Chapter 1

INTRODUCTION

OUTLINE

- Integral Equation
- Linear and Non-linear Equation
- Fredholm and Volterra's Integral Equations
- Singular Integral Equations
- Kernel – Separable, Transposed, Iterated and Resolvent
- Leibnitz's Rule

1.1 INTRODUCTION

An integral equation is an equation in which the unknown function occurs under the integral sign. The name 'integral equation' for any equation involving the unknown function u(x) under the integral sign was introduced by du Bois-Reymond in 1888. In 1782, Laplace used the integral transform

$$f(x) = \int_0^\infty e^{-xt} f(t)\, dt$$

to solve the linear difference equations and differential equations. In 1826, Abel solved the integral equation named after him having the form

$$f(x) = \int_0^x (x-t)^{-\alpha} u(t)\, dt$$

where $f(x)$ is a continuous function satisfying $f(a) = 0$ and $0 < \alpha < 1$. Huygens solved the Abel's integral equation for $\alpha = 1/2$. In 1826, Poisson obtained an integral equation of the type

$$u(x) = f(x) + \lambda \int_0^x k(x,t)\, u(t)\, dt$$

in which the unknown function $u(t)$ occurs outside as well as before the integral sign and the variable x appears as one of the limits of the integral. Dirichlet's problem, which is the determination of a function ψ having prescribed values over a certain boundary surface S and satisfying Laplace's equation $\nabla^2\psi = 0$ within the region enclosed by S, was shown by Heumann in 1870 to be equivalent to the solution of an integral equation. He solved the integral equation by an expansion in powers of a certain parameter λ. In 1896, Volterra gave the first general treatment of the solution of the class of linear integral equation bearing his name and characterized by the variables x appearing as the upper limit of the integral. In 1900, Fredholms have discussed a more general class of linear integral equation having the form

$$u(x) = f(x) + \int_a^b k(x,t)\, u(t)\, dt$$

GENERAL DEFINITIONS

(1) Integral Equation [MEERUT 2008]

An integral equation is an equation in which an unknown function appears under one or more integral signs.

For example : for $a \le x \le b,\ a \le t \le b$, the equations

$$f(x) = \int_a^x k(x,t)\, u(t) dt \quad \text{......(1)}$$

$$u(x) = f(x) + \lambda \int_a^x k(x,t)\, u(t) dt \quad \text{......(2)}$$

$$u(x) = \int_a^b k(x,t)\, [u(t)]^2 dt \quad \text{......(3)}$$

where the function $u(x)$ is the unknown function, while the functions f(x) and k(x, t) are known functions and λ, a and b are constants, are the integral equations.

Remarks

1. If the derivative of the function are involved in the equation, then it is called an integro-differential equations.
2. The function $f(x)$ and $k(x,t)$ may be complex valued functions of the real variables x and t.
3. The function u(x) is the unknown function, while the other functions are known.

(2) Linear and Non-Linear Integral Equations

An integral equation is called linear, if, only linear operations are performed in it upon the unknown function. On the other hand, an integral equation, which is not linear is known as a non-linear integral equations.

In the above definition, equations (1) and (2) are linear integral equation while equation (3) is non-linear integral equation.

General Form : The most general type of linear integral equation is of the form

$$v(x).u(x) = f(x) + \lambda \int_a k(x,t)\, u(t)\, dt \quad \text{......(1)}$$

where the upper limit may be either variable x or fixed. The functions f, v and k are known functions, while u is unknown, which is to be determined. λ is a non-zero real or complex parameter. The function $k(x,t)$ is known as kernel of the integral equation.

Types of the Linear Integral Equation

(i) If $v(x) \neq 0$, then (1) is known as linear integral equation of the third kind.

(ii) When $v(x) = 0$, then (1) reduces to

$$f(x) + \lambda \int_a k(x,t)\, u(t)\, dt = 0$$

which is known as linear integral equation of the first kind.

(iii) When $v(x) = 1$, then (1) reduces to

$$u(x) = f(x) + \lambda \int k(x,t)\, u(t)\, dt$$

which is known as linear integral equation of the second kind.

(3) Fredholm Integral Equation **[AMRITSAR-2004]**

A linear integral equation of the form

$$v(x).u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

where a, b both are constants, $f(x)$, $v(x)$ and $k(x,t)$ are known functions while u(x) is unknown functions and λ is a non-zero real or complex parameter, is called Fredholm integral equation of third kind.

SPECIAL CASES

(a) **Fredholm Integral Equation of the First Kind** : Put v(x) = 0 in (1), then integral equation of the form

$$f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt = 0 \qquad \ldots\ldots(2)$$

is known as Fredholm integral equation of the first kind. **[GARHWAL-2000,04]**

(b) **Fredholm Integral Equation of the Second Kind** : Put v(x) = 1 in (1), then integral equation of the form

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \ldots\ldots(3)$$

is known as Fredholm integral equation of the second kind. **[GARHWAL-2000,04]**

(c) **Homogeneous Fredholm Integral Equation of the Second Kind** : Put f(x) = 0 in (3), then a linear integral equation of the form

$$u(x) = \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \ldots\ldots(4)$$

is known as the homogeneous Fredholm integral equation of the second kind.

(4) Volterra Integral Equation **[KANPUR-1990]**

A linear integral equation of the form

$$v(x).u(x) = f(x) + \lambda \int_a^x k(x,t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

where a is constant, f(x), v(x) and k(x, t) are known functions, while u(x) is unknown function, λ is a non-zero real or complex parameter is called Volterra integral equation of third kind.

SPECIAL CASES

(a) Volterra Integral Equation of the First Kind : Put v(x) = 0 in (1), then a linear integral equation of the form

$$f(x) + \lambda \int_a^x k(x, t)\, u(t)\, dt = 0 \qquad \ldots\ldots(2)$$

is known as Volterra integral equation of the first kind. **[GARHWAL-2004]**

(b) Volterra Integral Equation of the Second Kind : Put v(x) = 1 in (1), then a linear integral equation of the form

$$u(x) = f(x) + \lambda \int_a^x k(x, t)\, u(t)\, dt \qquad \ldots\ldots(3)$$

is known as Volterra integral equation of the second kind. **[GARHWAL-2004]**

(c) Homogeneous Volterra Integral Equation of the Second Kind : A linear equation of the form

$$u(x) = \lambda \int_a^x k(x, t)\, u(t)\, dt \qquad \ldots\ldots(4)$$

is known as the homogeneous Volterra integral equation of the second kind.

(5) Singular Integral Equations **[MEERUT-2006,08, GARHWAL-2004]**

When one or both limits of integration becomes infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is known as singular integral equation.

For example :

(i) $$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-t|}\, u(t)dt \qquad \ldots\ldots(1)$$

(ii) $$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}\, u(t)dt, \quad 0 < \alpha < 1 \qquad \ldots\ldots(2)$$

(6) Convolution Type Integral Equation

If the kernel k(x, t) of the integral equation is a function of the difference (x – t), i.e.

$$k(x, t) = k(x - t)$$

where k is a certain function of one variable, then the integral equation

$$u(x) = F(x) + \lambda \int_a^x k(x - t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

and the corresponding Fredholm equation

$$u(x) = F(x) + \lambda \int_a^b k(x - t)\, u(t)\, dt \qquad \ldots\ldots(2)$$

are called integral equation of the convolution type.

Remarks

1. $k(x - t)$ is called the difference kernel.
2. The function defined by the integral

$$\int_0^x k(x-t)\, u(t)\, dt = \int_0^x k(t)\, u(x-t)\, dt$$

is called the convolution or the Faltung of the two functions k and u and is known as convolution integral.

1.2 TYPES OF KERNELS

(i) Symmetric Kernel : A kernel $k(x, t)$ is symmetric (or complex symmetric or Hermitian) if :

$$k(x, t) = \overline{k(x, t)}$$

where bar denotes the complex conjugate. A real kernel $k(x, t)$ is symmetric if

$$k(x, t) = k(t, x)$$

For example :

$\sin(x + t)$, e^{xt}, $x^3t^3 + x^2t^2 + xt + 1$ are all symmetric kernels.

(ii) Separable or Degenerate Kernel : [MEERUT 2007 BP]

A kernel which is particularly useful in solving the Fredholm equation has the form

$$k(x, t) = \sum_{i=1}^{n} a_i(x)\, b_i(t)$$

where n is finite and a_i, b_i are linearly independent sets of functions. Such a kernel is called separable or degenerate kernel.

Remark

❶ A degenerate kernel has a finite number of characteristic values.

(iii) Transposed Kernel : The kernel $k^T(x, t) = k(t, x)$ is called the transposed kernel of $k(x, t)$.

(iv) Iterated Kernels :

(a) Consider Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

Then, the iterated kernels $k_n(x, t)$, n = 1, 2, 3,..... are defined as follows

$$k_1(x, t) = k(x, t)$$

and $$k_n(x, t) = \int_a^b k(x, s) k_{n-1}(s, t)\, ds, \quad n = 2, 3, \ldots$$

(b) Consider Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^x k(x, t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

Then, the iterated kernels $k_n(x, t)$, n = 1, 2, 3,..... are defined as follows

$$k_1(x, t) = k(x, t)$$

and $\quad k_n(x,t) = \int_a^x k(x,s)k_{n-1}(s,t)\, ds, \; n = 2, 3, \ldots$

(v) Resolvent Kernel or Reciprocal Kernel :

Consider the integral equations

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

and $$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt \qquad \ldots\ldots(2)$$

Let the solution of (1) and (2) be given by

$$u(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \qquad \ldots\ldots(3)$$

and $$u(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\lambda)\, f(t)\, dt \qquad \ldots\ldots(4)$$

Then, $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ is called the resolvent kernel or reciprocal kernel.

1.3 EIGEN VALUES AND EIGEN FUNCTION

Consider the homogeneous Fredholm integral equation

$$u(x) = \lambda \int_a^b k(x,t)\, u(t)\, dt \qquad \ldots\ldots(1)$$

Then values of the parameter λ for which (1) has a non-zero solution $[(u(x) \neq 0)]$ are called eigen values of (1) or of the kernel $k(x,t)$, and every non-zero solution of (1) is called an eigen function corresponding to the eigen value λ.

Remarks

1. The eigen values are also known as characteristic values or characteristic numbers.
2. Eigen functions are also known as characteristic function or fundamental functions.
3. The number $\lambda = 0$ is not an eigen value, since for $\lambda = 0$, it follows from (1) that $u(x) = 0$.
4. If $u(x)$ is an eigen function of (1), then $C.u(x)$, where C is an arbitrary constant, is also an eigen function of (1), which corresponds to the same eigen value λ.
5. A homogeneous Fredholm integral equation of the second kind may, generally have no eigen value and eigen function, or it may not have any real eigen value or eigen function.

1.4 DIFFERENTIATION UNDER THE SIGN OF INTEGRATION (LEIBNITZ'S RULE)

Let $F(x,t)$ and $\dfrac{\partial F}{\partial x}$ be continuous functions of both x and t and let the first derivative of $G(x)$ and $H(x)$ be continuous. Then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\,dt + F[x,H(x)]\frac{dH}{dx} - F[x,G(x)]\frac{dG}{dx} \quad \text{......(1)}$$

Remark

Note ❶ If G and H are absolute constants, then (1) gives

$$\frac{d}{dx}\int_G^H F(x,t)\,dt = \int_G^H \frac{\partial F}{\partial x}\,dt$$

Conversion of multiple integral into a single ordinary integral

Consider the integral

$$I_n(x) = \int_a^x (x-t)^{n-1} f(t)\,dt \quad \text{......(1)}$$

where, t is a positive integer and a is a constant.

Differentiating (1) using Leibnitz's rule, we get

$$\frac{dI_n}{dx} = (n-1)\int_a^x (x-t)^{n-2} f(t)dt + [(x-t)^{n-1} f(t)]_{t=x}$$

$$\frac{dI_n}{dx} = (n-1)\, I_{n-1},\ n>1 \quad \text{......(2)}$$

From (1), we get

$$I_1(x) = \int_a^x f(t)\,dt \Rightarrow \frac{dI_1}{dx} = f(x) \quad \text{......(3)}$$

Now, differentiating (2) successively m times, we get

$$\frac{d^m I_n}{dx^m} = (n-1)(n-2)(n-3)\text{......}(n-m)\, I_{n-m},\ n>m$$

In particular, we have

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)!\, I_1(x) \Rightarrow \frac{d}{dx}\left(\frac{d^{n-1} I_n}{dx^{n-1}}\right) = (n-1)!\,\frac{dI_1}{dx}$$

i.e., $$\frac{d^n I_n}{dx^n} = (n-1)!\, f(x) \quad \text{......(4)}$$

Therefore, we have

$$I_1(x) = \int_a^x f(x_1)\,dx_1 \quad \text{[from (3)]}$$

and $$\frac{dI_2}{dx} = I_1 = \int_a^x f(x_1)dx_1 \Rightarrow I_2(x) = \int_a^x\int_a^{x_2} f(x_1)dx_1dx_2 \quad \text{[from (2)]}$$

In general, we have

$$I_n(x) = (n-1)!\int_a^x\int_a^{x_n}\text{..........}\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\text{.....}dx_{n-1}dx_n \quad \text{......(5)}$$

Using (1) and (5), we conclude that

$$\int_a^x\int_a^{x_n}\text{..........}\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\text{.....}dx_{n-1}dx_n$$

$$= \frac{1}{(n-1)!}\, I_n(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

On integrating $(n-1)$ times, we have

$$\int_a^x f(t)\, dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\, dt$$

1.5 CONNECTION WITH DIFFERENTIAL EQUATION

Observe that the first order differential equation

$$\frac{d\phi}{dx} = F(x, \phi) \qquad [a \le x \le b] \qquad \ldots\ldots(1)$$

can be written immediately as the Volterra integral equation of the second kind

$$\phi(x) = \phi(a) + \int_a^x F(t, \phi(t))\, dt \qquad \ldots\ldots(2)$$

As an interesting but simple example of the above, we consider the following problem, solved by J. Bernoulli for the n = 3 case.

To find the equation $y = \phi(x)$ of the curve joining a fixed origin O to a point P such that the area under the curve is $1/n^{th}$ of the arcs of the rectangle OXPY having OP as diagonal for all points P on the curve given in Fig. 1.1.

Clearly this problem is equivalent to solving the homogeneous Volterra integral equation of the second kind

$$\int_0^x \phi(t) dt = \frac{1}{n} x\, \phi(x) \qquad \ldots\ldots(3)$$

for the unknown function $\phi(x)$ with $\phi(0) = 0$.

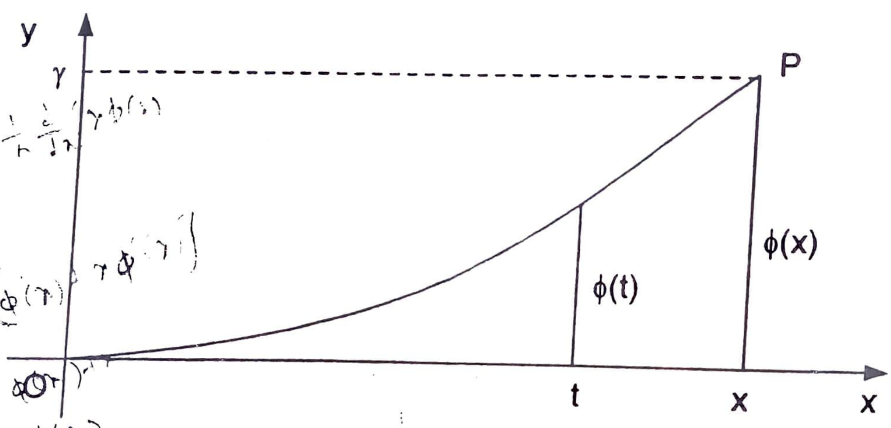

Fig. 1.1. Bernoulli's problem for n = 3, y = x²

This equation can be solved by converting it into the differential equation

$$\phi(x) = \frac{1}{n}\{\phi(x) + x\phi'(x)\}$$

or simply $\phi'(x) = \dfrac{n-1}{x}\phi(x)$ ......(4)

which has the solution

$$\phi(x) = Ax^{n-1} \qquad \ldots\ldots(5)$$

For the case n = 3, solved by Bernoulli, the curve is a parabola $y = Ax^2$.

Next, consider the second order differential equation

$$\frac{d^2\phi}{dx^2} = F(x,\phi) \qquad [a \le x \le b] \qquad \text{......(6)}$$

which can be expressed as

$$\phi(x) = \phi(a) + (x-a)\phi'(a) + \int_a^x (x-t)\, F[t,\phi(t)]\, dt \qquad \text{......(7)}$$

Taking a = 0 and $b - a = l$, equation (7) can be written as

$$\phi(x) = \phi(0) + \frac{\phi(l)-\phi(0)}{l}\, x + \int_0^x (x-t)\, F[(t,\phi(t)]\, dt + \frac{x}{l}\int_0^l (t-l)\, F[t,\phi(t)]\, dt \qquad \text{......(8)}$$

$$\left[\because l\phi'(0) = \phi(l) - \phi(0) + \int_0^l (t-l)\, F[t,\phi(t)]\, dt\right]$$

If the differential equation is linear and we write

$$F[x,\phi(x)] = r(x) - q(x)\,\phi(x) \qquad \text{......(9)}$$

We get the Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \int_0^l k(x,t)\, q(t)\, \phi(t)\, dt \qquad \text{......(10)}$$

where $f(x) = \phi(0) + \frac{\phi(l)-\phi(0)}{l} x - \int_0^l k(x,t)\, r(t)\, dt$ ......(11)

and $k(x,t) = \begin{cases} \dfrac{t(l-x)}{l} & (t \le x) \\ \dfrac{x(l-t)}{l} & (t \ge x) \end{cases}$ ......(12)

If we take $\phi(0) = \phi(l) = 0 \Rightarrow f(x) = 0$, then we get the homogeneous Fredholm integral equation

$$\phi(x) = \int_0^l k(x,t)\, q(t)\, \phi(t)\, dt$$

Remark

❶ An important distinction between the differential equation and the equivalent integral equation approach should be observed. In the case of differential equation formulation of a physical problem, the boundary conditions are imposed separately whereas the integral equation formulation contain the boundary conditions implicitly.

1.6 SOLUTION OF AN INTEGRAL EQUATION

Consider the linear integral equation

$$v(x)\,.\,u(x) = f(x) + \lambda\int_a^b k(x,t)\, u(t)\, dt \qquad \text{......(1)}$$

and $v(x)\,.\,u(x) = f(x) + \lambda\int_a^x k(x,t)\, u(t)\, dt$ ......(2)

Then, a solution of the integral equation (1) and (2) is a function u(x), which when substituted into the equation, reduces to an identity.

SOLVED EXAMPLES

EXAMPLE 1

Verify that the given function $u(x) = \frac{1}{2}$ is the solution of the integral equation

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \sqrt{x}$$

Solution : Here, the given integral equation is

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \sqrt{x}$$

Put $u(x) = \frac{1}{2}$ in the given equation, we get

$$\frac{1}{2}\int_0^x \frac{dt}{\sqrt{x-t}} = \sqrt{x}$$

i.e., $$\frac{1}{2}\left(-2\sqrt{x-t}\right)_0^x = \sqrt{x}$$

$\Rightarrow$ $$\sqrt{x} = \sqrt{x}$$

which is an identity in x. Hence, the function $u(x) = \frac{1}{2}$ is the solution of given integral equation.

EXAMPLE 2

Show that the function $u(x) = 1$ is a solution of the Fredholm integral equation

$$u(x) + \int_0^1 x(e^{xt} - 1)\, u(t)\, dt = e^x - x$$ [illegible]

Solution : Here, the given integral equation is

$$u(x) + \int_0^1 x(e^{xt} - 1)\, u(t)\, dt = e^x - x \qquad \ldots(1)$$

Also, given

$$u(x) = 1 \quad \Rightarrow \quad u(t) = 1$$

Then, LHS of (1), becomes

$$\text{LHS} = 1 + \int_0^1 x(e^{xt} - 1)\, dt = 1 + x\left[\frac{1}{x}e^{xt} - t\right]_0^1$$

$$= 1 + x\left(\frac{1}{x}e^x - 1 - \frac{1}{x}\right) = 1 + e^x - x - 1 = e^x - x = \text{RHS}$$

Hence, $u(x) = 1$ is the solution of given integral equation.

EXAMPLE 3

Show that the function $u(x) = (1 + x^2)^{-3/2}$ is a solution of the Volterra equation

$$u(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{(1+x^2)} u(t)\,dt$$

[MEERUT-2003,GARHWAL-2001,02,KANPUR-1995]

Solution : Here, the given integral equation is

$$u(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{(1+x^2)} u(t)\,dt \qquad \ldots\ldots(1)$$

Also, given

$$u(x) = (1+x^2)^{-3/2} \Rightarrow \quad u(t) = (1+t^2)^{-3/2}$$

The RHS of (1) is given by

$$RHS = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}(1+t^2)^{-3/2}\,dt$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^{x^2}(1+s)^{-3/2}.\frac{1}{2}\,ds \qquad \text{(Putting } t^2 = s \Rightarrow 2t\,dt = ds\text{)}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}.\frac{1}{2}\left[\frac{(1+s)^{-1/2}}{-1/2}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{(1+s)^{1/2}}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{(1+x^2)^{1/2}} - 1\right]$$

$$= \frac{1}{1+x^2} + \frac{1}{(1+x^2)^{3/2}} - \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

$$= u(x) = \text{LHS}$$

Hence, $u(x) = (1+x^2)^{-3/2}$ is the solution of given integral equation.

EXAMPLE 4

Show that the function $u(x) = xe^x$ *is a solution of the Volterra integral equation.*

$$u(x) = \sin x + 2\int_0^x \cos(x-t)\,u(t)\,dt$$

[MEERUT-1998, 2004 BP, GARHWAL-1999,KANPUR-2005]

Solution : The given integral equation is

$$u(x) = \sin x + 2\int_0^x \cos(x-t)\,u(t)\,dt \qquad \ldots\ldots(1)$$

Also, given $u(x) = xe^x \quad \Rightarrow \quad u(t) = te^t$

Then RHS of (1) is

$$= \sin x + 2\int_0^x t\{e^t \cos(x-t)\}\,dt$$

$$= \sin x + 2\int_0^x t\,\{e^t \cos(t-x)\}\,dt$$

$$= \sin\frac{\pi x}{2}\left\{1 - \frac{1}{2}(2-x) - \frac{x}{2}\right\} + \frac{x}{2} = \frac{x}{2} = RHS$$

Hence, $u(x) = \sin\left(\frac{\pi x}{2}\right)$ is the solution of given integral equation

EXAMPLE 10

Verify that the given function $u(x) = \frac{x}{(1+x^2)^{5/2}}$ *is a solution of the integral equation*

$$u(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-t}{(1+x^2)^2}\,u(t)\,dt$$ **[GARHWAL-2003]**

Solution : The given integral equation is

$$u(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-t}{(1+x^2)^2}\,u(t)\,dt \qquad \ldots\ldots(1)$$

Also, given that

$$u(x) = \frac{x}{(1+x^2)^{5/2}} \Rightarrow u(t) = \frac{t}{(1+t^2)^{5/2}}$$

Then RHS of (1) is

$$= \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-t}{(1+x^2)^2}\cdot\frac{t}{(1+t^2)^{5/2}}\,dt$$

$$= \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3}{(1+x^2)^2}\cdot\frac{t}{(1+t^2)^{5/2}}\,dt + \int_0^x \frac{1}{(1+x^2)^2}\cdot\frac{t^2}{(1+t^2)^{5/2}}\,dt$$

$$= \frac{3x+2x^3}{3(1+x^2)^2} + \frac{3x+2x^3}{3(1+x^2)^2}\left[\frac{1}{(1+t^2)^{3/2}}\right]_0^x + \frac{1}{3(1+x^2)^2}\left[\frac{t^3}{(1+t^2)^{3/2}}\right]_0^x$$

$$= \frac{3x+2x^3}{3(1+x^2)^2} + \frac{3x+2x^3}{3(1+x^2)^2}\left[\frac{1}{(1+x^2)^{3/2}} - 1\right] + \frac{1}{3(1+x^2)^2}\left[\frac{x^3}{(1+x^2)^{3/2}}\right]$$

$$= \frac{1}{(1+x^2)^{3/2}}\left[\frac{3x+2x^3}{3(1+x^2)^2} + \frac{x^3}{3(1+x^2)^2}\right] = \frac{x}{(1+x^2)^{5/2}} = LHS$$

Hence, $u(x) = \frac{x}{(1+x^2)^{5/2}}$ is the solution of given integral equation.

EXERCISE - 1

Verify or check that the given functions are solutions of the corresponding integral equations :

(1) $u(x) = x\,e^x$; $\qquad u(x) = e^x \sin x + 2\int_0^x \cos(x-t)\,u(t)\,dt$

(2) $u(x) = 3$; $\qquad \int_0^x (x-t)^2\,u(t)\,dt = x^3$

(3) $u(x) = x - \frac{x^3}{6}$; $\quad u(x) = x - \int_0^x \sinh(x-t)\, u(t)\, dt$

(4) $u(x) = e^x[\cos e^x - e^x \sin e^x]$;

$u(x) = (1 - xe^{2x})\cos 1 - e^{2x}\sin 1 + \int_0^x [1 - (x-t)e^{2x}]\, u(t)\, dt$

(5) $u(x) = \cos x$; $\quad u(x) - \int_0^\pi (x^2 + t)\cos t\, u(t)\, dt = \sin x$

(6) $u(x) = 1 - \frac{2\sin x}{(1 + (\pi/2))}$; $\quad u(x) = 1 + \int_0^\pi \cos(x+t)\, u(t)\, dt$

(7) $u(x) = xe^{-x}$; $\quad u(x) = (x-1)e^{-x} + 4\int_0^\infty e^{-(x+t)}\, u(t)\, dt$ [GARHWAL-2002]

(8) $u(x) = \sqrt{x}$; $\quad u(x) - \int_0^1 k(x,t)\, u(t)\, dt = \sqrt{x} + \frac{x}{15}(4x^{3/2} - 7)$

where $k(x,t) = \begin{cases} \frac{1}{2}x(2-t); & 0 \le x \le t \\ \frac{1}{2}t(2-x); & t \le x \le 1 \end{cases}$

1.7 CONVERSION OF DIFFERENTIAL EQUATIONS TO INTEGRAL EQUATIONS : INITIAL VALUE PROBLEM

Consider the following linear differential equation of order n

$$\frac{d^n y}{dx^n} + C_1(x)\frac{d^{n-1}y}{dx^{n-1}} + C_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + C_n(x).y = \phi(x) \qquad \ldots\ldots(1)$$

with continuous coefficients $C_i(x)$; i = 1, 2, ..., n and with initial conditions

$$y(a) = a_o,\; y'(a) = a_1 \ldots\ldots y^{(n-1)}(a) = a_{n-1} \qquad \ldots\ldots(2)$$

Let us introduce an unknown function $u(x)$ such that

$$\frac{d^n y}{dx^n} = u(x) \qquad \ldots\ldots(3)$$

Integrating both sides w.r.t. x from a to x of (3), we get

$$\left[\frac{d^{n-1}y}{dx^{n-1}}\right]_a^x = \int_a^x u(x)\, dx$$

$$\frac{d^{n-1}y}{dx^{n-1}} - y^{(n-1)}(a) = \int_a^x u(x)\, dx$$

Now using (2), we get

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(x)\, dx + a_{n-1} \qquad \ldots\ldots(4)$$

$$\Rightarrow \quad \frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(t)\, dt + a_{n-1} \qquad \ldots\ldots(5)$$

Again integrating both sides of equation (4) w.r.t. x from a to x, we have

$$\left[\frac{d^{n-2}y}{dx^{n-2}}\right]_a^x = \int_a^x u(x).\,dx^2 + a_{n-1}\int_a^x dx$$

$$\Rightarrow \quad \frac{d^{n-2}y}{dx^{n-2}} - y^{(n-2)}(a) = \int_a^x u(x)\,dx^2 + a_{n-1}[\,x\,]_a^x$$

$$\Rightarrow \quad \frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(x)\,dx^2 + (x-a)a_{n-1} + a_{n-2} \quad \ldots\ldots(6)$$

$$= \int_a^x u(t)\,dt^2 + (x-a)a_{n-1} + a_{n-2}$$

$$= \int_a^x (x-t)\,u(t)\,dt + (x-a)a_{n-1} + a_{n-2} \quad \ldots\ldots(7)$$

Integrating (6) w.r.t. x from a to x, we get

$$\left[\frac{d^{n-3}y}{dx^{n-3}}\right]_a^x = \int_a^x u(x)dx^3 + a_{n-1}\int_a^x (x-a)dx + a_{n-2}\int_a^x dx$$

$$\Rightarrow \quad \frac{d^{n-3}y}{dx^{n-3}} - y^{(n-3)}(a) = \int_a^x u(x)dx^3 + a_{n-1}\left[\frac{(x-a)^2}{2}\right]_a^x + a_{n-2}[x]_a^x$$

i.e.,
$$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(x)\,dx^3 + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}\frac{(x-a)}{1!} + a_{n-3} \quad \ldots\ldots(8)$$

$$= \int_a^x u(t)dt^3 + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}\frac{(x-a)}{1!} + a_{n-3}$$

$$= \int_a^x \frac{(x-t)^2}{2!}u(t)\,dt + a_{n-1}\frac{(x-a)^2}{2!} + a_{n-2}\frac{(x-a)}{1!} + a_{n-3} \quad \ldots\ldots(9)$$

and so on.

Finally, we get

$$\frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!}u(t)\,dt + a_{n-1}\frac{(x-a)^{n-2}}{(n-2)!} + a_{n-2}\frac{(x-a)^{n-3}}{(n-3)!} + \ldots\ldots + a_2(x-a) + a_1 \quad \ldots\ldots(10)$$

and

$$y = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)\,dt + a_{n-1}\frac{(x-a)^{n-1}}{(n-1)!} + a_{n-2}\frac{(x-a)^{n-2}}{(n-2)!} + \ldots\ldots + a_1(x-a) + a_0 \quad \ldots\ldots(11)$$

Now, multiplying (3), (5), ..., (10) and (11) by 1, $C_1(x)$, ..., $C_{n-1}(x)$ and $C_n(x)$ respectively, we get

$$\frac{d^ny}{dx^n} + C_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + C_n(x).y = u(x) + a_{n-1}C_1(x) + \{a_{n-2} + (x-a)a_{n-1}\}C_2(x) + \ldots\ldots\{a_0 + a_1(x-a) + \ldots\ldots + a_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\}C_n(x)$$

$$+\int_0^x [C_1(x) + (x-t)C_2(x) + \frac{(x-t)^2}{2!}C_3(x)$$

$$+ \ldots\ldots + \frac{(x-t)^{n-1}}{(n-1)!}C_n(x)]\, u(t)\, dt$$

$$\Rightarrow \quad \phi(x) = u(x) + g(x) - \int_0^x k(x,t)\, u(t)\, dt \qquad \ldots\ldots(12)$$

where $g(x) = a_{n-1}C_1(x) + \ldots\ldots + a_{n-1}\dfrac{(x-a)^{n-1}}{(n-1)!}C_n(x)$ ......(13)

and $k(x,t) = -[C_1(x) + (x-t)C_2(x) + \ldots\ldots \dfrac{(x-t)^{n-1}}{(n-1)!}C_n(x)]$ ......(14)

Now, let

$$\phi(x) - g(x) = f(x) \qquad \ldots\ldots(15)$$

$\therefore$ using (15), (12) gives

$$u(x) = f(x) + \int_0^x k(x,t)\, u(t)\, dt$$

which is the required integral equation of second kind.

SOLVED EXAMPLES

EXAMPLE 1

Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

with initial conditions $y(0) = 0,\ y'(0) = -1$. **[KANPUR-1991]**

Solution : Let us suppose that

$$\frac{d^2y}{dx^2} = u(x) \qquad \ldots\ldots(1)$$

Integrating (1) both sides w.r.t. x from 0 to x, we get

$$\frac{dy}{dx} - y'(0) = \int_0^x u(t)\, dt$$

$$= \int_0^x u(t)\, dt - 1 \qquad [\because y'(0) = -1] \qquad \ldots\ldots(2)$$

Again integrating, we have

$$y - y(0) = \int_0^x u(t)\, dt^2 - x$$

$$= \int_0^x (x-t)u(t)dt - x \qquad [\because y(0) = 0] \qquad \ldots\ldots(3)$$

Substituting the values of (1), (2) and (3), in the given equation, we get

$$u(x) - 5\left\{\int_0^x u(t)\, dt - 1\right\} + 6\left\{\int_0^x (x-t)u(t)dt - x\right\} = 0$$

$$\Rightarrow \quad u(x) = (6x-5) + \int_0^x (5 - 6x + 6t)\, u(t)\, dt$$

EXAMPLE 2

Form an integral equation corresponding to the differential equation given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0$$

with initial conditions $y(0) = 1, y'(0) = 0$. **[GARHWAL-2000]**

Solution : Here, the given differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0 \qquad \text{......(1)}$$

such that $y(0) = 1$ and $y'(0) = 0$ ......(2)

Let us suppose

$$\frac{d^2y}{dx^2} = u(x) \qquad \text{......(3)}$$

By integrating with respect to x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx \;\Rightarrow\; \frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \int_0^x u(x)\,dx \qquad [\because y'(0) = 0]$$

$$\text{or} \quad \frac{dy}{dx} = \int_0^x u(t)\,dt \qquad \text{......(4)}$$

Now integrating (4) w.r.t. x, we get

$$y(x) - y(0) = \int_0^x u(x)\,dx^2$$

$$\Rightarrow \quad y(x) - 1 = \int_0^x u(t)\,dt^2 \qquad [\because y(0) = 1]$$

$$\Rightarrow \quad y(x) = 1 + \int_0^x (x-t)\,u(t)\,dt \qquad \text{......(5)}$$

Finally, putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and $y(x)$ from (3), (4) and (5) in (1), we get

$$u(x) - 2x\int_0^x u(t)\,dt - 3\left[1 + \int_0^x (x-t)\,u(t)\,dt\right] = 0$$

$$\Rightarrow \quad u(x) - 3 - \int_0^x (2x)\,u(t)\,dt - \int_0^x 3(x-t)\,u(t)\,dt = 0$$

$$\Rightarrow \quad u(x) = 3 + \int_0^x (5x - 3t)\,u(t)\,dt$$

which is the required Volterra integral equation of second kind.

EXAMPLE 3

Form an integral equation corresponding to the differential equation given by

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

with initial conditions $y(0) = 1, y'(0) = 0$. **[MEERUT-1997, 98, 99, 2003]**

Solution : Let

$$\frac{d^2y}{dx^2} = u(x) \qquad \ldots\ldots(1)$$

Then $\frac{dy}{dx} - y'(0) = \int_0^x u(t)\,dt = \int_0^x u(t)\,dt \qquad [\because y'(0) = 0] \qquad \ldots\ldots(2)$

and $y - y(0) = \int_0^x (x-t)\,u(t)\,dt$

$$= \int_0^x (x-t)\,u(t)\,dt + 1 \qquad [\because y(0) = 1] \qquad \ldots\ldots(3)$$

Now putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from (1), (2) and (3) in the given equation, we get

$$u(x) + x\int_0^x u(t)\,dt + \left\{\int_0^x (x-t)\,u(t)\,dt + 1\right\} = 0$$

$$\Rightarrow \quad u(x) = -1 - \int_0^x (2x-t)\,u(t)\,dt$$

which is the required Volterra's integral equation of second kind.

EXAMPLE 4

Form an integral equation corresponding to the differential equation given by

$$y''' - 2xy = 0$$

with initial conditions $y(0) = 1/2, y'(0) = 1, y''(0) = 1$. **[KANPUR-1998]**

Solution : Here, the given differential equation is

$$\frac{d^3y}{dx^3} - 2xy = 0 \qquad \ldots\ldots(1)$$

with initial conditions

$$y(0) = \frac{1}{2},\ y'(0) = 1,\ y''(0) = 1 \qquad \ldots\ldots(2)$$

Let us suppose

$$\frac{d^3y}{dx^3} = u(x) \qquad \ldots\ldots(3)$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\left[\frac{d^2y}{dx^2}\right]_0^x = \int_0^x u(x)\,dx \Rightarrow \frac{d^2y}{dx^2} - y''(0) = \int_0^x u(x)\,dx$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 1 + \int_0^x u(x)\,dx \qquad \ldots\ldots(4)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 1 + \int_0^x u(t)\,dt \qquad \ldots\ldots(5)$$

Integrating (5) w.r.t. x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x dx + \int_0^x u(x)\,dx^2$$

$$\Rightarrow \quad \frac{dy}{dx} - y'(0) = x + \int_0^x u(x)\,dx^2$$

or
$$\frac{dy}{dx} = 1 + x + \int_0^x u(x)\,dx^2 \qquad \text{......(6)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + x + \int_0^x u(t)\,dt^2 \qquad \text{......(7)}$$

$$= 1 + x + \int_0^x (x-t)\,u(t)\,dt \qquad \text{......(8)}$$

Again, integrating (8) from 0 to x, we get

$$y(x) - y(0) = \int_0^x (1+x)\,dx + \int_0^x u(x)\,dx^3$$

$$\Rightarrow \quad y(x) = \frac{1}{2} + \left[x + \frac{x^2}{2}\right] + \int_0^x u(t)dt^3 = \frac{1}{2} + x + \frac{1}{2}x^2 + \int_0^x \frac{(x-t)^2}{2!}\,u(t)\,dt \qquad \text{......(9)}$$

Now, putting the value of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and y in the given equation (1), we get

$$u(x) - 2x\left[\frac{1}{2} + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2\,u(t)\,dt\right] = 0$$

$$\Rightarrow \quad u(x) = x(1 + 2x + x^2) + x\int_0^x (x-t)^2\,u(t)\,dt$$

$$= x(x+1)^2 + \int_0^x x(x-t)^2 u(t)\,dt$$

which is the required integral equation.

EXAMPLE 5

Form an integral equation corresponding to the differential equation given by

$$\frac{d^2y}{dx^2} + xy = 1, \quad y(0) = y'(0) = 0$$

[MEERUT-2004 BP]

Solution : Here, the given differential equation is

$$\frac{d^2y}{dx^2} + xy = 1 \qquad \text{......(1)}$$

with initial conditions

$$y(0) = y'(0) = 0 \qquad \text{......(2)}$$

Let us suppose

$$\frac{d^2y}{dx^2} = u(x) \qquad \text{......(3)}$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx \qquad \ldots\ldots(4)$$

$$\Rightarrow \quad \frac{dy}{dx} = \int_0^x u(x)\,dx \qquad [\because y'(0) = 0] \qquad \ldots\ldots(5)$$

$$\Rightarrow \quad \frac{dy}{dx} = \int_0^x u(t)\,dt$$

Integrating (5) w.r.t. x, we get

$$y(x) - y(0) = \int_0^x u(x)\,dx^2 \qquad \ldots\ldots(6)$$

$$\Rightarrow \quad y(x) = \int_0^x (x-t)\,u(t)\,dt \qquad [\because y(0) = 0]$$

Finally, putting the values of $\frac{d^2y}{dx^2}$ and y from above equation, in the given equation, we get

$$u(x) + x\int_0^x (x-t)\,u(t)\,dt = 1 \Rightarrow u(x) = 1 - \int_0^x x(x-t)\,u(t)\,dt \qquad \ldots\ldots(7)$$

which is the required integral equation.

EXAMPLE 6

Form an integral equation corresponding to the differential equation given by

$$\frac{d^2y}{dx^2} - \sin x.\frac{dy}{dx} + e^x.y = x$$

with the initial conditions $y(0) = 1, y'(0) = -1$. **[MEERUT-2002, 03, 04, 07(BP)]**

Solution : Here, the given differential equation is

$$\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} + e^x.y = x \qquad \ldots\ldots(1)$$

with initial conditions

$$y(0) = 1, y'(0) = -1 \qquad \ldots\ldots(2)$$

Let us suppose

$$\frac{d^2y}{dx^2} = u(x) \qquad \ldots\ldots(3)$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = -1 + \int_0^x u(x)\,dx \qquad [\because y'(0) = -1] \qquad \ldots\ldots(4)$$

or $\frac{dy}{dx} = -1 + \int_0^x u(t)\,dt$ ......(5)

Integrating (4) w.r.t. x, we get

$$[y(x)]_0^x = -[x]_0^x + \int_0^x u(x)\,dx^2$$

i.e., $y(x) - y(0) = -x + \int_0^x u(t)\,dt^2$

$\Rightarrow$ $y(x) = 1 - x + \int_0^x (x-t)\,u(t)\,dt$ $[\because y(0) = 1]$ ......(6)

Finally, putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from (3), (5) and (6) into (1), we get

$$u(x) - \sin x\left[-1 + \int_0^x u(t)\,dt\right] + e^x\left[1 - x + \int_0^x (x-t)\,u(t)\,dt\right] = x$$

$$\Rightarrow \quad u(x) = x - \sin x - e^x(1-x) + \int_0^x \sin x\, u(t)\,dt - \int_0^x e^x(x-t)\,u(t)\,dt$$

Hence, $u(x) = x - \sin x - e^x(1-x) + \int_0^x [\sin x - e^x(x-t)]\,u(t)\,dt$

which is the required integral equation.

EXAMPLE 7

Form an integral equation corresponding to the differential equation given by

$\frac{d^2y}{dx^2} + y = \cos x$ *with* $y(0) = 0, y'(0) = 1$. **[MEERUT-1996, KANPUR-1999]**

Solution : Here, the given differential equation is

$$\frac{d^2y}{dx^2} + y = \cos x \qquad ......(1)$$

with initial conditions

$$y(0) = 0,\ y'(0) = 1 \qquad ......(2)$$

Let us suppose

$$\frac{d^2y}{dx^2} = u(x) \qquad ......(3)$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\frac{dy}{dx} = \int_0^x u(x)\,dx + 1 \Rightarrow \frac{dy}{dx} = \int_0^x u(t)\,dt + 1 \qquad ......(4)$$

Again integrating, we get

$$y = \int_0^x u(t)\,dt^2 + x = \int_0^x (x-t)\,u(t)\,dt + x \qquad ......(5)$$

Finally, putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from (3), (4) and (5) in (1), we get

$$u(x) = (\cos x - x) - \int_0^x (x-t)\,u(t)\,dt$$

which is the required Volterra integral equation of second kind.

EXAMPLE 8

Form an integral equation corresponding to the differential equation given by

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + (x^2 - x)y = xe^x + 1$$

with initial conditions $y(0) = 1 = y'(0),\ y''(0) = 0$. **[GARHWAL-1999]**

Solution : Here, the given differential equation is

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + (x^2 - x)y = xe^x + 1 \qquad \text{......(1)}$$

with initial conditions

$$y(0) = 1 = y'(0),\ y''(0) = 0. \qquad \text{......(2)}$$

Let us suppose

$$\frac{d^3y}{dx^3} = u(x) \qquad \text{......(3)}$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\frac{d^2y}{dx^2} = \int_0^x u(t)\, dt \qquad \text{......(4)}$$

Again integrating w.r.t. x from 0 to x, we get

$$\frac{dy}{dx} = \int_0^x (x - t)\, u(t)\, dt + 1$$

Similarly we may get

$$y = \frac{1}{2!}\int_0^x (x - t)^2\, u(t)\, dt + x + 1 \qquad \text{......(5)}$$

Now, putting all these values in the given equation (1), we get

$$u(x) + x\left[\int_0^x u(t)\, dt\right] + (x^2 - x)\left[\frac{1}{2}\int_0^x (x - t)^2 u(t)\, dt + x + 1\right] = xe^x + 1$$

or $$u(x) = xe^x + 1 - x(x^2 - 1) - \int_0^x \left[x + \frac{1}{2}(x^2 - x)(x - t)^2\right] u(t)\, dt$$

which is the required integral equation.

EXAMPLE 9

Show that the linear differential equation of second order

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)$$

with initial conditions $y(0) = C_o$ *and* $y'(0) = C_1$ *can be transformed into non-homogeneous Volterra integral equation of second kind.*

Solution : Here, the given differential equation is

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \qquad \text{......(1)}$$

with initial conditions

$$y(0) = C_o \text{ and } y'(0) = C_1 \qquad \ldots\ldots(2)$$

Let us suppose that

$$\frac{d^2y}{dx^2} = u(x) \qquad \ldots\ldots(3)$$

Integrating (3) w.r.t. x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx \Rightarrow \frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

or $$\frac{dy}{dx} = C_1 + \int_0^x u(x)\,dx \Rightarrow \frac{dy}{dx} = C_1 + \int_0^x u(t)\,dt \qquad \ldots\ldots(4)$$

Integrating (4) w.r.t. x from 0 to x, we get

$$y(x) - y(0) = C_1 \int_0^x dx + \int_0^x u(x)\,dx^2$$

$$\Rightarrow \quad y(x) = y(0) + C_1 x + \int_0^x u(t)\,dt^2 = C_o + C_1 x + \int_0^x u(t)\,dt^2$$

$$= C_o + C_1 x + \int_0^x (x-t)\,u(t)\,dt \qquad \ldots\ldots(5)$$

Putting values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from (3), (4) and (5) in (1), we get

$$u(x) + a_1(x)\left[C_1 + \int_0^x u(t)dt\right] + a_2(x)\left[C_o + C_1 x + \int_0^x (x-t)\,u(t)\,dt\right] = F(x)$$

$$\Rightarrow \quad u(x) = F(x) - C_1 a_1(x) - [C_o + C_1 x]\,a_2(x) - \int_0^x a_1(x)u(t)dt - \int_0^x a_2(x)\,(x-t)u(t)dt$$

$$= F(x) - C_1 a_1(x) - (C_o + C_1 x)a_2(x) - \int_0^x [a_1(x) + a_2(x)(x-t)]\,u(t)\,dt$$

which is the required non-homogeneous Volterra's integral equation of second kind.

SOME SPECIAL TYPES OF PROBLEMS

EXAMPLE 10

Convert the following differential equation into an integral equation

$$y''(x) + \lambda y(x) = f(x), \quad y(0) = 1, \quad y'(0) = 0$$ [MEERUT-1998]

Solution : Here, the given integral equation is

$$y'' + \lambda y(x) = f(x) \qquad \ldots\ldots(1)$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0 \qquad \ldots\ldots(2)$$

From (1), we have

$$y''(x) = f(x) - \lambda y(x)$$

Integrating both sides of (3), w.r.t. x from 0 to x, we get

$$\int_0^x y''(x)dx = \int_0^x [f(x) - \lambda y(x)]\, dx$$

$\Rightarrow \quad [y'(x)]_0^x = \int_0^x [f(x) - \lambda y(x)]\, dx$

$\Rightarrow \quad y'(x) - y'(0) = \int_0^x [f(x) - \lambda y(x)]\, dx = \int_0^x [f(x) - \lambda y(x)]\, dx$

i.e., $\quad y'(x) = \int_0^x [f(x) - \lambda y(x)]\, dx \quad [\because y'(0) = 0]$ ......(3)

Integrating both sides of (3) from 0 to x, w.r.t. x, we get

$$\int_0^x y'(x)dx = \int_0^x [f(x) - \lambda y(x)]\, dx^2 \quad \text{or} \quad [y(x)]_0^x = \int_0^x [f(x) - \lambda y(x)]\, dx^2$$

i.e., $\quad y(x) = y(0) + \int_0^x [f(x) - \lambda y(x)]\, dx^2$

$$= 1 + \int_0^x [f(t) - \lambda y(t)]\, dt^2 = 1 + \int_0^x (x-t)[f(t) - \lambda y(t)]\, dt$$

which is the required integral equation.

EXAMPLE 11

Convert the following differential equation into an integral equation $y'' + y = 0$ *with initial conditions* $y(0) = y'(0) = 0$.

Solution : Here, the given differential equation is

$y'' + y = 0$ ......(1)

with initial conditions

$y(0) = y'(0) = 0$ ......(2)

From (1), we get

$y''(x) = -y(x)$ ......(3)

Integrating both sides of (3), w.r.t. x from 0 to x, we get

$$\int_0^x y''(x)dx = -\int_0^x y\,dx$$

$\Rightarrow \quad [y'(x)]_0^x = -\int_0^x y(x)\, dx$

i.e., $\quad y'(x) - y'(0) = -\int_0^x y(x)dx$

$\Rightarrow \quad y'(x) = -\int_0^x y(x)\, dx$ [using (2)] ......(4)

Again, integrating (4) w.r.t. x from 0 to x, we get

$$\int_0^x y'(x)dx = -\int_0^x y(x)\, dx^2$$

i.e., $\quad [y(x)]_0^x = -\int_0^x y(x)\, dx^2$

$\Rightarrow \quad y(x) - y(0) = \int_0^x [illegible]\, dt$

$\Rightarrow \quad y(x) = \int_0^x [illegible]\, dt$

$= \int_0^x (x - t)\, y(t)\, dt$ [illegible]

which is the required integral equation.

EXAMPLE 12

Convert $y''(x) - \sin x\, y'(x) + e^x y(x) = x$ with initial conditions $y(0) = 1$, $y'(0) = -1$ to a Volterra integral equation of second kind. Conversely, derive the original differential equation with the initial conditions from the integral equation obtained.

[MEERUT-2002, 04, 07]

Solution Here, the given differential equation is

$$y'' - \sin x\, y'(x) + e^x y(x) = x \qquad \ldots(1)$$

with initial conditions

$$y(0) = 1,\ y'(0) = -1 \qquad \ldots(2)$$

From (1), we get

$$y''(x) = x - e^x y(x) + \sin x\, y'(x) \qquad \ldots(3)$$

Integrating both sides of (3), w.r.t. x from 0 to x, we get

$$\int_0^x y''(x)\,dx = \int_0^x x\,dx - \int_0^x e^x y(x)\,dx + \int_0^x \sin x\, y'(x)\,dx$$

or $\quad [y'(x)]_0^x = \dfrac{x^2}{2} - \int_0^x e^x y(x)\,dx + [\sin x . y(x)]_0^x - \int_0^x \cos x . y(x)\,dx$

$\Rightarrow \quad y'(x) - y'(0) = \dfrac{x^2}{2} - \int_0^x e^x y(x)\,dx + \sin x . y(x) - \int_0^x \cos x\, y(x)\,dx$

$\Rightarrow \quad y'(x) = -1 + \dfrac{x^2}{2} - \int_0^x e^x y(x)\,dx + \sin x . y(x) - \int_0^x \cos x\, y(x)\,dx \qquad \ldots(4)$

Again, integrating both sides of (4) w.r.t. x from 0 to x, we get

$$\int_0^x y'(x)\,dx = \int_0^x \left(\frac{x^2}{2} - 1\right) dx + \int_0^x \sin x . y(x)\,dx - \int_0^x (e^x + \cos x)\, y(x)\,dx^2$$

$\Rightarrow \quad [y(x)]_0^x = \left[\dfrac{x^3}{6} - x\right]_0^x + \int_0^x \sin t . y(t)\,dt - \int_0^x (e^t + \cos t)\, y(t)\,dt^2$

$\Rightarrow \quad y(x) - y(0) = \dfrac{x^3}{6} - x + \int_0^x \sin t . y(t)\,dt - \int_0^x (x - t)(e^t + \cos t)\, y(t)\,dt$

$\Rightarrow \quad y(x) = 1 + \dfrac{x^3}{6} - x + \int_0^x \left[\sin t - (x - t)(e^t + \cos t)\right] y(t)\,dt \qquad \ldots(5)$

which is the required integral equation of second kind.

Conversely, to find the differential equation from (5), differentiating (5) w.r.t. x, we get

$$y'(x) = \frac{x^2}{2} - 1 + \frac{d}{dx}\int_0^x [\sin t - (x-t)(e^t + \cos t)]\, y(t)\, dt$$

$$= \frac{x^2}{2} - 1 + \int_0^x \frac{\partial}{\partial x}\left[\{\sin t - (x-t)(e^t + \cos t)\}\, y(t)\right] dt$$

$$+ \left[\sin x - (x-x)(e^x + \cos x)\right] y(x)\frac{dx}{dx}$$

$$-\left[\sin 0 - (x-0)(e^0 + \cos 0)\right] y(0)\frac{d0}{dx} \quad \text{[By Leibnitz's rule]}$$

$$\Rightarrow \quad y'(x) = \frac{x^2}{2} - 1 - \int_0^x (e^t + \cos t)\, y(t)\, dt + \sin x\, y(x) \quad \ldots\ldots(6)$$

Now, differentiating both sides of (6) w.r.t. x, we get

$$y''(x) = x + \cos x . y(x) + \sin x y'(x) - \frac{d}{dx}\int_0^x (e^t + \cos t)\, y(t)\, dt$$

$$= x + \cos x . y(x) + \sin x y'(x) - \left[\int_0^x \frac{\partial}{\partial x}\{(e^t + \cos t) y(t)\}\, dt\right.$$

$$\left. + (e^x + \cos x)\, y(x) . \frac{dx}{dx} - (e^0 + \cos 0)\, y(0) . \frac{d0}{dx}\right] \quad \text{[By Leibnitz rule]}$$

$$\Rightarrow \quad y''(x) = x + \cos x . y(x) + \sin x . y'(x) - \left[0 + (e^x + \cos x) y(x) + 0\right]$$

$$\Rightarrow \quad y''(x) - \sin x . y'(x) + e^x . y(x) = x \quad \ldots\ldots(7)$$

which is the same as given differential equation (1).

Also from (7), we can easily obtain

$$y(0) = 1 \text{ and } y'(0) = -1.$$

EXAMPLE 13

Convert the following initial value problem into an integral equation

$$\frac{d^2y}{dx^2} + A(x) . \frac{dy}{dx} + B(x) . y = f(x)$$

with initial condition

$$y(a) = y_o,\ y'(a) = y_o'.$$

Solution : Here, the given differential equation is

$$y''(x) + A(x) . y'(x) + B(x) . y(x) = f(x) \quad \ldots\ldots(1)$$

with initial conditions

$$y(a) = y_o,\ y'(a) = y_o'. \quad \ldots\ldots(2)$$

From (1), we get

$$y''(x) = f(x) - B(x)y(x) - A(x)y'(x) \quad \ldots\ldots(3)$$

Integrating both sides of (1), w.r.t. x from a to x, we get

$$[y'(x)]_a^x = \int_a^x [f(x) - B(x)y(x)]\, dx - \int_a^x A(x) . y'(x)\, dx$$

or $$y'(x) - y'(a) = \int_a^x [f(x) - B(x)y(x)]\,dx - \left\{ [A(x)y(x)]_a^x - \int_a^x A'(x)\,y(x)\,dx \right\}$$

$$\Rightarrow y'(x) - y_o' = \int_a^x [f(x) - B(x)y(x)]\,dx - \left\{ A(x)\,y(x) - A(a)\,y(a) - \int_a^x A'(x)\,y(x) \right\} dx \quad \text{......(4)}$$

Integrating (4) w.r.t. x from a to x, we get

$$\int_a^x y'(x)dx = \int_a^x [y_o' + y_o A(a)]\,dx - \int_a^x A(x)\,y(x)\,dx + \int_a^x \{f(x) - B(x)y(x) + A'(x)y(x)\}\,.dx^2$$

$$\Rightarrow [y(x)]_a^x = [y_o' + y_o A(a)](x-a) - \int_a^x A(x)y(x)\,dx + \int_a^x \{f(t) - B(t)\,y(t) + A'(t)\,y(t)\}\,dt^2$$

$$\Rightarrow y(x) - y(a) = [y_o' + y_o A(a)](x-a) - \int_a^x A(t)y(t)\,dt + \int_a^x (x-t)\{f(t) - B(t)\,y(t) + A'(t)\,y(t)\}\,dt$$

or $$y(x) = y_o + [y_o' + y_o A(a)](x-a) + \int_a^x (x-t)\,f(t)\,dt - \int_a^x [A(t) + (x-t)\{B(t) - A'(t)\}]\,y(t)\,dt$$

which is the required Volterra's integral equation.

EXAMPLE 14

Convert $y''(x) - 3y'(x) + 2y(x) = 4\sin x$ with initial conditions $y(0) = 1$, $y'(0) = -2$ into a Volterra integral equation of second kind. Conversely, derive the original differential equation with the initial conditions from the integral equation obtained.

[MEERUT-2001, 03, 04, 06, GARHWAL-2003]

Solution : Here, the given differential equation is

$$y''(x) - 3y'(x) + 2y(x) = 4\sin x \quad \text{......(1)}$$

with initial conditions

$$y(0) = 1,\ y'(0) = -2 \quad \text{......(2)}$$

From (1), we have

$$y''(x) = 4\sin x - 2y(x) + 3y'(x) \quad \text{......(3)}$$

Integrating both sides of (3), w.r.t. x from 0 to x, we get

$$\int_0^x y''(x)dx = 4\int_0^x \sin x\,dx - 2\int_0^x y(x)\,dx + 3\int_0^x y'(x)\,dx$$

$$\Rightarrow [y'(x)]_0^x = 4[-\cos x]_0^x - 2\int_0^x y(x)\,dx + 3[y(x)]_0^x$$

$$\Rightarrow y'(x) - y'(0) = 4(-\cos x + 1) - 2\int_0^x y(x)\,dx + 3\{y(x) - y(0)\}$$

$$\Rightarrow \quad y'(x) + 2 = 4 - 4\cos x - 2\int_0^x y(x)\,dx + 3y(x) - 3$$

$$\Rightarrow \quad y'(x) = -1 - 4\cos x + 3y(x) - 2\int_0^x y(x)\,dx \qquad \ldots\ldots(4)$$

Again integrating both sides of (4) w.r.t. x from 0 to x, we get

$$\int_0^x y'(x)dx = -\int_0^x dx - 4\int_0^x \cos x\,dx + 3\int_0^x y(x)\,dx - 2\int_0^x y(x)\,dx^2$$

$$\Rightarrow \quad [y(x)]_0^x = -x - 4[\sin x]_0^x + 3\int_0^x y(x)\,dx - 2\int_0^x y(t)\,dt^2$$

$$\Rightarrow \quad y(x) - y(0) = -x - 4\sin x + 3\int_0^x y(t)\,dt - 2\int_0^x (x-t)\,y(t)\,dt$$

$$\Rightarrow \quad y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2(x-t)]\,y(t)\,dt \qquad \ldots\ldots(5)$$

which is the required Volterra integral equation of second kind.

Conversely, we want to derive the given differential equation from integral equation (5),

Differentiating (5) w.r.t. x, we get

$$y'(x) = -1 - 4\cos x + \frac{d}{dx}\int_0^x [3 - 2(x-t)]\,y(t)\,dt$$

$$= 1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}[\{3 - 2(x-t)\}]\,y(t)\,dt + \left[\,[3 - 2(x-x)]\,y(x)\frac{dx}{dx}\right]$$

$$- [3 - 2(x-0)]\,y(0)\frac{d0}{dx} \qquad \text{[By Leibnitz rule]}$$

$$\Rightarrow \quad y'(x) = -1 - 4\cos x + \int_0^x (-2)\,y(t)\,dt + 3y(x)$$

$$\Rightarrow \quad y'(x) = -1 - 4\cos x + 3y(x) - 2\int_0^x y(t)\,dt \qquad \ldots\ldots(6)$$

Now, differentiating both sides of (6) w.r.t. x, we get

$$y''(x) = 4\sin x + 3y'(x) - 2\frac{d}{dx}\int_0^x y(t)\,dt$$

$$= 4\sin x + 3y'(x) - 2\left[\int_0^x \frac{\partial}{\partial x} y(t)\,dt + y(x)\frac{dx}{dx} - y(0).\frac{d0}{dx}\right]$$

$$= 4\sin x + 3y'(x) - 2[0 + y(x) - 0]$$

$\Rightarrow \quad y''(x) - 3y'(x) + 2y(x) = 4\sin x$, which is the required differential equation.

Now, putting x = 0 in (5), we get $y(0) = 1$.

Also, putting x = 0 in (6), we get

$$y'(0) = -1 - 4 + 3y(0) = -1 - 4 + 3 = -2.$$

EXERCISE - 2

(1) Reduce the following initial value problems to Volterra integral equation of second kind.

(i) $y'' + y = \cos x$ with initial conditions $y(0) = 0,\ y'(0) = 1$.

(ii) $y'' + y = 0$ with initial conditions $y(0) = 0,\ y'(0) = 1$.

Now using convolution theorem.

(4) $y'' = -xy' - y$. On integrating we have

$\Rightarrow \quad y'(x) - y'(0) = -\int_0^x xy'(x)\,dx - \int_0^x y\,dx$

$\Rightarrow \quad y'(x) = 1 - x.y(x)\,dx$

Again integrating, we have

$$y(x) - y(0) = x - x\int_0^x y(x)\,dx$$

$\Rightarrow \quad y(x) = 1 + x - \int_0^x xy(x)\,dx$

(5) Integrating both sides of the given equation two times.

Answers

(1) (i) $u(x) = \cos x - x - \int_0^x (x-t)\,u(t)\,dt$ (ii) $u(x) = -x - \int_0^x (x-t)\,u(t)\,dt$

(iii) $u(x) = 6x - 5 + \int_0^x (5 - 6x + 6t)\,u(t)\,dt$

(iv) $u(x) = \cos x - 2x(1+x^2) - \int_0^x (1+x^2)(x-t)\,u(t)\,dt$

(v) $u(x) = 4(x + \sin x - 2) + \int_0^x [3 - 2(x-t)]\,u(t)\,dt$

(2) $y(x) = \dfrac{x^2}{2} + \int_0^x t\,(t-x)\,y(t)\,dt$ (3) $y(x) = 1 + \int_0^x (x+t)\,y(t)\,dt$

(4) $y(x) = 1 + x - \int_0^x t.y(t)\,dt$ (5) $y(x) = y_0 + xy_0' + \int_0^x (x-t)y(t)\,dt$

1.8 BOUNDARY VALUE PROBLEM

A problem is said to be boundary value problem, in which an ordinary differential equation is to be solved under conditions involving dependent variable and its derivatives at two different values of independent variable.

We can define a relationship between a linear differential equation and a Fredholm integral equation, by using the method discussed in next example.

$$u''(x) = -\lambda u(x) \qquad \ldots\ldots(3)$$

Integrating both sides of (3) w.r.t. x from 0 to x, we have

$$\int_0^x u''(x)\,dx = -\lambda \int_0^x u(x)\,dx$$

$$\Rightarrow \quad [u'(x)]_0^x = -\lambda \int_0^x u(x)\,dx$$

or $$u'(x) - u'(0) = -\lambda \int_0^x u(x)\,dx \qquad \ldots\ldots(4)$$

Let $u'(0) = C$, a constant. ......(5)

Then, using (5) in (4), we get

$$u'(x) = C - \lambda \int_0^x u(x)\,dx \qquad \ldots\ldots(6)$$

Integrating both sides of (6) w.r.t. x, we get

$$\int_0^x u'(x)\,dx = C\int_0^x dx - \lambda \int_0^x u(x)\,dx^2$$

$$\Rightarrow \quad [u(x)]_0^x = Cx - \lambda \int_0^x u(t)\,dt^2$$

$$\Rightarrow \quad u(x) - u(0) = Cx - \lambda \int_0^x (x-t)\,u(t)\,dt$$

$$\Rightarrow \quad u(x) - 0 = Cx - \lambda \int_0^x (x-t)\,u(t)\,dt \qquad \ldots\ldots(7)$$

Now putting x = l in (7), we get

$$u(l) = Cl - \lambda \int_0^l (l-t)\,u(t)\,dt \quad \Rightarrow \quad 0 = Cl - \lambda \int_0^l (l-t)\,u(t)\,dt$$

$$\Rightarrow \quad C = \frac{\lambda}{l} \int_0^l (l-t)\,u(t)\,dt \qquad \ldots\ldots(8)$$

Putting this value in (7), we get

$$u(x) = \frac{\lambda}{l}.x\int_0^l (l-t)\,u(t)\,dt - \lambda \int_0^x (x-t)\,u(t)\,dt \qquad \ldots\ldots(9)$$

$$\Rightarrow \quad u(x) = \int_0^l \frac{\lambda x(l-t)}{l}\,u(t)\,dt - \lambda \int_0^x (x-t)\,u(t)\,dt$$

$$\Rightarrow \quad u(x) = \int_0^x \frac{\lambda x(l-t)}{l}\,u(t)\,dt + \int_x^l \frac{\lambda x(l-t)}{l}\,u(t)\,dt - \int_0^x \lambda(x-t)\,u(t)\,dt$$

$$= \lambda \int_0^x \left[\frac{x(l-t)}{l} - (x-t)\right] u(t)\,dt + \lambda \int_x^l \frac{x(l-t)}{l}\,u(t)\,dt$$

$$= \lambda \int_0^x \frac{x(l-t) - l(x-t)}{l}\,u(t)\,dt + \lambda \int_x^l \frac{x(l-t)}{l}\,u(t)\,dt$$

$$= \lambda \left[\int_0^x \frac{t(l-x)}{l}\,u(t)\,dt + \int_x^l \frac{x(l-t)}{l}\,u(t)\,dt\right]$$

$$= \lambda \int_0^l k(x,t)\,u(t)\,dt$$

where $k(x,t) = \begin{cases} \dfrac{t(l-x)}{l}; & \text{if } 0 < t < x \\ \dfrac{x(l-t)}{l}; & \text{if } x < t < l \end{cases}$

EXAMPLE 2

Obtain Fredholm integral equation of second kind corresponding to the boundary value problem is

$$\frac{d^2u}{dx^2} + \lambda u = x \text{ with boundary conditions}$$

$$u(0) = 0, \ u'(1) = 0.$$

Also, recover the boundary value problem from the integral equation you obtain.

[MEERUT-2000, 05, 05(BP), 06(BP),08]

Solution : Here, the given equation is

$$\frac{d^2u}{dx^2} + \lambda u = x \quad \text{......(1)}$$

with boundary conditions

$$u(0) = 0, \ u'(1) = 0. \quad \text{......(2)}$$

Integrating, both the sides of (1) (from 0 to x) w.r.t. x, we get

$$\int_0^x \frac{d^2u}{dx^2}\, dx = \int_0^x x\,dx - \lambda \int_0^x u(x)\, dx$$

$$\Rightarrow \quad u'(x) - u'(0) = \frac{1}{2}x^2 - \lambda \int_0^x u(t)\, dt$$

Let $u'(0) = C$, a constant

Then, we have

$$u'(x) = C + \frac{x^2}{2} - \lambda \int_0^x u(x)\, dx \quad \text{......(3)}$$

$$\Rightarrow \quad u'(x) = C + \frac{1}{2}x^2 - \lambda \int_0^x u(t)\, dt$$

Again integrating both sides of (3) w.r.t. x, we get

$$u(x) - u(0) = Cx + \frac{1}{6}x^3 - \lambda \int_0^x u(x)\, dx^2$$

$$\Rightarrow \quad u(x) = Cx + \frac{1}{6}x^3 - \lambda \int_0^x u(t)\, dt^2$$

$$= Cx + \frac{1}{6}x^3 - \lambda \int_0^x (x-t)\, u(t)\, dt \quad \text{......(4)}$$

Now, since $u'(1) = 0 \Rightarrow u'(1) = C + \frac{1}{2} - \lambda \int_0^1 u(t)\, dt$

$$\Rightarrow \quad C = -\frac{1}{2} + \lambda \int_0^1 u(t)\, dt$$

$$\Rightarrow \quad u(x) = -\frac{1}{2}x + \frac{1}{6}x^3 + \lambda\int_0^1 x\,u(t)\,dt - \lambda\int_0^x (x-t)\,u(t)\,dt \qquad \text{......5(a)}$$

$$= -\frac{1}{2}x + \frac{1}{6}x^3 + \lambda\left\{\int_0^x x\,u(t)\,dt + \int_x^1 x\,u(t)\,dt\right\} - \lambda\int_0^x (x-t)\,u(t)\,dt$$

or

$$u(x) = \frac{1}{6}(x^3 - 3x) + \lambda\left\{\int_0^x t u(t)\,dt + \int_x^1 x\,u(t)\,dt\right\}$$

$$= \frac{1}{6}(x^3 - 3x) + \lambda\int_0^1 k(x,t)\,u(t)\,dt \qquad \text{......5(b)}$$

where $k(x,t) = \begin{cases} t; & 0 < t < x \\ x; & x < t < 1 \end{cases}$

Conversely, we want to convert (5) to the boundary value problem.

Equation 5(a) can be written as

$$u(x) = -\left(\frac{1}{2}\right)x + \frac{1}{6}x^3 + \lambda\int_0^1 x\,u(t)\,dt - \lambda\int_0^x (x-t)\,u(t)\,dt \qquad \text{......(6)}$$

Differentiating (6) w.r.t. x, we have

$$u'(x) = -\frac{1}{2} + \frac{1}{2}x^2 + \lambda\frac{d}{dx}\int_0^1 x\,u(t)\,dt - \lambda\frac{d}{dx}\int_0^x (x-t)\,u(t\,dt$$

$$\Rightarrow \quad u'(x) = -\frac{1}{2} + \frac{1}{2}x^2 + \lambda\int_0^1 u(t)\,dt - \lambda\int_0^x u(t)\,dt \qquad \text{......(7)}$$

Again differentiating w.r.t. x, we get

$$u''(x) = x + \lambda\frac{d}{dx}\int_0^1 u(t)\,dt - \lambda\frac{d}{dx}\int_0^x u(t)\,dt$$

$$= x + \lambda\int_0^1 \frac{\partial}{\partial x}\{u(t)\}\,dt - \lambda\int_0^x \frac{\partial}{\partial x}\{u(t)\}\,dt - \lambda u(x) \qquad \text{[By Leibnitz's rule]}$$

$$\Rightarrow \quad u''(x) = x - \lambda u(x)$$

$$\Rightarrow \quad u''(x) + \lambda u(x) = x \qquad \text{......(8)}$$

Also, from (6) and (7), we have

$$u(0) = 0 \text{ and } u'(1) = -\frac{1}{2} + \frac{1}{2} = 0. \qquad \text{......(9)}$$

Hence, equation (8) is the required differential equation with boundary condition (9).

EXAMPLE 3

Convert $\frac{d^2y}{dx^2} + xy = 1,\ y(0) = 0, y(1) = 1$ *into an integral equation.*

[MEERUT-2001, 05BP]

Solution : Here, the given equation is

$$y''(x) + xy(x) = 1 \qquad \text{......(1)}$$

with boundary conditions $y(0) = 0, y(1) = 1.$ ......(2)

From (1), we have

$$y''(x) = 1 - xy(x) \qquad \text{......(3)}$$

Integrating, both the sides of (3) w.r.t. x from 0 to x, we get

$$\int_0^x y''(x)\,dx = \int_0^x dx - \int_0^x xy(x)\,dx$$

$$\Rightarrow \quad [y'(x)]_0^x = x - \int_0^x xy(x)\,dx$$

i.e., $$y'(x) - y'(0) = x - \int_0^x xy(x)\,dx \qquad \text{......(4)}$$

Let $y'(0) = C$, a constant. Put this value in (4), we get

$$y'(x) = C + x - \int_0^x x.y(x)\,dx \qquad \text{......(5)}$$

Now, integrating both sides of (5) w.r.t. x from 0 to x, we get

$$\int_0^x y'(x)\,dx = \int_0^x (C + x)\,dx - \int_0^x xy(x)\,dx^2$$

or $$[y(x)]_0^x = \left[Cx + \frac{1}{2}x^2\right]_0^x - \int_0^x t\,y(t)\,dt^2$$

i.e., $$y(x) - y(0) = Cx + \frac{1}{2}x^2 - \int_0^x (x - t)\,t\,y(t)\,dt$$

$$\Rightarrow \quad y(x) = Cx + \frac{1}{2}x^2 - \int_0^x (x - t)\,t\,y(t)\,dt \qquad \text{[using (2)]} \qquad \text{......(6)}$$

Now, putting x = 1 in (6), we get

$$y(1) = C + \frac{1}{2} - \int_0^1 (1 - t)\,y(t)\,.\,t\,.\,dt \quad \Rightarrow \quad 1 = C + \frac{1}{2} - \int_0^1 (1 - t)\,.\,t\,.\,y(t)\,dt$$

i.e., $$C = \frac{1}{2} + \int_0^1 (1 - t)\,ty(t)\,dt \qquad \text{[using (2)]} \qquad \text{......(7)}$$

Put this value of C in equation (6), we get

$$y(x) = x\left[\frac{1}{2} + \int_0^1 (1 - t)\,t\,y(t)\,dt\right] + \frac{1}{2}x^2 - \int_0^x (x - t).t\,y(t)\,dt$$

$$= \frac{1}{2}x(1 + x) + \int_0^1 xt(1 - t)\,y(t)\,dt - \int_0^x t(x - t)\,y(t)\,dt$$

$$= \frac{1}{2}x(1 + x) + \int_0^x xt(1 - t)\,y(t)\,dt$$

$$+ \int_x^1 xt(1 - t)\,y(t)\,dt - \int_0^x t(x - t)\,y(t)\,dt$$

$$= \frac{1}{2}x(1 + x) + \int_0^x t\,y(t)\,\{x - xt - x + t\}\,dt + \int_x^1 xt(1 - t)\,y(t)\,dt$$

$$= \frac{1}{2}x(1 + x) + \int_0^x t^2(1 - x)\,y(t)\,dt + \int_x^1 xt(1 - t)\,y(t)\,dt$$

$$= \frac{1}{2}x(1 + x) + \int_0^1 k(x, t)\,y(t)\,dt \qquad \text{......(8)}$$

where, $$k(x, t) = \begin{cases} t^2(1 - x); & 0 < t < x \\ xt(1 - t); & x < t < 1 \end{cases}$$

EXAMPLE 4

If $u(x)$ is continuous and satisfies

$$u(x) = \lambda \int_0^1 k(x, t)\, u(t)\, dt$$

where, $k(x, t) = \begin{cases} (1-t)\,x\,; & 0 \le x < t \\ (1-x)t\,; & t < x \le 1 \end{cases}$

then show that $u(x)$ is also the solution of boundary value problem.

$$\frac{d^2u}{dx^2} + \lambda u = 0,\ u(0) = 0,\ u(1) = 0$$

Solution : Here, we have given that

$$u(x) = \lambda \int_0^1 k(x, t)\, u(t)\, dt \qquad \text{......(1)}$$

where, $k(x, t) = \begin{cases} (1-t)\,x\,; & 0 \le x \le t \\ (1-x)t\,; & t \le x \le 1 \end{cases}$ ......(2)

Using (2), equation (1) can be written as

$$u(x) = \lambda \left[\int_0^x k(x, t)\, u(t)\, dt + \int_x^1 k(x, t)\, u(t)\, dt \right]$$

$$= \lambda \int_0^x (1-x)\, t\, u(t)\, dt + \lambda \int_x^1 (1-t)\, x\, u(t)\, dt \qquad \text{[Using (2)]}$$

$$= \int_0^x \lambda t(1-x)\, u(t)\, dt + \int_x^1 \lambda x(1-t)\, u(t)\, dt \qquad \text{......(3)}$$

Putting x = 0 and x = 1 successively in (3), we get

$$u(0) = 0 \text{ and } u(1) = 0 \qquad \text{......(4)}$$

Now, differentiating both sides of (3) w.r.t. x, we get

$$\frac{du}{dx} = \frac{d}{dx} \int_0^x \lambda t(1-x)\, u(t)\, dt + \frac{d}{dx} \int_x^1 \lambda x(1-t)\, u(t)\, dt$$

$$= \int_0^x \frac{\partial}{\partial x} \{\lambda t(1-x)\}\, u(t)\, dt + \lambda x(1-x)\, u(x) \frac{dx}{dx} - 0.\frac{d0}{dx}$$

$$+ \int_x^1 \frac{\partial}{\partial x} \{\lambda x(1-t)\}\, u(t)\, dt + 0.\frac{d(1)}{dx} - \lambda x(1-x)\, u(x) \frac{dx}{dx}$$

[By Leibnitz rule]

$$\Rightarrow \quad \frac{du}{dx} = -\int_0^x \lambda t\, u(t)\, dt + \int_x^1 \lambda(1-t)\, u(t)\, dt \qquad \text{......(5)}$$

Differentiating both sides of (5) w.r.t. x, we get

$$\frac{d^2u}{dx^2} = -\left[\int_0^x \frac{\partial}{\partial x} \{\lambda t\}\, u(t).dt + \lambda x\, u(x) \frac{dx}{dx} - 0.\frac{d(0)}{dx} \right]$$

$$+ \int_x^1 \frac{\partial}{\partial x} \{\lambda(1-t)\}\, u(t)\, dt + 0.\frac{d(1)}{dx} - \lambda(1-x).u(x)\frac{dx}{dx}$$

[By Leibnitz rule]

$$= \lambda x\, u(x) - \lambda(1-x)\, u(x)$$
$$= -\lambda x\, u(x) - \lambda u(x) + \lambda x\, u(x)$$

$$\therefore \frac{d^2u}{dx^2} + \lambda u = 0 \qquad \text{......(6)}$$

Hence, we conclude that if u(x) satisfies (1), then u(x) is also the solution of the boundary value problem.

$$\frac{d^2u}{dx^2} + \lambda u = 0, \; u(0) = u(1) = 0 .$$

EXERCISE - 3

(1) Reduce the following boundary value problem into an integral equation

$$\frac{d^2y}{dx^2} + \lambda y = 0, \; y(0) = 0, \; y'(1) + vy(1) = 0$$

(2) (a) If $y''(x) + \lambda y(x) = 0$ and u satisfies the boundary conditions $y(0) = 0, y(l) = 0$, show that

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t)\, y(t)\, dt - \lambda \int_0^x (x-t)\, y(t)\, dt$$

(b) Show that for result of part (a) can be written as

$$y(x) = \lambda \int_0^l k(x,t)\, y(t)\, dt$$

$$\text{where } k(x,t) = \begin{cases} \dfrac{t(l-x)}{l}, & when\; t < x \\ \dfrac{x(l-t)}{l}, & when\; t > x \end{cases}$$

(c) Verify directly that the expression obtained satisfies the prescribed differential equation and boundary conditions.

(3) Convert the boundary value problem $y'' + y = 0, y(0) = 1, y'(1) = 0$ into an integral equation.

(4) Convert the following boundary value problem $y'' + \lambda y = x, \; y(0) = y(\pi) = 0$ into an integral equation.

(5) (a) If $y''(x) = F(x)$ and y satisfies the end conditions $y(0) = 0, y(1) = 0$, show that

$$y(x) = \int_0^x (x-t)\, F(t)\, dt - x\int_0^1 (1-t)\, F(t)\, dt$$

(b) Show that the result of part (a) can be written as $y(x) = \int_0^1 k(x,t)\, F(t)\, dt$, where $k(x,t)$ is given by

$$k(x,t) = \begin{cases} t(x-1), & when\; t < x \\ x(t-1), & when\; t > x \end{cases}$$

(c) Verify directly that the expression obtained satisfies the prescribed differential equation and boundary conditions.