\int_0^{2\pi}(1+\cos 2t)\, dt$$

$$= -\frac{\lambda C_1}{4}[\cos 2t]_0^{2\pi} + \frac{\lambda C_2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\Rightarrow \quad C_1 = \lambda\pi C_2 \quad \Rightarrow \quad C_1 - \pi\lambda C_2 = 0 \quad \text{......(9)}$$

Similarly, from (6), we obtain

$$C_2 = \int_0^{2\pi} \sin t\,(\lambda C_1 \sin t + \lambda C_2 \cos t)\, dt$$

$$= \frac{\lambda C_1}{2}\int_0^{2\pi}(1-\cos 2t)\, dt + \frac{\lambda C_2}{2}\int_0^{2\pi} \sin 2t\, dt$$

$$\Rightarrow \quad C_2 = \lambda\pi C_1 \quad \text{or} \quad \lambda\pi C_1 - C_2 = 0 \quad \text{......(10)}$$

For non-trivial solution of (1), we must have

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad -1 + \lambda^2\pi^2 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{\pi} \quad \text{or} \quad -\frac{1}{\pi}$$

Therefore, the required eigen values are given by

$$\lambda_1 = \frac{1}{\pi} \quad \text{and} \quad \lambda = -\frac{1}{\pi} \quad \text{......(11)}$$

Determination of Eigen Functions

(i) Corresponding to $\lambda_1 = \frac{1}{\pi}$

Putting $\lambda = \lambda_1 = \frac{1}{\pi}$ in (9) and (10), we get

$$C_1 - C_2 = 0 \quad \text{and} \quad C_1 - C_2 = 0 \quad \Rightarrow \quad C_1 = C_2.$$

Then, from (7), the required eigen function $u_1(x)$ is given by

$$u_1(x) = \frac{C_1}{\pi}(\sin x + \cos x)$$

Setting $\frac{C_1}{\pi} = 1$. Therefore, $u_1(x) = \sin x + \cos x$

$\Rightarrow$ The corresponding normalized eigen function $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{u_1(x)}{\left[\int_0^{2\pi}\{u_1(x)\}^2\, dx\right]^{1/2}} = \frac{\sin x + \cos x}{\left[\int_0^{2\pi}(\sin x + \cos x)^2\, dx\right]^{1/2}}$$

$$= \frac{\sin x + \cos x}{\left[\int_0^{2\pi} (1 + \sin 2x)\,dx\right]^{1/2}} = \frac{\sin x + \cos x}{\left\{\left[x - \frac{\cos 2x}{2}\right]_0^{2\pi}\right\}^{1/2}}$$

$$= \frac{\sin x + \cos x}{\sqrt{2\pi}} \qquad \text{......(12)}$$

(ii) Corresponding to $\lambda_2 = -\frac{1}{\pi}$

Now, putting $\lambda = \lambda_2 = -\frac{1}{\pi}$ in (9) and (10), we get

$$C_1 + C_2 = 0 \quad \text{and} \quad -C_1 - C_2 = 0 \quad \Rightarrow \quad C_2 = -C_1$$

Then (7) gives

$$u_2(x) = \frac{C_1}{\pi}(\sin x - \cos x)$$

Let $\frac{C_1}{\pi} = 1$, then $u_2(x) = \sin x - \cos x$

Now, the corresponding normalized eigen function $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{u_2(x)}{\left[\int_0^{2\pi} \{u_2(x)\}^2\,dx\right]^{1/2}} = \frac{\sin x - \cos x}{\left[\int_0^{2\pi} (\sin x - \cos x)^2\,dx\right]^{1/2}}$$

$$= \frac{\sin x - \cos x}{\left[\int_0^{2\pi} (1 - \sin 2x)\,dx\right]^{1/2}} = \frac{\sin x - \cos x}{\left\{\left[x + \frac{\cos 2x}{2}\right]_0^{2\pi}\right\}^{1/2}}$$

$$= \frac{\sin x - \cos x}{\sqrt{2\pi}} \qquad \text{......(13)}$$

And $\quad f_1 = \int_0^{2\pi} f(x)\,\phi_1(x)\,dx = \int_0^{2\pi} \frac{x(\sin x + \cos x)}{\sqrt{2\pi}}.dx$

$$= \frac{1}{\sqrt{2\pi}}[\{x(\sin x - \cos x\}]_0^{2\pi} - \int_0^{2\pi} (\sin x - \cos x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[-2\pi - \{-\cos x - \sin x\}_0^{2\pi}\right] = -\sqrt{2\pi}$$

also, $\quad f_2 = \int_0^{2\pi} f(x)\,\phi_2(x)\,dx = \int_0^{2\pi} \frac{x(\sin x - \cos x)}{\sqrt{2\pi}}\,dx$

$$= \frac{1}{\sqrt{2}}\left[\{x(-\cos x - \sin x)\}_0^{2\pi} - \int_0^{2\pi} (-\cos x - \sin x)dx\right]$$

$$= \frac{1}{\sqrt{2}}\left[-2\pi - \{-\sin x + \cos x\}_0^{2\pi}\right] = -\pi\sqrt{2\pi}$$

Here, we observed that $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$.

Hence, (1) have the unique solution, which is given by

$$u(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda}\,\phi_m(x)$$