UNIT- V

Measure and Outer measure

Def:

The outer measure μ^* is define as a non-negative extended real valued funition defined on all subsets of a space X and having the following properties.

i) $\mu^* \phi = 0$

ii) $A \subset B \Rightarrow \mu^* A \leq \mu^* B$

iii) If $E \subset \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^* E \leq \sum_{i=1}^{\infty} \mu^* E_i$

Note:

i) The second property is called monotonicity and third is called countable subadditivity.

ii) In the view of (i) finite sub additivity follows from (iii)

ie) If $E \subset \bigcup_{i=1}^{\infty} E_i$ then take $E_{n+1}, \ldots = \phi$

Then using (iii)

$$\mu^* A \leq \sum_{i=1}^{\infty} \mu^* E_i = \sum_{i=1}^{n} \mu^* E_i$$

Def:

The outer measure μ^* is called finite if $\mu^*(E) < \infty$

Def:

A set E is said to be measurable with respect to μ^* if for every set A we have,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$$

Note:

Since μ^* is subaddivity it is only

Theorem: 1

The class $\mathcal{B}$ of μ^*. measurable sets is a σ-algebra. If $\bar{\mu}$ is μ^* restricted to $\mathcal{B}$, Then $\bar{\mu}$ is a complete measure on $\mathcal{B}$.

proof: Given that $\mathcal{B}$ is the classes of μ^* measurable sets.

To prove:

$\mathcal{B}$ is a σ-algebra clearly the empty set ϕ is μ^* measurable.

$\therefore \phi \in \mathcal{B}$

By the symmetric of the definition of measurability E and $\tilde{E}$ whenever E is μ^* measurable $\tilde{E}$ is also μ^* measurable.

ie) $E \in \mathcal{B}$ then $\tilde{E} \in \mathcal{B}$

Let E_1 & E_2 be μ^* measurable sets.

Since E_2 is μ^* measurable for every set A,

$$\mu^* A = \mu^*(A \cap E_2) + \mu^*(A \cap \tilde{E}_2)$$

We know that

$$\mu^*(A \cap \tilde{E}_2) = \mu^*(A \cap \tilde{E}_2 \cap E_1) + \mu^*(A \cap \tilde{E}_2 \cap \tilde{E}_1)$$

$$\mu^* A = \mu^*(A \cap E_2) + \mu^*(A \cap \tilde{E}_2 \cap E_1)$$

Now,

$$+ \mu^*(A \cap \tilde{E}_2 \cap \tilde{E}_1) \longrightarrow ①$$

$$\begin{aligned} A \cap (E_1 \cup E_2) &= (A \cap E_1) \cup (A \cap E_2) \\ &= (A \cap E_2) \cup (A \cap E_1 \cap R) \\ &= (A \cap E_2) \cup [(A \cap E_1) \cap (E_2 \cup \tilde{E}_2)] \\ &= (A \cap E_2) \cup [(A \cap E_1 \cap E_2) \\ &\qquad \cup (A \cap E_1 \cap \tilde{E}_2)] \\ &= (A \cap E_2) \cup (A \cap E_1 \cap \tilde{E}_2) \end{aligned}$$

$\mu^*(A\cap(E_1\cup E_2)) \le \mu^*(A\cap E_2)+\mu^*(A\cap E_1\cap \tilde{E}_2) \to ②$

using ② in ①.

$$\mu^* A \ge \mu^*[A\cap(E_1\cup E_2)] + \mu^*[A\cap\tilde{E}_2\cap\tilde{E}_1]$$

$$\ge \mu^*[A\cap(E_1\cup E_2)+\mu^*(A\cap\widetilde{(E_1\cup E_2)})]$$

Since $\therefore E_1\cup E_2$ is μ^* measurable.

$\Rightarrow E_1\cup E_2 \in \mathcal{B}$

Thus the union of μ^* measurable sets is measurable and by induction the union of any finite no of measurable sets is measurable.

$\therefore \mathcal{B}$ is an algebra of sets.

Assume that $E=\cup E_i$

Where $\{E_i\}$ is a disjoint sequence of measurable sets.

$$\text{let } G_n = \bigcup_{i=1}^{n} E_i$$

Then G_n is measurable and

$$\mu^* A = \mu^*(A\cancel{\mathcal{E}}\cap E_n)+\mu^*(A\cap\tilde{E}_n)$$

$$\Rightarrow \mu^* A \ge \mu^*(A\cap E_n)+\mu^*(A\cap\tilde{E}) \to ③$$

Since $\tilde{E}\subset\tilde{G}_n$. Now $G_n\cap E_n = E_n$ and $G_n\cap\tilde{E}_n = G_{n-1}$ and by the measurability of E_n, we have,

$$\mu^* A = \mu^*(A\cap E_n)+\mu^*(A\cap\tilde{E}_n) \quad [\because G_n\subset E \;\; \exists\; \tilde{G}_n\supset\tilde{E}]$$

Replace A by $A\cap G_n$

$$\mu^*(A\cap G_n) = \mu^*(A\cap G_n\cap E_n)+\mu^*(A\cap G_n\cap\tilde{E}_n)$$

$$= \mu^*(A\cap E_n)+\mu^*(A\cap G_{n-1})$$

$\therefore$ By induction

~~$\mu^*(A\cap G_n) = \mu^*(A\cap G_n\cap E_n)+\mu^*(A\cap G_n\cap\tilde{E}_n$~~)

$\mu^*(A\cap G_n) = \mu^*(A\cap E_n) + \mu^*(A\cap E_{n-1}) + \dots + \mu^*(A\cap E_1)$

$\mu^*(A\cap G_n) = \sum_{i=1}^{n} \mu^*(A\cap E_i) \longrightarrow (4)$

using (4) in (3)

$$\mu^* A \geq \sum_{i=1}^{n} \mu^*(A\cap E_i) + \mu^*(A\cap \tilde{E})$$

$$\geq \mu^*(A\cap \bigcup_{i=1}^{n} E_i) + \mu^*(A\cap \tilde{E})$$

$$\mu^* A \geq \mu^*(A\cap E) + \mu^*(A\cap \tilde{E})$$

$\therefore$ E is μ^* measurable.

by definition

$\tilde{E}$ is also μ^* measurable.

Hence $\mathcal{B}$ is a σ-algebra

Suppose $\bar{\mu}$ is a μ^* restricted to $\mathcal{B}$ that is

$$\bar{\mu} = \mu^*/\mathcal{B}$$

Let E_1 & E_2 are disjoint measurable set

$\therefore \bar{\mu}(E_1 \cup E_2) = \mu^*(E_1\cup E_2) \cap (E_2 \cap \tilde{E}_2)$

$\Rightarrow \bar{\mu}(E_1\cup E_2) = \mu^*[(E_1\cup E_2)\cap E_2] + \mu^*[(E_1\cup E_2)\cap \tilde{E}_2]$

$= \mu^*[(E_1\cap E_2)\cup(E_2\cap \tilde{E}_2)] + \mu^*[(E_1\cap \tilde{E}_2)\cup(E_2\cap \tilde{E}_2)]$

$= \mu^*[(E_1\cap E_2)\cup E_2] + \mu^*[E_1\cap \tilde{E}_2]$

$= \mu^*(E_2) + \mu^*(E_1)$

$\mu^*(E_1\cup E_2) = \bar{\mu}(E_1) + \bar{\mu}(E_2)$

$\therefore$ finite addivity follows by induction

If E is the disjoint union of μ^* measurable sets $\{E_i\}$

ie) $E = \bigcup_{i=1}^{\infty} E_i$

Then

$$\bar{\mu} E \geq \bar{\mu}\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \bar{\mu}(E_i)$$

$$\Rightarrow \bar{\mu} E \geq \sum_{i=1}^{\infty} \bar{\mu}(E_i) \longrightarrow (5)$$

Now,

$$\mu^* E \leq \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\bar{\mu} E = \mu^* E \leq \sum_{i=1}^{\infty} \mu^* E_i = \sum_{i=1}^{\infty} \bar{\mu} E_i$$

ie) $\bar{\mu} E \leq \sum_{i=1}^{\infty} \bar{\mu} E_i \longrightarrow (6)$

From (5) & (6)

$$\bar{\mu} E = \sum_{i=1}^{\infty} \bar{\mu} E_i$$

Hence $\bar{\mu}$ is countable additive

Also $\bar{\mu}$ is non-negative and

$\bar{\mu}\phi = \mu^*\phi = 0$

$\therefore$ It is measure on $\mathcal{B}$.

To prove:

μ is complete measure.

A measure space $(X, \mathcal{B}, \mu)$ said to Complete if $\mathcal{B}$ contains all subsets if sets of measure zero.

ie) if $B \in \mathcal{B}$, $\mu B = 0$, & $A \subset B$

Then $A \in \mathcal{B}$

Let $B \in \mathcal{B}$, $\mu B = 0$, & $A \subset B$

ie) To prove: $A \in \mathcal{B}$

Since $\mu = \mu^*$ we have $\bar{\mu} B = \mu^* B = 0$

$\mu^* B = 0$

Since $A \subset B \Rightarrow \mu^* A \leq \mu^* B = 0$

$\Rightarrow \mu^* A = 0$

$\therefore A$ is μ^* measurable

$\Rightarrow A \in B$

Hence $\bar{\mu}$ is a Complete measurable.

Hence the proof.

Def:

A measure on an algebra is a non-negative extended real valued set function μ defined on an algebra τ^6 of sets such that,

i) $\mu(\phi) = 0$

ii) If $\{A_i\}$ is a disjoint sequence of sets in a^a whose union is also in a. Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu A_i$$

Note:

A measure μ on an algebra a is a measure iff a is σ-algebra.

Def:

The set function μ^* is defined as

$$\mu^* E = \inf \sum \mu A_i$$

Where $\{A_i\}$ ranges over an sequence from a such that

$$E \subset \bigcup_{i=1}^{\infty} A_i$$

Lemma: 2

If $A \in a$ and if $\{A_i\}$ is any sequence of sets in a such that $A \subset \bigcup_{i=1}^{\infty} A_i$

Then $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$

Proof:

Given $A \in a$ and $\{A_i\}$ a sequence of sets in a such that

$$A \subset \overset{\infty}{\underset{i=1}{\cup}} A_i$$

To prove:

$$\mu A \le \sum_{i=1}^{\infty} \mu A_i$$

set $B_n = A \cap A_n \cap (\tilde{A}_{n-1} \cap \dots \cap \tilde{A}_1)$

$= A \cap A_n \cap (\bigcap_{i=1}^{n-1} \tilde{A}_i)$

since $\mathcal{a}$,

$A \in \mathcal{a}$, $A_i \in \mathcal{a}$, $\forall i$ and

$\mathcal{a}$ is an algebra we get $B_n \in \mathcal{a}$ $\forall n$

Also $B_n \subset A_n$ $\forall n$

Let $m \neq n$ suppose $m < n$

Then $B_m \subset A_m$

$\Rightarrow B_m \cap B_n \subset A_m \cap B_n$

$A_m \cap B_n = A_m \cap (A \cap A_n \cap \tilde{A}_1 \cap \dots \tilde{A}_{n-1})$

$= A \cap A_n \cap \tilde{A}_1 \cap \dots A_m \cap \tilde{A}_m \dots \cap \tilde{A}_{n-1}$

$A_m \cap B_n = \phi$

|||ly

$B_m \cap B_n = \phi$ for $n < m$

$B_m \cap B_n = \phi$ for $m \neq n$

$\therefore$ B_n's are disjoint

Since $B_n \subset A$, $\forall n$

$\Rightarrow \overset{\infty}{\underset{n=1}{\cup}} B_n \subset A$ ——— (1)

Let $x \in A$, since $A \subset \overset{\infty}{\underset{n=1}{\cup}} A_n$

$x \in A_n$ for some n

Let m be the smallest value of n

Such that $x \in A_m$ and $x \in \tilde{A}_i$, $i = 1, 2, \dots m$

$x \in A_m \cap \tilde{A}_1 \cap \dots \cap \tilde{A}_{m-1}$

$\Rightarrow x \in B_m$

$x \in \bigcup_{m=1}^{\infty} B_n$

$A \subset \bigcup_{n=1}^{\infty} B_n$ —— ②

From ① & ②

$A = \bigcup_{n=1}^{\infty} B_n$

$\mu A = \sum_{n=1}^{\infty} \mu B_n$

A is the disjoint union of the sequence $\langle B_n \rangle$

$A = \bigcup_{n=1}^{\infty} B_n$ where B_n's are disjoint

By countable subaddivity,

$\Rightarrow \mu A = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu B_n$

$\Rightarrow \mu A \leq \sum_{n=1}^{\infty} \mu A_n \quad [\because B_n \subset A_n \ \forall n]$

Hence the proof.

Corollary: 3

If $A \in \mathcal{a}$ then $\mu^* A = \mu A$

proof:

Since $\mu^* A = \inf \sum_{i=1}^{\infty} \mu A_i$

$\mu^* A \leq \sum \mu A_i$ —— ①

Where $\{A_i\}$ ranges over all sequene from $\mathcal{a}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$

If $A \in \mathcal{a}$ then by the above Lemma

$\mu A \leq \sum_{i=1}^{\infty} \mu A_i$ —— ②

From ① & ②

We have, $\mu A = \mu^* A$

Hence the proof.

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Lemma : 4 The set function μ^* is an outer measure.

proof :

W.K.T.

$$\mu^* E = \inf \sum \mu A_i$$ where $\{A_i\}$ ranges over all sequence in a such that

$$E \subset \bigcup_{i=1}^{\infty} A_i$$

Clearly, μ^* is a monotone non-negative set function defined for all set and

$$\mu^* \phi = 0$$

To prove :

We have only to show that μ^* is is countable subaddivity.

Let $E \subset \bigcup_{i=1}^{\infty} E_i$

If $\mu^* E_i = \infty$ for any i.

Then we have

$$\mu^* E \leq \sum_{i=1}^{n} \mu^* E_i = \infty$$

If not given $\epsilon > 0$ there is a sequence $\langle A_{ij} \rangle_{j=1}^{\infty}$ for each i of sets in a such that,

$$E_i \subset \bigcup_{j=1}^{\infty} A_{ij} \text{ and}$$

$$\sum_j \mu A_{ij} < \mu^* E_i + \frac{\epsilon}{2^i} \quad \text{--- } (1)$$

Since $E_i \subset \bigcup_{j=1}^{\infty} A_{ij}$

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i,j}^{\infty} A_{ij}$$

$$E \subset \bigcup_i E_i \subset \bigcup_{i,j} A_{ij}$$

$$E \subset \bigcup_{i,j} A_{ij}$$

$$\Rightarrow \mu^* E = \inf \sum_{i,j} \mu A_{ij} \leq \sum_{i,j} \mu A_{ij}$$

$$\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \mu A_{ij} \right),$$

$$< \sum_{i=1}^{\infty} \left(\mu^* E_i + \epsilon/2^i \right)$$

$$< \sum_{i=1}^{\infty} \mu^* E_i + \epsilon$$

Since ϵ is an arbitrary positive number,

$$\mu^* E \leq \sum_{i=1}^{\infty} \mu^* E_i$$

$\therefore \mu^*$ is an countable subaddivity.

$\therefore \mu^*$ is an outer measure.

5M Lemma: 5

If $A \in a$ then A is measurable with respect to μ^*

(or)

Every set in an algebra is measurable with respect to μ^*

Proof:

Given $A \in a$

To prove:

A is measurable with respect to μ^*

Let E be an arbitrary set of finite outer measure and ϵ be +ve number.

Then there is a sequence $\langle A_i \rangle$ from a such that $E \subset \cup A_i$

$$\sum \mu A_i < \mu^* E + \epsilon \quad \text{--- (1)}$$

Now, $A_i = (A_i \cap A) \cup (A_i \cap \tilde{A}) \ \forall i$

By the addivity of μ on a we have,

Then $\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap \tilde{A}) \ \forall i$

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Hence

$$\sum \mu(A_i) = \sum \mu(A_i \cap A) + \sum \mu(A_i \cap \tilde{A})$$

$$\mu^* E + \epsilon > \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A})$$

$$> \sum_{i=1}^{\infty} [\mu(A_i \cap A) + \mu(A_i \cap \tilde{A})]$$

$$> \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A})$$

$$\mu^* E + \epsilon > \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$$

$[\because E \subseteq \cup A_i$

$E \cap A \subset \cup (A_i \cap A)$

$E \cap \tilde{A} \subset \cup (A_i \cap \tilde{A})]$

Since ϵ is arbitrary.

$$\mu^* E \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A});$$

$\therefore$ A is measurable with respect to μ^*

Hence the proof.

Note:

The outer measure μ^* in lemma (2) is called the outer measure induced by μ.

For a given algebra $\mathcal{A}$ of a set we use $\mathcal{A}_\sigma$ to denote for those sets that are countable union of sets of $\mathcal{A}$ and we use $\mathcal{A}$ of σ to denote these sets are countable intersections of sets in $\mathcal{A}_\sigma$.

Def:

An outer measure μ^* said to be regular if given any subset E of X and any $\epsilon > 0$. There is a μ^* measurable set A with $E \subset A$ and $\mu^* A \leq \mu^* E + \epsilon$

10M
·X· State and prove extension theorem (or) Caratheodory extension theorem:

Let μ be a measure on an algebra $\mathcal{a}$ and μ^* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets in an extension of μ to σ-algebra containing $\mathcal{a}$. If μ is finite (or σ-finite) so is $\bar{\mu}$. If μ is σ-finite Then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{a}$ which is a extension of μ.

proof:

Given μ is an σ-algebra $\mathcal{a}$ and μ^* is a outer measure induced by μ.

To prove:

The restriction $\bar{\mu}$ of μ^* to the μ^* measurable sets is an extension of μ to a σ-algebra containing $\mathcal{a}$.

Let $\mathcal{B}$ be a σ-algebra containing $\mathcal{a}$ and the measure $\bar{\mu}$ is μ^* restricted to the μ^* measurable set of $\mathcal{a}$.

Then by thm:-1

"The classes of $\mathcal{B}$ of μ^* measurable set is a σ-algebra. If $\bar{\mu}$ is μ^* restricted to $\mathcal{B}$. Then $\bar{\mu}$ is complete measure on $\mathcal{B}$"

We have $\bar{\mu}$ is complete measure on $\mathcal{B}$. Now by the lemma,

" If $A \in \mathcal{a}$ then A is measurable w.r.to μ^*.

$\therefore A \in \mathcal{B}$ and

$$\bar{\mu}A = \begin{cases} \mu^* A & \text{if } A \in \mathcal{a} \\ 0 & \text{otherwise} \end{cases}$$

$[\because$ if $A \in \mathcal{a}$ then $\mu^* A = \mu A]$

$\therefore \bar{\mu}$ is an extension of μ to $\mathcal{B}$.

ie) $\bar{\mu}$ is an extension of μ to a σ-algebra containing $\mathcal{a}$.

Also, $\bar{\mu}$ is the finite (or) σ-finite whenever μ. Now to p.T. If μ is a σ-finite then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{E}$ which is an extension of μ.

Let $\mathcal{B}$ be the smallest σ-algebra containing $\mathcal{a}$ and $\bar{\mu}$ be some measure on $\mathcal{B}$ that agrees μ on $\mathcal{a}$.

i.e) $$\tilde{\mu}A = \begin{cases} \mu A & \text{if } A \in \mathcal{a} \\ 0 & \text{otherwise} \end{cases}$$

Since each set in $\mathcal{a}_\sigma$ can be expressed as a disjoint countable union of sets in $\mathcal{a}$ the measure $\tilde{\mu}$ must agree with $\bar{\mu}$ on $\mathcal{a}_\sigma$.

ie) If $A \in \mathcal{a}_\sigma$ then $A = \cup A_i$, where $A_i \in \mathcal{a}$.

$$\tilde{\mu}A = \mu A = \mu(\cup A_i) = \Sigma \mu A_i$$

$$= \Sigma \bar{\mu} A_i \quad (A_i \in \mathcal{E},\ \tilde{\mu} = \bar{\mu})$$

ie) $\tilde{\mu}A = \bar{\mu}A$, if $A \in \mathcal{a}_\sigma$.

Let B be any set in $\mathcal{B}$ with finite outer measure.

Then by preposition : 6

There is a set A in $\mathcal{a}_\sigma$ such that

$B \subset A$.

$\mu^* A \leq \mu^* B + \epsilon \longrightarrow (1)$

For given $\epsilon > 0$

since $B \subset A$, $\bar{\mu} B \leq \bar{\mu} A = \mu^* A \leq \mu^* B + \epsilon$

i.e) $\bar{\mu} B \leq \mu^* B + \epsilon$

Since ϵ is arbitrary positive number.

We have $\bar{\mu} B \leq \mu^* B \longrightarrow (2)$ for each $B \in \mathcal{B}$.

Since the class of sets measurable w.r.to μ^* is a σ-algebra containing $\mathcal{a}$ each B in $\mathcal{B}$ must be measurable.

If B is measurable and A is $\mathcal{a}_\sigma$, with B is contained in A and

$\mu^* A \leq \mu^* B + \epsilon$

Then $A = B \cup (A \sim B)$

$\Rightarrow \mu^* A = \mu^* B + \mu^* (A \sim B)$

$\therefore \mu^* A = \mu^* B + \mu^* (A \sim B)$

$\therefore \bar{\mu} (A \sim B) = \mu^* A - \mu^* B \longrightarrow (3)$

$\bar{\mu} (A \sim B) \leq \mu^* (A \sim B)$

$\leq \mu^* A - \mu^* B \leq \epsilon$

$\bar{\mu} (A \sim B) \leq \epsilon \longrightarrow (4)$ if $\mu^* B < \infty$

Hence $\mu^* B \leq \mu^* A$

$= \bar{\mu} A$ $[\because B \subset A)$

$= \bar{\mu} B + \bar{\mu} (A \sim B)$

$= \bar{\mu} B + \epsilon$ [by (4)]

Since ϵ is arbitrary,

$\mu^* B \le \bar{\mu} B \rightarrow$ (5)

From (2) & (5)

$\mu^* B = \bar{\mu} B$

If μ is a σ-finite measure.

Let $\{X_i\}$ be a countable disjoint collection of sets in $\mathcal{a}$ with $X = \cup X_i$ and μX_i finite.

If B is any set in $\mathcal{B}$. Then $B = \cup (X_i \cap B)$ and this is a countable disjoint union of sets in $\mathcal{B}$.

$\therefore \tilde{\mu} B = \bar{\mu} (\cup \subset X_i \cap B)$

$= \Sigma \, \bar{\mu} (X_i \cap B)$

$= \bar{\mu} (\cup (X_i \cap B))$

$\tilde{\mu} B = \bar{\mu} B$

$\therefore \bar{\mu}$ is unique.

This μ is the only measure on the smallest σ-algebra containing $\mathcal{a}$ which is an extending of μ.

Hence the proof.

Product measure:

Direct product:

Let $(X, \mathcal{a}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces. Then and consider the direct product of $X \times Y$ of X and Y is defined as the set consisting of all ordered pairs (x, y). where $x \in X$ and $y \in Y$.

ie) $X \times Y = \{(x, y) / x \in X, y \in Y\}$

Rectangle:

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two Complete measure spaces. Let $A \subset X$ and $B \subset Y$ then $A \times B \subset X \times Y$ we called $A \times B$ a rectangle.

Measurable Rectangle: $(\mathcal{R})$

If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we called $A \times B$ is measurable rectangle.

Note:

i) The family of measurable rectangle is donoted $\mathcal{R}$.

ii) The collection of $\mathcal{R}$ of measurable rectangle is a semi-algebra.

iii) If $A \times B$ be a measurable rectangle we get $\lambda(A \times B) = \mu A \cdot \nu B$.

Semi-Algebra:

A collection $\mathcal{C}$ of subsets of X is a semi-algebra of sets. If the intersection of two sets in $\mathcal{C}$ is again in $\mathcal{C}$ and the complement of any set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$.

ie) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \in \mathcal{R}$.

and $\sim(A \times B) = (\tilde{A} \times B) \cup (A \times \tilde{B}) \cup (\tilde{A} \times \tilde{B})$

Where $\tilde{A} \times B$, $A \times \tilde{B}$, $\tilde{A} \times \tilde{B}$ are all in $\mathcal{R}$.

Lemma: (A)

Let $\{A_i \times B_i\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Then

$$\lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i)$$

proof :

Fix a point $x \in A$

Then for every $y \in B$, (x, y) belongs to exactly one rectangle $A_i \times B_i$ only

Thus B is the disjoint union of these B_i such that x is in the corresponding A_i

Since ν is countably additive.

$\nu(\cup B_i) = \Sigma \nu B_i$ $[\because \nu B(1) = \Sigma \nu B_i(1)]$

$\nu B\, \psi_A(x) = \Sigma \nu B_i\, \psi_{A_i}(x)$ [$\because$ by definition of characteristic function]

Then by the corollary of monotone convergence.

"Let $\{f_n\}$ be a sequence of non-negative measurable functions then

$$\int \Sigma f_n = \Sigma f_n$$

We have,

$$\Sigma \int \nu B_i\, \psi_{A_i}\, d\mu = \int \nu B\, \psi_A\, d\mu$$

i.e; $\Sigma \nu B_i\, \mu A_i = \nu B \mu A$

$\Rightarrow \Sigma \lambda(A_i \times B_i) = \lambda(A \times B)$

[$\because$ By note $\lambda(A \times B) = \mu A\, \nu B$]

Hence the proof.

Def :

If E is any subset of $X \times Y$ and x is a point of X we define x Cross section E_x by

$$E_x = \{y \langle x, y \rangle \in E\}$$

Similarly, For the y cross section for y in Y.

Thus B is the disjoint union of these B_i such that x is in the corresponding A_i

Since ν is countably additive.

$\nu(\cup B_i) = \Sigma \nu B_i$ [∵ $\nu B(1) = \Sigma \nu B_i(1)$]

$\nu B \psi_A(x) = \Sigma \nu B_i \psi_{A_i}(x)$ [∵ by definition of characteristic function]

Then by the corollary of ~~mon~~ monotone convergence.

"Let $\{f_n\}$ be a sequence of non-negative measurable functions then

$\int \Sigma f_n = \Sigma f_n$"

We have,

$$\Sigma \int \nu B_i \psi_{A_i} d\mu = \int \nu B \psi_A d\mu$$

i.e. $\Sigma \nu B_i \mu A_i = \nu B \mu A$

$\Rightarrow \Sigma \lambda (A_i \times B_i) = \lambda (A \times B)$

Hence the proof. [∵ By note $\lambda(A \times B) = \mu A \nu B$]

Def:

If E is any subset of $X \times Y$ and x is a point of X we define x cross section E_x by.

$$E_x = \{y ; \langle x, y \rangle \in E\}$$

Similarly, For the y cross section for y in Y.

The characteristic function of E_x is related to that of E by,

$$\psi_{E_x}(y) = \psi_E(x, y)$$

We also have,

$(\tilde{E})x = \sim E_x$ and

$(\cup E_\alpha)_x = \cup (E_\alpha)x$

Lemma: 5

Let x be a point of x and E be a set in R_σ, then E_x is measurable subset of Y.

proof:

If E is in the class R of measurable rectangles then by the lemma is trivial.

Next we show that, it is true for E in R_σ.

Let $E = \bigcup_{i=1}^{\infty} E_i$

Where each E_i is measurable rectangle

Then $\psi_{E_x}(y) = \psi_E(x, y)$

$= \sup \psi_{E_i}(x, y)$

$= \sup_i (\psi_{E_i})_x (y)$

Since each E_i is measurable rectangle $\psi(E_i)$ is a measurable function of y and so ψ_{E_x} must also be measurable function.

$\therefore E_x$ is a measurable subset of y.

Suppose Now that $E = \bigcap_{i=1}^{\infty} E_i$ Where $E_i \in R_{\sigma\delta}$

$$\psi_{E_x}(y) = \psi_E(x, y) = \inf_i \psi_{E_i}(x, y)$$

$$= \inf_i \psi_{E_i}{}_x(y) \quad \psi_{(E_i)_x}(y)$$

Since each $\psi_{(E_i)_x}(y)$ is measurable, we have $\psi_{E_x}(y)$ is measurable.

Thus E_x is measurable for any $E \in R_{\sigma\delta}$

$$\because \left\{ \begin{array}{l} \psi_{A\cup B}(x) = \max\{\psi_A(x), \psi_B(x)\} \\ \psi_{A\cap B}(x) = \max\{\psi_A(x), \psi_B(x)\} \end{array} \right\}$$

Hence the proof.

Lemma: 6.

Let E be a set in $R_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$ then the function g defined by $g(x) = \nu E_x$ is a measurable function of x & $\int g \, d\mu = \mu \times \nu(E)$

Proof:

If E is a measurable rectangle then the lemma is trivially true.

Next we note that any set in R_σ is a disjoint union of measurable rectangle.

Let $\{E_i\}$ be a disjoint sequence of measurable rectangle and let $E = \cup E_i$

Set $g_i(x) = \nu[(E_i)_x \, x]$

Then each g_i is a non-negative measurable function and

$\Sigma g_i = \Sigma \nu [(E_i)_x]$ [$\because$ By countable additive to ν]

$= \nu \cup (E_i)_x{}^x$

$= \nu (\cup E_i)_x{}^x$

$= \nu E_x = g$

i.e) $g = \Sigma g_i$

Thus "g" is measurable and

$\int g \, d\mu = \Sigma \int g_i \, d\mu$

$= \Sigma \mu \times \nu (E_i)$

$= \mu \times \nu (E)$

$\therefore$ The lemma holds for $E \in R_\sigma$.

Let E be a set of finite measure in $R_{\sigma\delta}$. Then there is a sequence E_i of sets in R_σ.

Such that $E_{i+1} \subset E_i$ & $E = \cap E_i$

we may take $\mu \times \nu [E_i) < \infty$

Let $g_i(x) = \nu [(E_i)(x)]$

Since $\int g_1 \, d\mu = \mu \times \nu (E_1) < \infty$

$\therefore$ We have $g_1(x) < \infty$ for almost all of

for all x with $g_1(x) < \infty$. We have

$[(E_i)x)$ a decreasing sequence of measurable sets of finite measure whose intersection is E_x.

i.e) $E_x = \cap (E_i)_x{}^x$

$\nu(E_x) = [\nu \cap (E_i)x]$

By the preposition,

"If $E_i \in \mathcal{B}$ and $\mu E_1 < \infty$ and $E_1 \supset E_{i+1}$ then

$$\mu(\cap E_i) = \lim_{n\to\infty} \mu E_n" \longrightarrow (*)$$

We have,

$$\nu[\cap(E_i)x] = \lim_{n\to\infty} \nu[(E_n)x]$$

$$= \lim_{n\to\infty} \nu[(E_i)_x]$$

$$= \lim_{i\to\infty} g_i(x)$$

$$\lim_{i\to\infty} g_i(x) = \nu(E_x) = g(x)$$

i.e) $g_i \to g$ a.e as $i \to \infty$

$\therefore$ g is measurable

Since $0 \le g_i < g$ then Lebesgue Convergence theorem

$$\Rightarrow \int g\, d\mu = \lim \int g_i\, d\mu$$

$$= \lim_{i\to\infty} \mu \times \nu(E_i)$$

$$= \mu \times \nu(E) \text{ by } (*)$$

ie) $\int g\, d\mu = \mu \times \nu(E)$

where $E \in R_{\sigma\delta}$

Hence the proof.

Lemma: 7

Let E be a set for which $\mu \times \nu(E) = 0$ then for almost all x we have $\nu(E_x) = 0$

proof:

By the preposition

"Let μ be a measure on algebra $\mathcal{A}$ and μ^* be outer measure induced by μ and E by any set. Then there is a set $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^* E = \mu^* B$.

We have there is a set δ in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $\mu \times \nu(F) = \mu \times \nu(E) = 0$

Then by lemma : 6

for ~~lemma~~ almost all x.

We have $\nu(F_x) = 0$

But $E_x \subset F_x$ and so $\nu(E_x) \leq \nu(F_x) = 0$ almost all x.

$\Rightarrow \nu(E_x) = 0$ for almost all x, since ν is complete.

preposition : 3

Let E be a measurable subset $X \times Y$ such that $\mu \times \nu(E)$ is finite. Then for almost all x. The set E_x is a measurable subset of Y. The function g defined by $g(x) = \nu(E_x)$ is a measurable function defined for almost all x and

$$\int g \, d\mu = \mu \times \nu(E)$$

Proof:

By the preposition: 1

There is a set F in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $\mu \times \nu(F) = \mu \times \nu(F)$

Let $G = F \triangle E$

Since $E \cup F$ are measurable, G is measurable and $\mu\times\nu(E) = \mu\times\nu(E) + \mu\times\nu(G)$

Since $\mu\times\nu(E)$ is finite and is equal to $\mu\times\nu(F)$

We have $\mu\times\nu(G) = 0$

Then By lemma (7) we have

$\nu(G_x) = 0$ for almost all x

Hence $g(x) = \nu(E_x) = \nu(F_x)$ a.e

Then by lemma (6)

g is measurable function and

$\int g\,d\mu = \mu\times\nu(E)$

Hence the proof.

10M

Let $(X, \mathcal{A}, \mu)$ & $(Y, \mathcal{B}, \nu)$ be two complete measure space and f is an integrable function on $X \times Y$. Then

i) For almost all x the function f_x defined by $f_x(y) = f(x,y)$ is an integrable function on Y.

ii) For almost ~~all~~ all y the function f_y defined by $f_y(x) = f(x,y)$

ii) $\int_Y f(x,y)\,d\nu(y)$ is an integrable function on X.

iii) $\int_X f(x,y)\,d\mu(x)$ is an integrable function on Y.

(iii) $\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_{X \times Y} f\, d(\mu_x \nu)$

$= \int_Y \left[\int_X f\, d\mu\right] d\nu$

proof:

Because of symmetric between X and Y it satisfies to proof (i) & (ii) and the first half of (iii)

If the conclusion of the theorem holds f on each of two functions, it also holds for their difference, and hence it is sufficient to consider the case when f is non-negative.

Then by preposition (3) the theorem is true.

If f is characteristic function of a measurable set of finite measure.

Hence the thm must be true if f is simple function which vanishes outside a set of finite measure.

Then by preposition,

Let f be an non-ve measurable function. There is a sequence $\{\phi_n\}$ of simple function with $\phi_{n+1} \geq \phi_n$ such that $f = \lim \phi_n$ at each point of X.

If X is defined on a σ-finite measure space. Then we may choose the functions ψ_n so that each vanishes a set of finite measure.

Assume that f is the limit of an increasing sequence ϕ_n of non-ve simple function.

Since ϕ_n is integrable and simple it must vanish outside a set of finite measure.

Thus f_x is the limit point of increasing sequence $\{(\phi_n)_x\}$ and is measurable By monotone convergence th

$$\int_Y f(x,y)\, d\nu(y) = \lim \int_Y \phi_n(x,y)\, d\nu(y)$$

and so this integral is a measurable function of X.

Again by the monotone converg thm,

$$\int_X \left[\int_Y f\, d\nu \right] d\mu = \lim \int_X \left[\int_Y \phi_n\, d\nu \right] d\mu$$

$$= \lim \int_{X \times Y} \phi_n\, d(\mu \times \nu)$$

$$= \int_{X \times Y} f\, d(\mu \times \nu)$$

Hence the proof.

(7) Tonelli's thm:

Replace integral function by measurable function.

Let $(X, \mathcal{A}, \mu)$ & $(Y, \mathcal{B}, \nu)$ be two σ-finite measure space and let f be a non-negative measurable function on $X \times Y$.

Then (i) before statement.