UNIT – III

CONNECTED SPACES.

Defn: Let X be a T.S. A speration of x is a pair U, V of disjoint nonempty open subsets of X whose union is X.

The space X is said to be connected if there does not exist a separation of X.

Theorem: A space X is connected iff the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof:- If A is a nonempty proper subset of X that is both open and closed in X, then the sets U = A and V = X – A constitute a separation of X, since they are open, disjoint and nonempty and their union is X. Hence X is connected.

[Conversely, we shall prove that X is connected implies the only subsets of X that are both open and closed in X are the empty set and X itself.

By contrapositive statement,

Assume that] X.

Conversely, Assume that U and V form a separation of X.

Then U is nonempty and different from X

Since V is open and $X - V = U$ is closed.

Hence U is both open and closed in X.

Lemma: If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected iff there exists no separation of Y.

Proof:- By contrapositive statement, we shall prove that,

Y is not connected iff there exists a separation of Y.

Suppose first that Y is not connected.

Then there exists a pair A, B of disjoint open subsets of Y whose union is Y.

This implies A is both open and closed in Y.

Let $\bar{A}$ be the closure of A in X.

Then $\bar{A} \cap Y$ is the closure of A in Y

Since A is closed in Y, $A = \bar{A} \cap Y$

$$\Rightarrow A \cap B = \bar{A} \cap Y \cap B$$

$$\Rightarrow \phi = \bar{A} \cap B \quad \{\because B \subset Y \,\&\, A \cap B = \phi\}$$

Since $\bar{A}$ is the union of A and its limit points, B contains no limit point of A.

Similarly we can show that A contains no limit point of B.

Hence A, B form a separation of Y as a subspace of X.

Conversely,

Suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other.

Then $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

Then
$$\bar{A} \cap Y = \bar{A} \cap (A \cup B)$$
$$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$$
$$= A \cup \phi = A.$$

(i) $\bar{A} \cap Y = A$

Similarly $\bar{B} \cap Y = B$.

$\therefore$ A and B are closed in Y.

Since $A = Y - B$ and $B = Y - A$ they are open in Y.

Hence A & B are two disjoint nonempty open subsets of Y whose union is Y.

$\therefore$ Y is not connected.

Example: 1 Let X denote a two point space in the indiscrete topology.

(ie) $\tau = \{X, \phi\}$.

Then there is no separation of X, so X is connected.

Example: 2

Let $Y = [-1, 0) \cup (0, 1]$ be the subspace of $\mathbb{R}$.

Let $A = [-1, 0)$ and $B = (0, 1]$.

A and B are open in Y, since

$A = (-2, 0) \cap Y$ and $B = (0, 2) \cap Y$.

Also A and B are nonempty and disjoint.

Hence A an B form a separation of Y.

$\therefore Y$ is not connected.

Alternatively,

$\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$ and $Y = A \cup B$.

$\therefore A$ & B form a separation of Y as a subspace.

$\therefore Y$ is not connected.

Example: 3

Let $X = [-1, 1]$ be the subspace of the real line $\mathbb{R}$.

Let $A = [-1, 0]$ & $B = (0, 1]$

A & B are nonempty and disjoint.

But they donot form a separation of X, because A is not open.

Alternatively, $A \cap \bar{B} = \{0\} \neq \phi$.

$\therefore$ There exists no separation of X, so X is connected.

Example: 4

The rationals $\mathbb{Q}$ is not connected.

Let $p, q \in \mathbb{Q}$.

We can choose an irrational number a such that $p < a < q$.

Let $A = \{x \in \mathbb{Q} / -\infty < x < a\}$ &

$B = \{y \in \mathbb{Q} / a < y < \infty\}$

Then $A = \mathbb{Q} \cap (-\infty, a)$ and $B = \mathbb{Q} \cap (a, \infty)$

$\therefore$ A and B are open and disjoint.

$\therefore$ A & B form a separation of $\mathbb{Q}$.

$\therefore$ $\mathbb{Q}$ is not connected.

The only connected subspaces of $\mathbb{Q}$ are one point sets.

Example: 5 Consider the subset X of the plane $\mathbb{R}^2$. $X = \{x \times y / y = 0\} \cup \{x \times y / x > 0 \,\&\, y = 1/x\}$

Then X is not connected.

Because the two sets form a separation of X since neither contains the limit point of the other.

Lemma: If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely with either C or D.

Proof: Given that the sets C & D form a separation of X.

$\therefore$ C and D are both open and nonempty.

(ie) $X = C \cup D$ and $C \cap D = \phi$.

Since C and D are both open in X, the sets $A = C \cap Y$ and $B = D \cap Y$ are open in Y.

Also $A \cup B = (C \cap Y) \cup (D \cap Y)$
$= (C \cup D) \cap Y = X \cap Y = Y$.

$A \cap B = (C \cap Y) \cap (D \cap Y)$
$= (C \cap D) \cap Y = \phi$.

$\therefore$ A and B form a separation of Y.

Since Y is connected, one of them must be empty.

(ie) either $A = C \cap Y = \phi$
$\Rightarrow Y \subset D$.

(or) $B = D \cap Y = 0 \Rightarrow Y \subset C$.

Hence Y must lie entirely in C or in D.

Theorem:- The union of a collection of connected subspaces of X that have a point in common is connected.

Proof:-

Let $\{A_\alpha\}$ be a collection of subspaces of a space X.

Let p be a point of $\cap A_\alpha$.

We prove that the space $Y = \cup A_\alpha$ is connected.

Suppose that $Y = C \cup D$ is a separation of Y.

Since $p \in Y$, either $p \in C$ (or) $p \in D$.

Suppose $p \in C$.

Since A_α is a connected subspace of Y and C & D form a separation of Y, we have A_α must lie entirely either C or D.

Since the point $p \in C$, and $C \cap D = \phi$, A_α cannot lie in D.

Hence $A_\alpha \subset C$ for every α.

This implies $Y = \cup A_\alpha \subset C$.

This contradicting the fact that $Y = C \cup D$ is a separation of Y and C & D are nonempty.

$\therefore Y = \cup A_\alpha$ is connected.

Theorem:- Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is connected.

Proof :-

"If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected".

Let A be a connected supspace of X

and let $A \subset B \subset \bar{A}$.

We shall prove that B is connected.

Suppose that $B = C \cup D$ is a separation of B

Then by the lemma

"If the sets C & D form a separation of X and if Y is a connected subspace of X, then Y lies entirely either in C or D",

we have,

A must lie entirely either in C or D.

Suppose $A \subset C$.

Then $\bar{A} \subset \bar{C}$.

Since $\bar{C} \cap D = \phi$, $\bar{A} \cap D = \phi$.

Since $B \subset \bar{A}$, $B \cap D = \phi$.

This contradicts the fact that D is a nonempty subset of B.

$\therefore$ B is connected.

Theorem:- The image of a connected space under a continuous map is connected.

Proof:-

Let $f : X \to Y$ be a continuous map.

Let X be connected.

We shall prove that the image space $Z = f(X)$ is connected.

Since the map obtained from f by restricting its range to the space Z is also continuous,

the map $g : X \to Z$ is a continuous surjective map.

To prove that $f(X)$ is connected, it suffices to prove $g(X) = Z$ is connected.

Suppose that $Z = A \cup B$ is a separation of Z.

(ie) A and B are two nonempty disjoint open subsets of Z.

Since g is continuous $g^{-1}(A)$ and $g^{-1}(B)$ are open in X.

Also $g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B)$

$= g^{-1}(Z) = g^{-1}(g(X)) = X$.

Hence $X = g^{-1}(A) \cup g^{-1}(B)$ form a separation of X

This contradicts the fact that X is connected.

$\therefore$ Z is connected.

(i.e) $f(X)$ is connected.

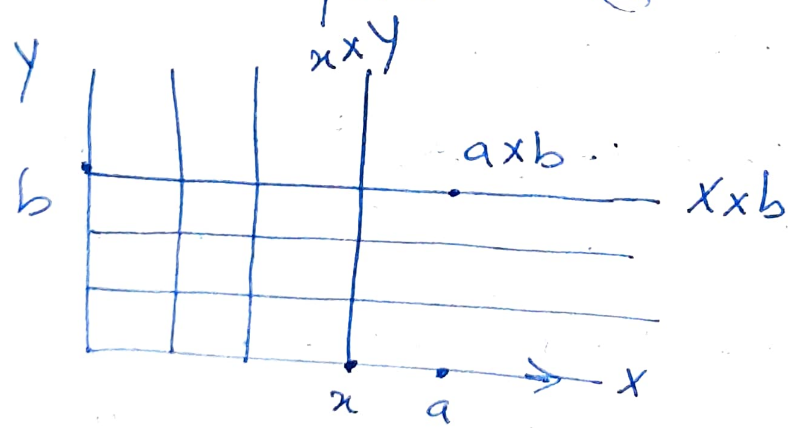
Theorem:- A finite cartesian product of connected spaces is connected.

Proof:-

First we prove the theorem for the product of two connected spaces X and Y.

To prove $X \times Y$ is connected.

Choose a base point $a \times b$ in the product $X \times Y$.



Since the horizontal slice $X \times b$ is homeomorphic with X, and the vertical slice $x \times Y$ is homeomorphic with Y, they are connected.

Consider the T-shaped space

$$T_x = (X \times b) \cup (x \times Y).$$

Then $x \times b \in (X \times b) \cap (x \times Y)$

$\therefore T_x$ is connected, because the union of two connected spaces that have a point in common is connected.

Now form the union $\bigcup_{x \in X} T_x$ of all T-shaped spaces.

This union is connected, because it is the union of a collection of connected spaces that have the point $a \times b$ in common.

Since $X \times Y = \bigcup_{x \in X} T_x$, $X \times Y$ is connected.

Next we shall prove that the finite ~~pr~~ cartesian product of connected spaces is connected.

Let $X_1, X_2, \cdots X_n$ be connected spaces.

We shall prove that $X_1 \times X_2 \times \cdots \times X_n$ is connected by induction on n.

If $n=1$, the result is obvious.

Assume that $X_1 \times X_2 \times \cdots \times X_{n-1}$ is connected.

Then by the first part,

$(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ is connected.

Since $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$, $X_1 \times X_2 \times \cdots \times X_n$ is connected.

Example:- $\mathbb{R}^\omega$ in the box topology is not connected.

Proof:-

Let A be the collection of all bounded sequences of real numbers.

and let B be the collection of all unbounded sequences of real numbers.

Clearly A and B are nonempty and disjoint.

Let $a \in \mathbb{R}^\omega$ where $a = (a_1, a_2, a_3, \dots)$

consider the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$$

If $a \in A$, That is if a is bounded,

then U consists entirely of bounded sequences.

$\therefore a \in U \subset A$.

Hence A is open.

If $a \in B$, That is if a is unbounded,

Then U consists entirely of unbounded sequences.

$\therefore a \in U \subset B$.

Hence B is open.

Hence $\mathbb{R}^\omega = A \cup B$ ~~form~~ is a separation of $\mathbb{R}^\omega$.

Hence $\mathbb{R}^\omega$ is not connected in the box topology.

Example:- $\mathbb{R}^\omega$ is connected in the product topology.

Proof:- Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $x = (x_1, x_2, \dots)$ such that $x_i = 0$ for $i > n$.

The space $\tilde{\mathbb{R}}^n$ is homeomorphic to $\mathbb{R}^n$.

Since $\mathbb{R}^n$ is connected, $\tilde{\mathbb{R}}^n$ is connected.

Let $\mathbb{R}^\infty = \bigcup_{n \in I} \tilde{\mathbb{R}}^n$

Since $0 = (0, 0, 0, \dots) \in \tilde{\mathbb{R}}^n$ $\forall n \in I$, and $\tilde{\mathbb{R}}^n$ is connected, we have

$\mathbb{R}^\infty$ is connected.

We shall prove that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is connected.

Let $a = (a_1, a_2, \dots) \in \mathbb{R}^\omega$.

Let $U = \prod U_i$ be a basis element for the product topology that contains a.

We show that U intersects $\mathbb{R}^\infty$.

There is an integer N such that $U_i = \mathbb{R}$ for $i > N$.

Then the point,

$x = (a_1, a_2, \dots a_n, 0, 0, 0, \dots)$ of $\mathbb{R}^\infty$

belongs to U, since $a_i \in U_i$ for all i, and $0 \in U_i$ for $i > N$.

$\therefore$ $\mathbb{R}^\omega \subset \tilde{\mathbb{R}}^\infty$, but $\tilde{\mathbb{R}}^\infty \subset \mathbb{R}^\omega$

$\therefore$ $\mathbb{R}^\omega = \tilde{\mathbb{R}}^\infty$, hence $\mathbb{R}^\omega$ is connected.

CONNECTED SUBSPACES OF THE REAL LINE.

Defn: A simply ordered set L having more than one element is called a linear continuum if the following hold. (3)

(1) L has the least upper bound property

(2) If $x<y$, there exists z such that $x<z<y$.

Theorem: If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.

Proof:

A subspace Y of L is said to be convex if for every pair of points a, b of Y with $a<b$, the entire interval $[a,b]$ of points of L lies in Y.

We prove that if Y is a convex subspace of L then Y is connected.

Suppose that $Y = A \cup B$ is a separation of Y.

(i) A & B are disjoint nonempty open subsets of whose union is Y.

Choose $a \in A$ and $b \in B$.

For convenience let $a<b$.

Clearly $a, b \in Y$ & $a<b$.

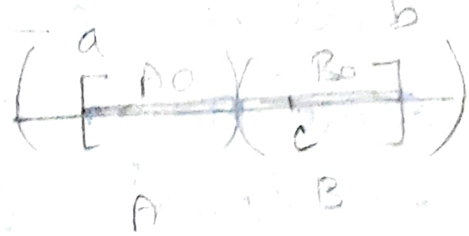
Since Y is convex, $[a,b] \subset Y$.

Let $A_0 = A \cap [a,b]$ and $B_0 = B \cap [a,b]$.

Then A_0 & B_0 are open in $[a,b]$ in the subspace topology which is same as the order topology

Since $a \in A_0$ and $b \in B_0$, A_0 & B_0 are nonempty.
Also $A_0 \cup B_0 = [a, b]$ and $A_0 \cap B_0 = \phi$.
$\therefore [a, b] = A_0 \cup B_0$ is a separation of $[a, b]$ (4)

Let $c = \text{Sup } A_0$.



claim: $c \notin A_0$ and $c \notin B_0$.

case (i)

Suppose $c \in B_0 = B \cap [a, b]$.

Then $c \neq a$, since $a \in A_0$ & $A_0 \cap B_0 = \phi$.

$\therefore$ either $c = b$ (or) $a < c < b$.

In either case, since B_0 is open in $[a, b]$, we can find an interval $(d, c]$ such that $(d, c] \subset B_0$.

If $c = b$, then d is an upper bound for A_0, and $d < c$.
which contradicts the fact that $c = \text{Sup} A_0$.
$\therefore c \neq b$.

If $c < b$, then $(c, b] \cap A_0 = \phi$, because c is an upper bound for A_0.
Then $(d, b] = (d, c] \cup (c, b]$ and $(d, b] \cap A_0 = \phi$.
Again d is an upper bound for A_0 and $d < c$.
Which is a contradiction.
$\therefore c \notin B_0$

Case (ii) Suppose $c \in A_0$.

Then $c \neq b$, since $b \in B_0$ & $A_0 \cap B_0 = \phi$.

$\therefore$ either $c = a$ (or) $a < c < b$.

Since A_0 is open in $[a, b]$, we can find an interval $[c, e) \subset A_0$.

By the linear continuum property of L, we can find a point z of L such that $c < z < e$.

$\Rightarrow z \in A_0$.

which is a contradiction because $c = \sup A_0$ and $z > c$.

$\therefore c \notin A_0$.

Hence $c \notin A_0$ & $c \notin B_0$.

which is a contradiction, because $A_0 \cup B_0 = [a, b]$

Hence Y is connected.

Corollary Since the linear continuum L itself is convex, L is connected.

The Real line $\mathbb{R}$ is connected and so are the intervals & rays in $\mathbb{R}$.

Proof: Since the Real line $\mathbb{R}$ is a linear continuum in the order topology, by the above theorem, $\mathbb{R}$ is connected & so interval

Intermediate value Theorem (6)

Theorem: Let $f: X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a & b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof: Given X is a connected space & Y is an ordered set in the order topology.

Also $a, b \in X$ & $r \in Y$, $f(a) < r < f(b)$.

Consider the open rays $(-\infty, r)$ & (r, ∞) in Y.

Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$.

Clearly, $f(a) \in A$ and $f(b) \in B$.

$\therefore$ A and B are nonempty.

Since $(-\infty, r)$ & (r, ∞) are open in Y & $f(X) \subset Y$,

A & B are open in $f(X)$.

Also
$$A \cup B = f(X) \cap \{(-\infty, r) \cup (r, \infty)\}$$
$$= f(X) \cap (Y - \{r\})$$
$$= f(X) \quad \text{if } r \notin f(X).$$

and
$$A \cap B = f(X) \cap \{(-\infty, r) \cap (r, \infty)\}$$
$$= f(X) \cap \phi = \phi.$$

Hence if there we no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ is a separation of $f(X)$. Which contradicts that $f(X)$ is connected.

Hence there exists a point $c \in X$ s.t $f(c) = r$.

Hence the theorem.

Hence the connected.

Def: Path & Path connected.

Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a,b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

A space X is said to be path connected if every pair of points of X can be joined by a path X.

Lemma: A path connected space X is connected.

Proof: Let X be path connected.

To prove X is connected.

Suppose that $X = A \cup B$ is a separation of X.

(i.e) A & B are disjoint nonempty subsets of X whose union is X.

Let $f : [a,b] \to X$ be any path in X.

Since the continuous image of a connected ~~space~~ set is connected, $f([a,b])$ is connected.

Then $f([a,b]) \subset A$ or $f([a,b]) \subset B$.

Hence, there is no path in X joining a point of A to a point of B.

This is a contradiction to the assumption that X is path connected.

Hence $\exists$ no separation of X.

$\therefore X$ is connected.