Closed sets and Limit points

Defn:- A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Ex:1 (i) The subset $[a,b]$ of $\mathbb{R}$ is closed, since

$$\mathbb{R} - [a,b] = (-\infty, a) \cup (b, +\infty) \text{ is open.}$$

(ii) The subset $[a, +\infty)$ is closed, since

$$\mathbb{R} - [a, +\infty) = (-\infty, a) \text{ is open.}$$

(iii) The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Ex:2 In the plane $\mathbb{R}^2$, the set $\{x \times y \,/\, x \geq 0 \,\&\, y \geq 0\}$ is closed, since

$$\mathbb{R}^2 - \{x \times y \,/\, x \geq 0 \,\&\, y \geq 0\} = (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0).$$

since $(-\infty, 0) \times \mathbb{R}$ & $\mathbb{R} \times (-\infty, 0)$ are the product of open sets of $\mathbb{R}$, they are open in $\mathbb{R}^2$.

Ex:3 In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

Theorem:-

Let X be a topological space. Then the following conditions hold.

(1) ϕ and X are closed

(2) Arbitrary intersections of closed sets are closed

(3) Finite unions of closed sets are closed.

Proof:-

(1) Since ϕ and X are open,

$\phi^c = X$ is open

$X^c = \phi$ is open.

$\therefore$ ϕ & X are closed.

(2) Let $\{A_\alpha\}_{\alpha \in J}$ be an arbitrary collection of closed sets of X.

(ie) A_α is closed for each $\alpha \in J$.

$\Rightarrow$ $X - A_\alpha$ is open for each $\alpha \in J$.

$\therefore$ By De Morgan's Law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

Since arbitrary union of open sets is open,

$\bigcup_{\alpha \in J} (X - A_\alpha)$ is open.

Hence $\bigcap_{\alpha \in J} A_\alpha$ is closed.

(ie) Arbitrary intersections of closed sets is closed

(3) Let $A_1, A_2, \dots A_n$ be closed.

(ie) A_i is closed for each $i = 1, 2, \dots n$.

$\Rightarrow$ $X - A_i$ is open, for each $i = 1, 2, \dots n$.

Now $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$

Since finite intersection of open set is open

$\bigcap_{i=1}^{n}(X - A_i)$ is open.

Hence $X - \bigcup_{i=1}^{n} A_i$ is open.

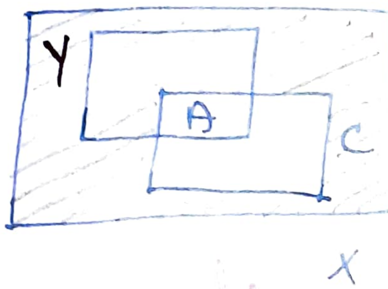
$\therefore \bigcup_{i=1}^{n} A_i$ is closed.

(ii) ~~Arbitrary~~ Finite unions of closed sets are closed.

Theorem: Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.

Proof:-

Assume that $A = C \cap Y$, where C is closed in X.



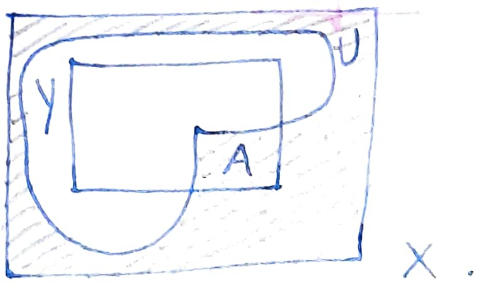
Then $X - C$ is open in X.

$\therefore (X-C)\cap Y$ is open in Y, by the definition of the subspace topology.

But $(X-C)\cap Y = Y - A$.

Hence $Y - A$ is open in Y.

$\therefore A$ is closed in Y.



Conversely,

Assume that A is closed in Y.

Then $Y - A$ is open in Y.

By definition, $Y - A$ is the intersection of an open set U of X with Y.

(ie) $Y - A = U \cap Y$.

Since U is open in X, $X - U$ is closed in X.

Also $A = Y \cap (X - U)$.

$\therefore$ A equals the intersection of a closed set with Y.

Theorem:- Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof:- Given that A is closed in Y.

Then $A = C \cap Y$, where C is closed in X.

Since both C and Y are closed, then $C \cap Y$ is also closed.

$\therefore$ A is closed in X.

Closure and Interior of a Set.

Defn:- Given a subset A of a topological space the interior of A is defined as the union of open sets contained in A. and the closure of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $\operatorname{Int} A$ and the closure of A is denoted by $\operatorname{Cl} A$ or $\bar{A}$.

clearly $\operatorname{Int} A$ is an open set and orem: Let Y be a subspace of X; let A be a subset of Y; Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof :-

Let B denote the closure of A in Y.

The set $\bar{A}$ is closed in X.

Then by the theorem

"A set A is closed in Y iff it equals the intersection of a closed set of X with Y", ——★

$\bar{A} \cap Y$ is closed in Y.

Also $A \subset \bar{A} \cap Y$ $\left\{ \because \begin{array}{l} A \subset \bar{A} \\ A \subset Y \end{array} \right\}$

By definition, B equals the intersection of all closed subsets of Y containing A.

$\therefore B \subset \bar{A} \cap Y$. —— ①.

On the other hand,

We know that B is closed in Y.

Hence by the same theorem (★) stated above

$B = C \cap Y$, where C is closed in X.

Since $A \subset B$, $A \subset C \cap Y$

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$\Rightarrow A \subset C$ & $A \subset Y$.

$\therefore C$ is a closed set of X containing A.

Because $\bar{A}$ is the intersection of all closed sets containing A, we conclude that $\bar{A} \subset C$.

Then $\bar{A} \cap Y \subset C \cap Y = B$.

(ie) $\bar{A} \cap Y \subset B$ —— ②.

From ① & ② $B = \bar{A} \cap Y$.

(ie) closure of A in Y equals $\bar{A} \cap Y$.

* Theorem: Let A be a subset of the topological space X.

(a) Then $x \in \bar{A}$ iff every open set U contain[ing x] intersects A.

(b) Supposing the topology of X is given by a [basis] then $x \in \bar{A}$ iff every basis element B conta[ining] x intersects A.

Proof:-

We shall prove (a) by its contrapo[sitive] statement.

(i) We shall prove that

$x \notin \bar{A} \Leftrightarrow$ there exists an open set U con[taining] x does not intersect A.

Let us assume that $x \notin \bar{A}$.

Then $x \in X - \bar{A}$.

Since $\bar{A}$ is closed, $X - \bar{A}$ is open.

Let $X - \bar{A} = U$

$\Rightarrow x \in U$ and $U \cap A = \phi$.

$\therefore$ There exists an open set U containing [x] does not intersect

Conversely,

Assume there exists an open set U such that $x \in U$ and $U \cap A = \phi$.

To prove $x \notin \bar{A}$.

Since U is open, $X - U$ is closed.

Since $U \cap A = \phi$ & $A \subset X$, $A \subset X - U$.

$\therefore (X - U)$ is closed and $A \subset X - U$.

By the definition of $\bar{A}$, $\bar{A} \subset X - U$.

Now $x \in U \Rightarrow x \notin X - U$.

(ie) $x \notin \bar{A}$.

(b) Let $x \in \bar{A}$.

By (a), Every open set containing x intersects A.

$\therefore$ Every basis element B containing x intersects A, because B is an open set.

Conversely,

Assume every basis element ~~B~~ containing x intersects A.

Since for every open set U containing x, there exists a basis element B such that $x \in B \subset U$, $U \cap A \neq \phi$.

(ie) Every open set U containing x intersects A.

$\Rightarrow x \in \bar{A}$. {by (a)}.

Note: "U is an open set containing x" means
"U is a neighborhood of x".

Ex: Let X be the real line $\mathbb{R}$.
If $A = (0,1]$, then $\bar{A} = [0,1]$, for every neighborhood of 0 intersects A, while every point outside $[0,1]$ has a neighborhood disjoint from A.

If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then $\bar{B} = \{0\} \cup B$.

If $C = \{0\} \cup (1,2)$ then $\bar{C} = \{0\} \cup [1,2]$.

If $\mathbb{Q}$ is the set of rationals then $\bar{\mathbb{Q}} = \mathbb{R}$.

Also $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+$ and $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{0\}$.

Limit point

Defn:- If A is a subset of the topological space and if x is a point of X, we say that x is a limit point (or cluster point or accumulation point) of A if every neighborhood of x intersects A in some point other than x itself.

(i.e) x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Ex:- Consider the real line $\mathbb{R}$.
If $A = (0,1]$, then the point 0 is a limit point and so is the point $1/2$

If $B = \{1/n, n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B.

If $C = \{0\} \cup (1,2)$, then the limit points of C are the points of the interval $[1,2]$.

If $\mathbb{Q}$ is the set of rationals then every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

If $\mathbb{Z}_+$ is the set of positive integers, no point of $\mathbb{R}$ is a limit point of $\mathbb{Z}_+$.

If $\mathbb{R}_+$ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}_+$ is a limit point of $\mathbb{R}_+$.

Theorem:- Let A be a subset of the topological space X; Let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Proof:- First let us prove $\bar{A} \supset A \cup A'$.

If $x \in A'$.

Then every neighborhood of x intersects A in a point different from x.

By the theorem

"$x \in \bar{A}$ iff every open set U containing x intersects A"

we have $x \in \bar{A}$.

Hence $A' \subset \bar{A}$.

If $x \in A$, then by definition $A \subset \bar{A}$.

$\therefore x \in \bar{A}$, it follows that $A' \subset \bar{A}$.

Hence $A \cup A' \subset \bar{A}$. —— ①

On the other hand,

We shall prove that $\bar{A} \subset A \cup A'$.

Let $x \in \bar{A}$.

Since $A \subset \bar{A}$, $x \in A$ (or) $x \notin A$.

If $x \in A$, then clearly $x \in A \cup A'$.

Suppose $x \notin A$.

Since $x \in \bar{A}$, we know that every neighborhood of x intersects A.

Because $x \notin A$, the set U intersect A a point different from x.

$\Rightarrow x$ is a limit point of A.

$\Rightarrow x \in A'$.

$\therefore x \in A \cup A'$.

Hence $\bar{A} \subset A \cup A'$ —— ②

From ① & ② $\bar{A} = A \cup A'$

Corollary:- A subset of a topological space is closed iff it contains all its limit points.

Proof:-

W.K.T A subset A of X is closed iff $A = \bar{A}$.

Let A be closed.

Then $A = \bar{A}$.

By the above theorem $\bar{A} = A \cup A'$.

(ii) $A = A \cup A'$

$\Rightarrow A' \subset A$.

Hence A contains all its limit points.

Conversely,

Let $A \supset A'$.

Then $A \cup A' = A$.

$\therefore A \cup A' = \bar{A}$.

Hence $A = \bar{A}$.

$\therefore A$ is closed.

Hausdorff space.

Defn:- A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

Theorem:- Every finite point set in a Hausdorff space X is closed.

Proof:-

It is sufficient to prove that every one-point set $\{x_0\}$ is closed.

If x is a point of X different from x_0.

Since X is Hausdorff, x and x_0 have disjoint neighborhoods U and V respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$

$\therefore$ The closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

Hence $\{x_0\}$ is closed.

Every finite point set is a finite union of one-point sets.

Since one point sets of X are closed and finite union of closed sets is closed, we have

Every finite point set in a Hausdorff space X is closed.

Ex:-

If every finite point sets are is closed in a space X, then X need not be Hausdorff.

The real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point-sets are closed.

T_1 - axiom:- If X is a topological space, such every finite point set of X is closed, then

X is said to satisfy T_1-axiom.

Theorem: Let X be a space satisfying the T_1-axiom. Let A be a subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points of A.

Proof:-

Let every neighborhood of x contains infinitely many points of A.

Hence every neighborhood of x intersects A in some point other than x itself.

$\therefore$ x is a limit point of A.

Conversely,

Suppose that x is a limit point of A, and suppose some neighborhood U of x intersects A in only finitely many points of A.

Since, x is a limit point of A, U also intersects $A - \{x\}$, in finitely many points.

Let $\{x_1, x_2, \dots x_m\}$ be the points of $U \cap (A - \{x\})$.

Since X satisfies, T_1-axiom, the set $\{x_1, x_2, \dots x_m\}$ is closed.

$\therefore$ $X - \{x_1, x_2, \dots x_m\}$ is an open set of X.

Also $U \cap (X - \{x_1, \dots x_m\})$ is a neighbourhood of
x that intersects $A - \{x\}$ not at all.

$\therefore x$ is not a limit point of A.

This is a contradiction

$\therefore$ Every nbd of x contains infinitely many points of A.