Unit - IV

Measure and Integration

Def:

By a measurable space we mean a couple $(X, \mathcal{B})$ consisting of a set X and σ-algebra $\mathcal{B}$ of subsets of X. A subset A of X is called measurable (or measurable with respect to $\mathcal{B}$) if $A \in \mathcal{B}$.

Def:

By a measure μ on a measurable space $(X, \mathcal{B})$ we mean a nonnegative set function defined for all sets of $\mathcal{B}$ and satisfying $\mu(\phi) = 0$ and

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu E_i \quad \text{for any}$$

sequence $\{E_i\}$ of disjoint measurable sets. By a measure space (X, ϕ, μ) we mean a measurable space (X, ϕ) together with a measure μ defined on $\mathcal{B}$.

Def:

A set E is said to be of finite measure; if $E \in \mathcal{B}$ and $\mu E < \infty$

A set E is said to be of σ-finite measure if E is the union of a countable collection of measurable sets of finite measure.

Preposition: 1

If $A \in \mathcal{B}$, $B \in \mathcal{B}$ and $A \subset B$ then $\mu A \leq \mu B$

proof:

Since $B = A \cup (B \sim A)$ is a disjoint union we have $\mu B = \mu A + \mu(B \sim A)$

$$\mu B \geq \mu A$$

Hence the proof.

Preposition: 2

If $E_i \in \mathcal{B}$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$ then $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$.

proof:

Set $E = \bigcap_{i=1}^{\infty} E_i$

Then $E_1 = E \cup \bigcup_{i=1}^{\infty} (E_i \sim E_{i+1})$

and this is a disjoint union.

Hence $\mu E_1 = \mu E + \sum_{i=1}^{\infty} \mu(E_i \sim E_{i+1})$

Since $E_i = E_{i+1} \cup (E_i \sim E_{i+1})$ is a disjoint we have,

$$\mu(E_i \sim E_{i+1}) = \mu E_i - \mu E_{i+1}$$

Hence,

$$\mu E_1 = \mu E + \sum_{i=1}^{\infty} (\mu E_i - \mu E_{i+1})$$

$$= \mu E + \lim_{n\to\infty} \sum_{i=1}^{n-1} (\mu E_i - \mu E_{i+1})$$

$$\mu E_1 = \mu E + \mu E_1 - \lim \mu E_n$$

Preposition: 3

If $E_i \in \mathcal{B}$ then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu E_i$

proof:

Let $G_n = E_n \sim \left[\bigcup_{i=1}^{n-1} E_i\right]$

Then $G_n \subset E_n$ and sets G_n are disjoint

Hence $\mu G_n \leq \mu E_n$

$$\mu(\cup E_i) = \sum_{n=1}^{\infty} \mu E_n$$

Def:

A measure μ is called finite if $\mu(X) < \infty$. It is called σ-finite, if there is a sequence $\{X_n\}$ of measurable sets in $\mathcal{B}$ such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu X_n < \infty$.

Theorem: (Fatou's lemma)

Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge almost everywhere on a set E to a function f. Then $\int_E f \leq \underline{\lim} \int_E f_n$.

proof:

Without loss of generality, we may assume that $f_n(x) \to f(x)$ for each $x \in E$.

From the def. of $\int f$ it suffices to show that. If ϕ is any non-negative simple function with $\phi \leq f$ then

$$\int_E \phi \leq \underline{\lim} \int_E f_n$$

Case(i)

If $\int_E \phi = \infty$ then there is a measurable set $A \subset E$ with $\mu A < \infty$ such that $\phi > a > 0$ on A. ie) $A = \{x \in E : \phi(x) > a\}$ — ①

Set $A_n = \{x \in E : f_k(x) > a \;\; \forall \; k \geq n\}$ — ②

Thus, A_n is measurable. Since each f_k is measurable.

Then $\{A_n\}$ is an increasing sequence measurable sets whose union contains A

Since $\phi \leq \lim f_n$

Thus $\lim \mu A_n = \infty$

Since $\int_E f_n \geq a \mu A_n$

we have $\int_E f_n < \infty = \int_E \phi$

Case (ii)

if $\int_E \phi < \infty$ then the set

$A = \{x \in E : \phi(x) > 0\}$ is a measurable set of finite measure.

Let M be the maximum of ϕ, let ϵ be given positive number and set

$A_n = \{x \in E : f_k(x) > (1-\epsilon)\phi(x)$ for all $k \geq n\}$

Then $\{A_n\}$ is an increasing sequence of sets whose union contains A and so $\{A \sim A_n\}$ is a decreasing sequence of sets whose ~~intereasing~~ intersection is empty.

By preposition : 2

$\lim \mu(A \sim A_n) = 0$ and so we can find an n such that

$\mu(A \sim A_k) < \epsilon$ for all $k \geq n$

Thus for $k \geq n$

$$\int_E f_k \geq \int_{A_k} f_k \geq (1-\epsilon)\int_{A_k} \phi$$

$$\geq (1-\epsilon)\int_E \phi - \int_{A \sim A_k} \phi$$

$$\geq \int_E \phi - \epsilon\left[\int_E \phi + \mu M\right]$$

$$\underline{\lim} \int_E f_n \geq \int_E \phi - \epsilon\left[\int_E \phi + M M\right]$$

Since ϵ is arbitrary.

$$\underline{\lim} \int_E f_n \geq \int_E \phi$$

$$\therefore \int_E f \leq \underline{\lim} \int_E f_n$$

Monotone Convergence theorem:

Let $\{f_n\}$ be a sequence of non-negative measurable function which converge a.e of a function f and suppose that $f_n \leq f$ for all n. Then $\int f = \lim \int f_n$.

proof:

Since $f_n \leq f$

We have $\int f_n \leq \int f$

Thus by Fatou's lemma, we get

$$\int f \leq \underline{\lim} \int f_n \leq \overline{\lim} \int f_n \leq \int f$$

Lemma:

Let E be a measurable set such that $0 < \nu E < \infty$. Then there is a positive set A contained in E with $\nu A > 0$

proof:

Either E itself is a positive set or it contains sets of negative measure.

In the latter case let n_1 be the smallest positive integer such that there is a measurable set $E_1 \subset E$ with $\nu E_1 - 1/n$

proceeding inductively,

if $E \sim \bigcup_{j=1}^{k-1} E_k$ is not already a positive set.

Let n_k be the smallest positive integer for which there is a measurable set E_k such that

$$E_k \subset E \sim \left[\bigcup_{j=1}^{k-1} E_j\right] \text{ and}$$

$$\nu E_k < -\frac{1}{n_k}$$

If we set

$$A = E \sim \bigcup_{k=1}^{\infty} E_k$$

Then $E = A \cup \left[\bigcup_{k=1}^{\infty} E_k\right]$

Since this is a disjoint union we get,

$$\nu E = \nu A + \sum_{k=1}^{\infty} \nu E_k$$

With the series on the right absolutely Convergent.

Since νE is finite

Thuse $\sum_k \frac{1}{n_k}$ Converges and we have $n_k \to \infty$.

Since $\nu E_k \leq 0$ and $\nu_E > 0$

We must have $\nu_A > 0$. To show that A is a positive set. Let $\epsilon > 0$ be given

Since $n_k \to \infty$

We may choose k so large that $(n_k^{-1})^{-1} < \epsilon$

Since $A \subset E \sim \left[\bigcup_{j=1}^{k} E_j\right]$

A can contain no measurable sets with measure less that $-(n_k-1)^{-1}$.

Which is greater than $-\epsilon$.

Thus A contains no measurable sets of measure less than $-\epsilon$.

Since ϵ is an arbitrary positive number.

It follows that A can Contain no set of negative measure and so must be a positive set.

Lemma : 4.8

Suppose that to each α in dense set D of real numbers. There is assigned a set $B_\alpha \in \mathcal{B}$ Such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable extended real valued function f on x, such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$.

proof:

Given that D is a dense set of real numbers and to each $\alpha \in D$ $\exists$ $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$.

To prove:

There is a unique measurable extended real valued function f on x such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$.

For each $x \in X$ define

$f(x) = \inf \{\alpha \in D / x \in B_\alpha\}$

Where as usual $\inf \phi = \infty$

If $x \in B_\alpha$ Then $f(x) \leq \alpha$ —— (1)

If $x \in X - B_\alpha$ Then $x \in X$ & $x \notin B_\alpha$

$x \notin B_\alpha \Rightarrow x \notin B_\beta$ for each $\beta < \alpha$.

$\{\because \beta < \alpha \Rightarrow B_\beta \subset B_\alpha\}$ —— (2)

$\therefore f(x) \geq \alpha \quad \forall \ x \in X \sim B_\alpha$.

Thus from (1) & (2)

There exists a real valued function f on X, such that $f \leq \alpha$ on B_α & $f \geq \alpha$ on $X \sim B_\alpha$. It remains to prove that f is measurable.

Since D is dense in R. $\overline{D} = R$

$\therefore$ For any $\lambda \in R$, we can choose a sequence $\{\alpha_n\}$ in D with $\alpha_n < \lambda$ and $\lambda = \lim \alpha_n$

Claim:

$$\{x : f(x) < \lambda\} = \bigcup_{n=1}^{\infty} B_{\alpha_n}$$

Let $x \in \{x : f(x) < \lambda\} \Rightarrow f(x) < \lambda$

$\Rightarrow f(x) < \alpha_n$ for some n.

$\Rightarrow x \in B_{\alpha_n}$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} B_{\alpha_n}$

On the other hand,

Let $x \in \bigcup_{n=1}^{\infty} B_{\alpha_n} \Rightarrow x \in B_{\alpha_n}$ for some n.

$\Rightarrow f(x) \leq \alpha_n < \lambda$

$\Rightarrow x \in \{x : f(x) < \lambda\}$

Hence $\{x : f(x) < \lambda\} = \bigcup_{n=1}^{\infty} B_{\alpha_n}$

Since each n, $B_{\alpha_n} \in \mathcal{B}$ we have B_{α_n} is measurable for each n.

$\therefore \bigcup_{n=1}^{\infty} B_{\alpha_n}$ is measurable.

$\Rightarrow \{x : f(x) < \lambda\}$ is measurable.

$\therefore f$ is measurable.

To prove the uniqueness of f

Let g be any extended real value function with $g < \alpha$ on B_α and $g \geq \alpha$ on $X \sim B_\alpha$

ie) $x \in B_\alpha \Rightarrow g(x) \leq \alpha$

$\therefore \{\alpha \in D : x \in B_\alpha\} \subset \{\alpha \in D : \alpha \geq g(x)\}$

Since $g(x) \leq \alpha \Rightarrow x \in B_\alpha$ we get

$\{\alpha \in D : \alpha > g(x)\} \subset \{\alpha \in D : x \in B_\alpha\}$

Now,

$g(x) = \inf \{\alpha \in D : \alpha > g(x)\}$

$= \inf \{\alpha \in D : \alpha \geq g(x)\}$ $\{\because D$ is dense in $R\}$

$= \inf \{\alpha \in D : x \in B_\alpha\}$

i.e) $g(x) = f(x)$ for every $x \in X$.

Hence f is unique.

proposition: 4.9

Suppose that for each α in a dense set D of real numbers, there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. Then there is a measurable function f such that $f \leq \alpha$ a.e on B_α and $f \geq \alpha$ a.e on $X \sim B_\alpha$. If g is any B_α other ~~f~~ function with this property then $g = f$ a.e.

proof:

Given that D is dense in R.

And for each $\alpha \in D$, there is a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$.

To prove:

There is a measurable function f such $f \leq \alpha$ a.e on B_α and $f \geq \alpha$ a.e on $X \sim B_\alpha$.

Let C be a countable dense subset of D and

Let $N = \cup (B_\alpha \sim B_\beta)$ for $\alpha, \beta \in C$ with $\alpha < \beta$.

Then $\mu_N = \mu(\cup(B_\alpha \sim B_\beta))$

$= \Sigma \mu (B_\alpha \sim B_\beta) = 0$

$\therefore N$ is a set of measure zero.

Let $B_\alpha' = B_\alpha \cup N$.

For $\alpha, \beta \in C$ with $\alpha < \beta$.

$B_\alpha' \sim B_\beta' = (B_\alpha \cup N) \sim (B_\beta \cup N)$

$B_\alpha' - B_\beta' = (B_\alpha \sim B_\beta) \sim N$

$= (B_\alpha \sim B_\beta) \sim \cup (B_\alpha \sim B_\beta)$

$B_\alpha' - B_\beta' = \phi$

$\Rightarrow B_\alpha' \subset B_\beta'$ for $\alpha < \beta$.

Then by the previous theorem

There is a measurable function 'f' such that $f \le \gamma$ on B_γ' and $f \ge \gamma$ on $\tilde{B}$. Since c is dense in D, for each $\alpha \in D$, we can choose a sequence $\{\gamma_n\}$ from c with $\alpha < \gamma_n$ and $\alpha = \lim \gamma_n$.

claim:

$B_\alpha \sim B\gamma_n' \subset B_\alpha \sim B\gamma_n$

Let $x \in B_\alpha \sim B\gamma_n'$

$\Rightarrow x \in B_\alpha$ & $x \notin B\gamma_n'$

$x \notin B\gamma_n \cup N$

$\Rightarrow x \notin B\gamma_n$ & $x \notin N$

$\Rightarrow x \in B_\alpha$ & $x \notin B\gamma_n$

$\therefore B_\alpha \sim B\gamma_n' \subset B_\alpha \sim B\gamma_n$

Also, $B_\alpha \sim B\gamma_n' = B_\alpha \sim (B\gamma_n \cup N)$

$= (B_\alpha \sim B\gamma_n) \cap (B_\alpha \sim N)$

$= (B_\alpha \sim B\gamma_n) \cap \phi$

$= \phi$

Thus $P = \bigcup_n (B_\alpha - B\gamma_n')$ is a countable union of null sets and so is a null set.

Let $A = \cap (B\gamma_n') = \cap B\gamma_n'$

Then $f \le \inf \gamma_n = \alpha$ on A and

$A \sim B_\alpha \subset P$

$\therefore f \leq \alpha$ a.e. on B_α.

lly we can prove that $f \geq \alpha$ a.e on $\tilde{B}_\alpha$

Suppose that g be an extended real valued function with $g \leq \gamma$ a.e on B_γ and $g \geq \gamma$ a.e on $X \sim B_\gamma$ for each $\gamma \in C$.

Then $g \leq \gamma$ on B_γ' except for x in a null set Q_γ.

Let $Q = \cup Q_\gamma$

Then Q is also a null set.

$\therefore$ We have $f = g$ on $X \sim Q$

$\Rightarrow f = g$ a.e on X.

4.2. Integration:

Signed Measure:

Def:

By a signed measure on the measurable space (X, β) we mean an extended real-valued set function γ defined for the sets of $\mathcal{B}$ ie) $\gamma : \mathcal{B} \to \mathbb{R} \cup \{-\infty, \infty\}$ satisfying the following conditions:

(i) γ assume at most one of the values $-\infty, \infty$.

(ii) $\gamma(\phi) = 0$

(iii) $\gamma\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \gamma E_i$

Eg: $(\mathbb{R}, \mu, -m)$

Def:

We say that a set A positive set w.r.to a signed measure γ if A is measurable and for any measurable subset E of A, we have $\gamma(A) \geq 0$.

Def:
A set that is both positive & negative w.r. to ν is called null set.

Note:
(1) A measurable set is a null set if every measurable subset of it has ν measure zero.
(2) All measurable sets are positive.
(3) Countable subsets are negative.

Lemma: 19
Every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.

proof:
Let A be the union of sequence $\{A_n\}$ of positive sets if E is any measurable subset of A,
define: $E_n = E \cap A_n \cap A_{n-1}^c \cap \dots A_2^c \cap A_1^c$.
Thus $E_n \subset A_n$ measurable & so $\nu E_n \geq 0$
Since E_n are disjoint & $E = \bigcup_{n=1}^{\infty} E_n$, we have
$$\nu E = \sum_{n=1}^{\infty} \nu E_n \geq 0$$
Thus A is a positive set.

Lemma: 20
Let E be a measurable set such that $0 < \nu E < \infty$. Then there is a positive set $A \subset E$ with $\nu A > 0$

Proof:
If E itself is a positive set, we are done otherwise.
E contains sets of negative measure

ie) There is a measurable subset E_1 of E with $\nu E_1 < 0$

Let n_1 be the smallest the integer $\ni$ $\nu E_1 < -\frac{1}{n}$

Since $\nu E_1 < 0$

$\Rightarrow -\nu E_1 > 0$, $N(-\nu E_1) > 1$

$\Rightarrow -\nu E_1 > \frac{1}{N}$,

$\nu E_1 < -\frac{1}{N}$.

Now consider $E - E_1$. If $(E-E_1)$ is a positive set.

We are done $E = (E-E_1) \cup E_1$

$$\underset{>0}{\nu E} = \nu(E-E_1) + \underset{<0}{\nu E_1}$$

$$\therefore \nu(E-E_1) \underset{<\infty}{> 0}$$

$$0 < \nu(E-E_1) < \infty$$

Otherwise continue the process inductively

ie) There is a measurable subset E_2 of $(E-E_1)$ with $\nu E_2 < 0$

Let n_2 be the smallest positive integer $\ni$

$\nu E_2 < -1/n_2$

Now, consider $E-(E_1 \cup E_2)$ Again if it is positive we are done otherwise. Continue the process inductively & in $(k-1)$th step if $(E - \bigcup_{i=1}^{k} E_i)$ is not already a positive set,

Let n_k be the smallest positive integer for which there is a measurable set E_k such that $E_k \subset (E - \bigcup_{i=1}^{k-1} E_i)$.

& $\nu E_K < -\frac{1}{n_K}$

Now, we set $A = E - \bigcup_{K=1}^{\infty} E_K$,

Then $E = A \cup \left(\bigcup_{K=1}^{\infty} E_K\right)$

Now if we set $A = \bigcup_{K=1}^{\infty} E_K$,

Then $E = A \cup \left[\bigcup_{K=1}^{\infty} E_K\right]$

$\nu E = \nu A + \sum_{K=1}^{\infty} \nu E_K$ $[\because \nu E_K < 0 \ \forall K]$

Thus $\sum 1/n_K$ converges, $n_K \to \infty$

Since $\nu E_K \not< 0$, $\nu E > 0$,

we must have $\nu A > 0$

$\nu E_K < -1/n_K$, $\sum \frac{1}{n_K}$ converges by

Comparisntest

$\sum_{K=1}^{\infty} |\nu E_K|$ converges

$\sum_{K=1}^{\infty} (-\nu E_K)$ converges $\left(0 < \frac{1}{n_K} < -\nu E_K\right)$

To see that A is a positive set.

Let $\epsilon > 0$ is given

Since $n_K \to \infty$, we may choose K so large that $1/n_{K-1} < \epsilon$ since $A \subset E - \left[\bigcup_{i=1}^{K-1} E_i\right]$

A can contain no measurable set with measure less. Than $-1/n_{K-1}$ which is greater than $-\epsilon$ and A contains no measurable sets ~~with measure~~ of measure less than $-\epsilon$.

Since ϵ is an arbitrary positive numbers.

It follows that A can contain no sets of negative measure & so A must be positive.

Hahn Decomposition Theorem:

Let ν be a signed measure on the measurable space $(X, \mathcal{B})$. Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$

proof:

W.L.O.G (without loss of generality)

We may assume that ν does not take $-\infty$

$\lambda = \sup \{ \nu P \mid P \text{ is a +ve set w.r.to } \nu \}$

$\nu(\phi) = 0, \quad \lambda \geq 0$

~~$\nu(\phi) = 0, \quad \lambda \geq 0$~~

~~$\nu(A)$~~ Let $\{A_i\}$ of $\pm$ve set which converges to λ.

$$\lim_{i \to \infty} \nu A_i = \lambda$$

$$A = \bigcup_{i=1}^{\infty} A_i$$

$$\nu A \leq \lambda \quad \text{——} (1)$$

$$A - A_i \subset A$$

$$A - A_i \in \mathcal{B}$$

$$\nu(A - A_i) \geq 0$$

$$\nu(A) = \nu(A_i) + \nu(A - A_i)$$

$$\geq \nu A_i$$

$$\geq \lim \nu A_i$$

$$\nu(A) \geq \lambda \quad \text{——} (2)$$

From (1) & (2) $\quad \nu A = \lambda$

Let $\quad B = X - A \; (\sim A)$.

Suppose if E is a +ve subset of $\mathcal{B}$

$E \cup A =$ +ve set.

disjoint

$\lambda \geq \nu(E \cup A)$

$= \nu(E) + \nu A$

$= \nu E + \lambda$

$\lambda \geq \nu E + \lambda$

$\nu E = 0$ since $0 \leq \lambda < \infty$

Thus B contains no positive subset of positive measure.

Thus $X = A \cup B$

$X = (A \cup N) \cup (B - N)$

and $A \cap B = \phi$

Eg: (R, μ, m)

$A = I, \; B = Q$

$R = Q \cup I$

$Q \cap I = \phi$

Mutually singular:

Two measure μ and ν defined on $(X, \mathcal{B})$ are said to be mutually singular $(\mu \perp \nu)$.

if there are disjoint measurable sets A and B with $X = A \cup B$ ∋ $\mu A = 0 = \nu B$

Eg: m, o

(R, μ), $mQ = 0$

$A = I, \; B = Q$, $OI = 0$

$R = I \cup Q$

$I \cap Q = \phi$

Jordon Decomposition:

Let ν be a signed measure on the measurable space $(X, \mathcal{B})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{B})$ such that $\nu = \nu^+ - \nu^-$. moreover, there is only one such pair of mutually singular measures.

proof:

$$X = A \cup B$$

A - the set. B - ve set

$\nu^+ : \mathcal{B} : [0,\infty) \cup \{\infty\}$

$$\nu^+(E) = \nu(E \cap A)$$

$\nu^- : \mathcal{B} : [0,\infty) \cup \{\infty\}$

$$\nu^-(E) = -\nu(E \cap B)$$

$$\nu^+(\phi) = \nu(\phi \cap A) = \nu(\phi) = 0$$

$$\nu^-(\phi) = -\nu(\phi \cap B) = -\nu(\phi) = 0$$

Let $\{E_i\}$ be a sequence of measurable sets.

$$\nu^+\left(\bigcup_{i=1}^{\infty} E_i\right) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \cap A$$

$$= \nu\left(\bigcup_{i=1}^{\infty} (E_i \cap A)\right)$$

$$= \sum_{i=1}^{\infty} \nu(E_i \cap A)$$

$$= \sum_{i=1}^{\infty} \nu^+(E_i)$$

$$(\nu^+ - \nu^-)(E) = \nu^+(E) - \nu^-(E)$$

$$= \nu(E \cap A) + \nu(E \cap B)$$

$$= \nu[(E \cap A) \cup (E \cap B)]$$

$$= \nu(E \cap X)$$

$$= \nu(E)$$

$$\nu^+ \perp \nu^-$$

$$\nu^+(B) = \nu(B \cap A)$$
$$= \nu(\phi)$$
$$= 0$$
$$\nu^-(A) = -\nu(A \cap B) = -\nu(\phi) = 0$$

Note:

The measure $|\nu|$ defined by, $|\nu E| = \nu^+(E) + \nu^-(E)$ is called the absolute value of total variation of ν.

Def:

A measure ν is said to be the absolutely continuous w.r. to measure μ if $\nu(A) = 0$ for each set A for which $\mu(A) = 0$

We use the symbole $\nu << \mu$ to say ν is absolutely continuous.

Note:

(i) In case of signed measure μ and ν. we say $\nu << \mu$.

if $|\nu| << |\mu|$ and $\nu \perp \mu$

if $|\nu| \perp |\mu|$

ii) Also for $\mu << \mu$

$\nu = c\mu$, $c \neq 0$, $\nu << \mu$

iii) $0 << \mu$, But $\mu \not<< 0$

for a non-zero measure μ.

The Radon Nikodym Theorem:

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{B}$ which is absolutely continuous with respect to μ. Then there is a non-negative measurable function f such that for each set E in $\mathcal{B}$

we have

$$\nu E = \int_E f d\mu$$

The function f is unique in the sense that if g is any measurable function with this property. Then $g=f$ a.e $[\mu]$.

proof:

lemma:

{Suppose that to each α in a done set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$.

such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable extended real-valued function f on X such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$.

Then there is a measurable function f such that $f \leq \alpha$ a.e on B_α and $f \geq \alpha$ a.e on $X \sim B_\alpha$. If g is any other function with this property.

Then $g = f$ a.e }

proof:

$(X, \mathcal{B}, \mu)$ to be a σ-finite measurable space. We still assume μ to be finite.

$D = \mathbb{Q}$, $\alpha \in \mathbb{Q}$

$\nu : \mathcal{B} \to [0, \infty) \cup \{\infty\}$

$\mu : \mathcal{B} \to [0, \infty)$

$(\nu - \alpha\mu)$ to be a signed measure

$\forall \alpha \in \mathbb{Q}$

$A_\alpha \cup B_\alpha = X$, $A_\alpha \cap B_\alpha = \phi$

$A_0 = X$, $B_0 = \phi$

$B_\alpha - B_\beta \Rightarrow B_\alpha \cap B_\beta^c = B_\alpha \cap A_\beta$

$B_\alpha \cap B_\beta \quad \beta > \alpha$

B_α - -ve set w.r.to $\nu - \alpha\mu$

A_β - +ve set w.r.to $\nu - \beta\mu$

$(\nu - \alpha\mu)(B_\alpha - B_\beta) \le 0$ —— ①

$(\nu - \beta\mu)(B_\alpha - B_\beta) \ge 0$ —— ②

- eqn ②

$-(\nu - \beta\mu)(B_\alpha - B_\beta) \le 0$ —— ③

① + ③

$(\beta\mu)(B_\alpha - B_\beta) - \alpha\mu(B_\alpha - B_\beta) \le 0$

$(\beta\mu - \alpha\mu)(B_\alpha - B_\beta) \le 0$

$\underset{\le 0}{(\beta - \alpha)}\, \underset{\ge 0}{\mu(B_\alpha - B_\beta)} \le 0$

$$\left.\begin{array}{l} \mu(B_\alpha - B_\beta) \le 0 \\ \mu(B_\alpha - B_\beta) \ge 0 \end{array}\right\} = 0$$

$f \le \alpha$ on B_α

$f \ge \alpha$ on $X - B$

$\alpha = 0, \quad \nu - \alpha\mu = \nu$

$X = A_\alpha \cup B_\alpha, \quad A_\alpha \cap B_\alpha = \phi$

$A_0 = X, \quad B_0 = \phi$

$f \ge 0$

$$\left\{\begin{array}{l} f \le 0 \text{ a.e on } B_0 = \phi \\ f \ge 0 \text{ a.e on } A_0 = X \end{array}\right.$$

Let E is any arbitrary set in $\mathcal{B}$.

$E_k = E \cap (B_{\frac{k+1}{N}} \sim B_{k/N})$

$E_\infty = E \sim \cup B_{k/N}$

Then $E = E_\infty \cup \bigcup_{k=0}^{\infty} E_k$

choose $x \in E_k$

$\rightarrow x \in B_{\frac{k+1}{N}}$

$\frac{k}{N} \leq f(x) \leq \frac{k+1}{N}$ a.e on E_k.

$\int_{E_k} \frac{k}{N} d\mu \leq \int_{E_k} f d\mu \leq \int_{E_k} \frac{k+1}{N} d\mu$

$\frac{k}{N} \mu E_k \leq \nu E_k \leq \frac{k+1}{N} \mu E_k$

$0 \leq \left| \int_{E_k} f d\mu - \nu E_k \right| \leq \frac{1}{N} \mu E_k \longrightarrow 0$

as $n \to \infty$.

Hence $\nu << \mu$

$\nu E = 0$ for each set E $\mu E = 0$

$f = \infty$ on E_∞

i) $\mu E_\infty = 0$

$\int_{E_\infty} f d\mu = 0$

$\nu << \mu$, $\mu E_\infty = 0$

$\nu E_\infty = 0$

$\int_{E_\infty} f d\mu = \nu E_\infty$

ii) $\mu E_\infty > 0$

$E_\infty = E - \bigcup_{k=0}^{\infty} B_{k/N}$

$x \in E_\infty$, $x \in \bigcap_{k=0}^{\infty} A_{k/N}$, $k > 0$

Scanned with CamScanner

$x \in A_{k/N} \;\forall k$

$\&\; f \geq \frac{k}{N}, \quad k \geq 0$

$f = \infty \Rightarrow f\,d\mu = \infty$

$E_\infty \subseteq A_{\frac{k}{N}}\; E_\infty$

$(\nu - \frac{k}{N}\mu)(E_\infty) \geq 0$

$\nu E_\infty \geq \frac{k}{N}\mu E_\infty$

$\mu \nu E_\infty = \infty$

$\int_{E_\infty} f\,d\mu = \nu E_\infty$

$\nu E - \frac{1}{N}\mu E \leq \int_E f\,d\mu \leq \int \nu E + \frac{1}{N}\mu E$

μE is finite, $\quad \nu E = \int_E f\,d\mu$

denote $\left[\frac{d\nu}{d\mu}\right]$

N - arbitrary

Hence the proof