$= f^{-1}(U \cap V)$ is open.

Hence f is continuous.

THE PRODUCT TOPOLOGY.

Defn:- Let J be an index set. Given a set X, we define a J-tuple of elements of X to be a function $(x_\alpha) : J \to X$. If α is an element of J, we often denote the value of (x_α) at α by x_α. We calls it the αth coordinate of $(x_\alpha)_{\alpha \in J}$.

We denote the set of all J-tuples of elements of X by X^J.

Defn: Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets, Let $X = \bigcup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family, denoted by

$\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

That is, it is the set of all functions

$x : J \to \bigcup_{\alpha \in J} A_\alpha$ such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Box topology:-

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

Let as take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$, the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α, for each $\alpha \in J$.

The topology generated by this basis is called the box topology.

Result:- This collection forms a basis:

Since $\prod_{\alpha \in J} X_\alpha$ itself is a basis element, the

- first condition for basis follows.

Since, the intersection of two basis element is again a basis element,

(i.e.) $\left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} V_\alpha\right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$

the second condition for basis follows.

Defn: Projection mapping

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \longrightarrow X_\beta$ be the function assigning to each element of the product space to its βth coordinate, $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$. It is called the projection mapping associated with the index β.

Product Topology:-

Let S_β denote the collection

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) \,/\, U_\beta \text{ is open in } X_\beta\},$$

and let S denote the union of these collections

$$S = \bigcup_{\beta \in J} S_\beta .$$

The topology generated by the subbasis S is called the product topology.

In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Comparision of Box and Product topologies:

The box topology on $\prod X_\alpha$ has as basis of all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α.

The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Proof:-

Let $\mathcal{B}$ be the basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

By the definition of box topology, $\mathcal{B}$ consists of all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α.

Let $\mathcal{B}'$ be the basis that $\mathcal{S}$ generates. Clearly, $\mathcal{B}'$ consists of all finite intersections of elements of $\mathcal{S}$.

If we intersect elements belonging to the same $\mathcal{S}_\beta$, that is

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta) \text{ is}$$

again an element of $\mathcal{S}_\beta$.

If we intersect elements from different sets $\mathcal{S}_\beta$,

The typical element of the basis $\mathcal{B}$ can be described as follows:

Let $\beta_1, \beta_2, \cdots \beta_n$ be a finite set of distinct indices from the index set J, and let U_{β_i} be an open set in X_{β_i} for $i = 1, 2, \cdots n$.

Then $B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$ is the typical element of $\mathcal{B}$.

Now a point $x = (x_\alpha)$ is in B iff its β_1th coordinate is in U_{β_1}, its β_2th coordinate is in U_{β_2}, and so on.

There is no restriction whatever on the αth coordinate of x if $\alpha \neq \beta_i$, $i = 1, 2, \dots n$.

Hence B as the product,

$B = \prod_{\alpha \in J} U_\alpha$, where U_α denotes the entire space X_α if $\alpha \neq \beta_1, \beta_2, \dots \beta_n$.

<u>Theorem:-</u> Let $\{X_\alpha\}$ be an indexed family of spaces. Let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the box ~~to~~ or product topology then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$.

<u>Proof:-</u>

<u>To prove $\prod \bar{A}_\alpha \subset \overline{\prod A_\alpha}$,</u>

Let $x = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$.

We shall prove that $x = (x_\alpha) \in \overline{\prod A_\alpha}$.

Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $x = (x_\alpha)$.

Since $x_\alpha \in \overline{A_\alpha}$ for each α, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α.

Then $y = (y_\alpha) \in U$ and $y = (y_\alpha) \in \prod A_\alpha$.

Since U is an arbitrary neighbourhood of $x = (x_\alpha)$, and $U \cap \prod A_\alpha \neq \phi$, we have

$$x = (x_\alpha) \in \overline{\prod A_\alpha}.$$

Hence $\prod \overline{A_\alpha} \subset \overline{\prod A_\alpha}$ ——— (1).

Conversely,

Suppose $x = (x_\alpha) \in \overline{\prod A_\alpha}$, in either topology.

To prove $x = (x_\alpha) \in \prod \overline{A_\alpha}$, we shall prove that for any given index β,

$$x_\beta \in \overline{A_\beta}.$$

Let V_β be an arbitrary open set of X_β containing x_β.

Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, and $x = (x_\alpha) \in \pi_\beta^{-1}(V_\beta)$,

$\pi_\beta^{-1}(V_\beta)$ is an open set containing $x = (x_\alpha)$ in $\prod X_\alpha$.

But since $x = (x_\alpha) \in \overline{\prod A_\alpha}$,

There exists $y = (y_\alpha) \in \pi_\beta^{-1}(V_\beta) \cap \prod A_\alpha$.

This implies $y_\beta \in V_\beta \cap A_\beta$.

Hence $x_\beta \in \overline{A_\beta}$, for each β

$\Rightarrow x = (x_\alpha) \in \prod \overline{A_\alpha}$.

Thus $\overline{\prod A_\alpha} \subset \prod \overline{A_\alpha}$ ——— (2)

From (1) & (2) $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

<u>Theorem:-</u>

Let $f : A \longrightarrow \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$,

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each func f_α is continuous

<u>Proof:-</u> Let. π_β be the projection of the product onto its βth factor.

(ii) $\pi_\beta : \prod X_\alpha \longrightarrow X_\beta$.

Then π_β is continuous, for if U_β is open in X_β then $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$.

Now, Suppose that $f : A \to \prod X_\alpha$ is continuous

since $f_\beta : A \longrightarrow X_\alpha$, and $\pi_\beta : \prod X_\alpha \to X_\beta$,

we have $f_\beta = \pi_\beta \circ f$.

Since both π_β & f are continuous, and the composition of two continuous functions is continuous, we have

f_β is continuous for each β.

Conversely,

Suppose that each coordinate function f_α is continuous.

To prove $f : A \longrightarrow \prod\limits_{\alpha \in J} X_\alpha$ is continuous, it suffices to P.T the inverse image under f for each <u>subbasis</u> element of $\prod\limits_{\alpha \in J} X_\alpha$ is open in A.

A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$, where U_β is open in X_β, for some index β.

Now,

$$f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = f_\beta^{-1}(U_\beta),$$

because $f_\beta = \pi_\beta \circ f$.

Since f_β is continuous, $f_\beta^{-1}(U_\beta)$ is open in A.

Hence $f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open in A.

This implies $f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Note:- This theorem fails if we use the box topology.

THE METRIC TOPOLOGY

Defn:- A metric on a set X is a fun $d : X \times X \to \mathbb{R}$ having the following properties.

(1) $d(x,y) \geq 0$ for all $x, y \in X$, equality holds iff $x = y$.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$

(3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Note:- 1. Given a metric d on X, the number $d(x,y)$ is called the distance b/w x and y in the metric d.

2. Given $\epsilon > 0$, consider the set

$$B_d(x;\epsilon) = \{y \mid d(x,y) < \epsilon\}$$ of all points y whose distance from x is less than ϵ.

It is called the ϵ-ball centered at x.

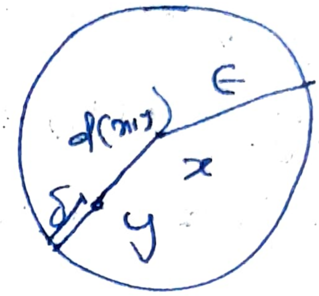
Defn:- If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x;\epsilon)$ for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the Metric topology induced by d.

Result:- If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x;\epsilon)$ form a basis for a topology on X.

Soln:

The first condition for a basis is trivial, since $x \in B(x;\epsilon)$, for any $\epsilon > 0$.

<u>To prove the second condition</u>



Let $y \in B(x;\epsilon)$.

Let $\delta = \epsilon - d(x,y) > 0$.

Then $B(y,\delta) \subset B(x;\epsilon)$.

Let $z \in B(y,\delta)$, then

$$d(y,z) < \delta = \epsilon - d(x,y).$$

By triangle inequality,

$$d(x,z) \le d(x,y) + d(y,z)$$
$$< \cancel{\epsilon+\epsilon}\ \cancel{d(x,y)} + \epsilon - \cancel{d(x,y)}$$
$$= \epsilon.$$

$\Rightarrow z \in B(x;\epsilon)$

$\therefore$ <u>$B(y,\delta) \subset B(x;\epsilon)$.</u>

Let B_1 & B_2 two elements in that collection, and $y \in B_1 \cap B_2$.

By the result just we have proved, we can choose positive numbers δ_1 and δ_2 so that

$$B(y,\delta_1) \subset B_1 \;\&$$
$$B(y,\delta_2) \subset B_2$$

Let $\delta = \min(\delta_1, \delta_2)$

Then $B(y,\delta) \subset B_1 \cap B_2$.

Let $B_3 = B(y,\delta)$,

This implies $y \in B_3 \subset B_1 \cap B_2$.

Hence the condition (ii).

<u>Defn</u>:- A set U is open in the metric topology induced by a metric d iff for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

<u>Ex:1</u> Given a set X, define $d : X \times X \to \mathbb{R}$, by

$$d(x,y) = 1, \text{ if } x \neq y$$
$$= 0 \quad \text{if } x = y.$$

clearly d is a metric on X.

The topology it induces is the <u>discrete topology</u>.

For, if $x \in X$, then $\{x\} = B(x; 1)$ which is a basis element.

Hence every singleton set is open, so that every subset of X is open.

<u>Ex:2</u> The standard metric on the ~~ts~~ real numbers $\mathbb{R}$ is defined by

$$d(x,y) = |x - y|.$$

The topology it induces is the order topology.

Because, each basis element (a,b) for the order topology is a basis element for the metric topology.

For, $(a,b) = B(x, \epsilon)$ where $x = \frac{a+b}{2}$ and $\epsilon = b - a/2$.

Conversely, each ϵ-ball $B(x;\epsilon)$ equals an open interval $(x-\epsilon, x+\epsilon)$.

<u>Defn:-</u> If X is a topological space, X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

A metric space is a metrizable space, together with a specific metric d that gives the topology of X.

<u>Defn:-</u> Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number M such that

$$d(a_1, a_2) \leq M \text{ for every pair } a_1, a_2$$

of points of A.

If A is bounded and nonempty, the diameter of A is defined to be the number

$$\text{diam } A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

<u>Theorem:-</u> Let X be a M.S with metric d. Define $\bar{d}: X \times X \to \mathbb{R}$ by the equation

$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof:-

First we shall prove $\bar{d}$ is a metric.

(i) Since $d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$,

we have $\bar{d}(x,y) \geq 0$ and

$\bar{d}(x,y)=0 \Leftrightarrow d(x,y)=0 \Leftrightarrow x=y$.

(ii) Since $d(x,y) = d(y,x)$,

$\bar{d}(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\}$

$= \bar{d}(y,x)$

(iii) Triangle inequality.

~~To~~ prove, $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$ for all $x,y,z \in X$,

consider the case, if either $d(x,y) \geq 1$ (or) $d(y,z) \geq 1$.

Then $\bar{d}(x,y) + \bar{d}(y,z) \geq 1$. and

$\bar{d}(x,z) \leq 1$, by definition.

$\therefore \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$ holds.

If $d(x,y) < 1$ and $d(y,z) < 1$

Then $d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$

Since $\bar{d}(x,z) \leq d(x,z)$ by definition,

$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$ holds.

Hence $\bar{d}$ is a metric on X.

Now consider the collection of all ϵ-balls with $\epsilon < 1$.

This is a basis for the metric topology.

For, if for every $x \in X$, $\exists$ $\epsilon > 0$ $\ni$ $x \in B_d(x;\epsilon)$.

$\Rightarrow$ $x \in B_d(x; \epsilon < 1) \subseteq B_d(x;\epsilon)$ and

if $x \in B_d(x;\epsilon_1 < 1) \cap B_d(x;\epsilon_2 < 1)$, then

$x \in B_d(x, \epsilon < 1) \subset B_d(x, \epsilon_1 < 1) \cap B_d(x;\epsilon_2 < 1)$

where $\epsilon = \min.\{\epsilon_1, \epsilon_2\}$.

Since $B_d(x;\epsilon < 1\} = \{y \in X \,/\, d(x,y) < \epsilon < 1\}$

$= \{y \in X \,/\, \bar{d}(x,y) < \epsilon\}$

$= B_{\bar{d}}(x;\epsilon)$.

This implies, the collection of ϵ-balls with $\epsilon < 1$ under d and $\bar{d}$ are the same collection,

d and $\bar{d}$ induces the same topology on X.