UNIT - I [LEBEGUE MEASURE]

INTRODUCTION: The length $l(I)$ of an inval is defined to be the differences of the end points of the interval length is an example of set function that associates an extended real number to each set.

EXAMPLE: IN Some Collection of sets we define the length of an set be Some of the intervals of the which is composed.

LEBEGUE MEASURE: We define a set function m that assigns to each set E in some collection m of sets of real number a non-negative extended real number mE called the measure of E, Satisfying the following properties

i) mE is defined for each set E of real number that is

$$m = P(R).$$

ii) For an interval I, $mI = l(I)$.

iii) If $\langle E_n \rangle$ is a sequence of disjoint sets (for which m is defined).
$$m(\cup E_n) = \Sigma mE_n$$

iv) m is translation invariant:

If E is a set for which m is defined and if $E+y$ is That set $\{x+y : x \in E\}$ obtained by replacing each point x in E by the point $x+y$ Then $m(E+y) = mE$.

σ ALGEBRA: An algebra a of Sets is called σ-algebra (or) a Borel holds.

If every union of a countable collection of sets in "a" is again in "a" such that,

if $\langle A_i \rangle$ is a sequence of sets then $\cup A_i$ must again in a.

[NOTE: $A \subseteq P(X)$ is called a σ-algebra

a) $\phi, X \in A$

b) $A \in A \Rightarrow A^c = X \setminus A \in A$

c) $A_i \in A$; $i \in A \Rightarrow \bigcup_{i=1}^{\infty} A_i \in A$

$A \in A$ is called a A measurable set

EXAMPLE: i) $A = \{\phi, X\}$

ii) $A = P(X)$]

OUTER MEASURE: For each set A of real numbers consider the countable collections $\{I_n\}$ of open intervels that cover A.

(ie) collection for which A is a subset of $\cup I_n$ and for each such collection. Consider, the sum of the length of the intervals in the collection the outer measure m^*A.

A is to be the infrimum of all such sums.

(ie) $m^*A = \inf \Sigma l(I_n)$. where $l(I_n)$ elements $A \subset \cup I_n$ the length of I_n.

(or)

Let A be a set of real numbers the outer measure m^*A of A is defined as

$$m^*A = \inf_{A \subset \cup I_n} \sum l(I_n).$$

NOTE: 1) If A is a set of all real numbers we can covers A by a countable collection of open intervals

2) $m^*A \geq 0 \; \forall A$

3) Since $m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$.

Given $\epsilon > 0$ $\exists$ an $A \subset \cup I_n$ and $\sum l(I_n) < m^*A + \epsilon$

4) The outer measure m^* is called a lebegue outer measure.

RESULT: 1 If A is subset of $A \subset (a,b)$ then $m^*A \leq b-a$

W.k.T $m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$.

Here A is a subset of (a,b) and

$l(a,b) = b-a$. (i-e) $m^*A \leq b-a$.

MONOTONICITY PROPERTY ON OUTER MEASURE:

If A and B are two sets in m with $A \subset B$ then $mA \leq mB$ this property called monotonicity.

RESULT: 2 If $A \subset B$ Then $m^*A \leq m^*B$.

Let $A \subset \cup I_n$, $B \subset \cup I_n$

(4) Since $A \subset B$. Then $\cup I_n \subset \cup I_n$.

Then $m^*A \le \sum l(I_n)$ and $m^*B \le \sum l(I_n)$

We get from this $m^*A \le m^*B$.

RESULT: 3 $m^*\{x\}=0$ if x is real since $x \in R$ there exist $\epsilon>0$ such that

$$\{x\} \subset (x-\epsilon, x+\epsilon).$$

$$m^*x \le \cancel{x}+\epsilon - \cancel{x}+\epsilon \le 2\epsilon \text{ for all } \epsilon>0$$

$$m^*x < 0.$$

$$m^*\{x\}=0.$$

RESULT: 4 Null set of sets with 0 under measure

$m^*\phi = 0$.

We have ϕ is a subset of a set

Let ϕ is subset of $\{x\}$

(i.e) $\phi \subset \{x\}$

$m^*\phi \le m^*\{x\}$

$m^*\phi < 0$.

PREPOSITION: (10m (x))

The outer measure of a interval is its length.

Proof: Case (i)

Let I be a finite closed interval say $[a,b]$.

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since $I \subset (a-\epsilon, b+\epsilon)$

We have $m^*I \le b+\epsilon - a+\epsilon$

$$\le b-a+2\epsilon$$

$$m^*I \le b-a \longrightarrow (1)$$

Next we have to prove $m^*I \ge b-a$.

Let $I \subset \bigcup_{n=1} I_n$ by Heine borel theorem.

There exists a finite subcollection from a sequence I_n say $I_1, I_2 \ldots I_m$.

which covers $I \subset \bigcup_{n=1}^{m} I_n$

Also we have

$$\sum_{n=1}^{m} l(I_n) \le \sum_{n=1}^{\infty} l(I_n)$$

Since $I = [a,b]$ be a subset of $\cup I_n$

We have $A \subset I_n$ for some $1 \le n \le m$.

Let $I_n = (a_1, b_1)$ where $a_1 < a < b < b_1$

$\therefore l(I_l) > b-a$

$$\Rightarrow \sum_{n=1}^{m} l(I_l) > b-a$$

If $a \le b$ then $b_1 \in [a,b]$ $b_1 \in I_l$

$\therefore b_1 \in I_t$ for some t.

Let $I_t = (a_2, b_2)$ then $a_2 < b_1 < b_2$ if $b_2 \le b$

Continuing in this way. We obtain a sequences (6)
$(a_1, b_1) \ldots (a_k, b_k)$ from the collection $\{I_n\}$ such that
$a_i < b_{i-1} < b_i$.

Since collection processed must terminates with same interval (a_k, b_k) where $k \le m$. But it terminates only $b \in (a_k, b_k)$. Thus,

$$\sum_{i=1}^{k} l(I_i) = b_1 - a_1 + b_2 - a_2 + \cdots + b_k - a_k$$
$$= b_k - a_k + b_{k-1} - a_{k-1} + \cdots + b_1 - a_1$$
$$= b_k + (b_{k-1} - a_k) + \cdots + (b_2 - a_3) + (b_1 - a_2) - a_1$$

$$\sum_{i=1}^{k} l(I_i) > b_k - a_1 \longrightarrow (2)$$

$$\{\text{since } a_i < b_{i-1}\}$$

But $b_k > b$ and $a_1 < a$

$$\therefore a_1 < a < b < b_k$$

$$\Rightarrow b_k - a_1 > b - a \Rightarrow \sum_{i=1}^{k} l(I_i) > b - a$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) > \sum_{i=1}^{k} l(I_i) > b - a$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) > b - a$$

$$\Rightarrow \inf \sum_{n=1}^{\infty} l(I_n) > b - a$$

$$\Rightarrow m^* I \ge b - a \longrightarrow (3)$$

$$m^* I = b - a.$$

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Case (ii) Let I be any finite interval

(i.e) $[a,b)$ $(a,b]$ (or) (a,b) for given $\epsilon>0$

$$J = [a+\epsilon/4,\ b-\epsilon/4].\ [J \subset I]$$

$$l(J) = b-\epsilon/4 - a - \epsilon/4 = b-a-\epsilon/2$$

Now $l(I)+\epsilon/2 > l(I)-\epsilon/2 \ \forall\ \epsilon>0$.

$\therefore\ l(I) > l(I)-\epsilon \ \forall\ \epsilon>0$.

Hence $l(I)-\epsilon\ l(J)$

Since $J \subset I$

$$l(J) = m^*J \le m^*I \le m^*\bar{I} = l(\bar{I}) \le l(I).$$

where $\bar{I} = [a,b]$

Now eqn $\Rightarrow$ (4) $l(I)-\epsilon < l(J) \le l(I)\ \forall\ \epsilon>0$

$\Rightarrow l(I)-\epsilon < m^*I < l(I)\ \forall\ \epsilon>0$.

$\Rightarrow m^*I = l(I)$.

Case (iii) Let I be an infinite interval given any real number Δ, There is a closed interval $J \subset I$ with $l(J)$.

Hence $m^*I \ge m^*J = l(J) = \Delta$

Since $m^*I \ge \Delta$ for each Δ

$m^*I \ge \infty = l(I)$

Hence proved.

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(8) **PREPOSITION: 2** Countable subadditivity of m^*

Let $\{A_n\}$ be a countable collection of sets of real number then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^* A_n$.

Proof: case (i) In $m^* A_n = \infty$ for some n.

Then the inequality holds trivially.

case (ii): Assume that $m^* A_n < \infty$ for an n.

Given $\epsilon > 0$ there is a countable collection $\{I_{n,i}\}_i$ of open intervals such that

$$A_n \subset \bigcup_{i=1}^{\infty} I_{n,i} \text{ and } \sum_i l(I_{n,i}) < m^* A + 2^{-n} \quad \longrightarrow (A)$$

Such countable number of countable collection in again a countable. We have the collection

$\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable. Also,

$$\bigcup A_n \subset \bigcup_{n=1}^{\infty} \{I_{n,i}\}_i$$

$$m^*(\bigcup A_n) \le m^*\left(\bigcup_{n=1}^{\infty} \{I_{n,i}\}_i\right)$$

$$m^*(\bigcup A_n) \le \sum l(I_{n,i})$$

$$\le \sum_n \sum_i l(I_{n,i})$$

$$< \sum m^* A_n + 2^{-n}\epsilon = \sum m^* A_n + \epsilon$$

Since ϵ was an arbitrary positive number $m^*(\bigcup A_n) \le \sum m^* A_n$.

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NOTE: 1) Every subset of a countable set is countable.

2) Set of all, rational number is countable.

3) The union of countable collection of countable set is countable.

COROLLARY: Outer measure of a countable set of zero. If A is countable the $m^*A = 0$.

Proof: Let $A = \{x_1, x_2, \ldots x_n\}$

$$= \bigcup_{i=1}^{n} x_i$$

$$m^*A = m^*\left(\bigcup_{i=1}^{n} x_i\right)$$

$$\leq \sum m^* x_i \quad (\because m^*\{x\} = 0)$$

$$\leq 0.$$

Since m^* is non-negative.

$$\therefore m^*A = 0.$$

COROLLARY: A set $[0,1]$ is uncountable:

Proof: Suppose a set $[0,1]$ is countable.

$$\therefore m^*[0,1] = 0 \longrightarrow (1)$$

$\because$ a countable set has zero measure.

By proposition (1)

$$m^*[0,1] = l[0,1] = b-a = 1-0 = 1 \longrightarrow (2)$$

 From (1) and (2)

which is a contradiction.

$\therefore [0,1]$ is uncountable.

NOTE: 1) A set of all rational numbers $\mathbb{Q}$ is countable

$m^*\mathbb{Q} = 0.$

2) By Preposition (2)

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} m^* A_i$$

This is also called countable subadditivity of m^*.

PREPOSITION: 3 Given any set A and any $\epsilon > 0$ Then there $\exists$ a open set O such that $A \subset O$ (A is subset of U). and m^*U is $m^*U \le m^*A + \epsilon$. There is a set $G \in G_\delta$ such that A is a subset of G and $m^*A = m^*G$.

(G_δ - special Borel set).

Proof: Let A be a set of real numbers then there exists a open interval I_n such that $A \subset \cup I_n$.

$$\sum l(I_n) \le m^* A + \epsilon \longrightarrow (1)$$

Let, $O = \cup I_n$ There U is a open set and $A \subset O$

$m^*O = m^*(\cup I_n)$

$m^* O \leq \sum m^*(I_n)$ (By preposition -2)

$m^* O \leq \sum l(I_n) \longrightarrow (2)$

From (1) and (2) we get

$m^* O \leq m^* A + \epsilon \longrightarrow (3)$.

G_δ set take $\epsilon = 1/n$ for each n.

There exists a O_n such that $A \subset U_n$ and

$m^* U_n \leq m^* A_n + 1/n \longrightarrow (4)$

Let $G = \cap O_n$

Then G is a G_δ set and G is super set of A $[G \supset A]$.

Also $m^* G \leq m^* U_n$.

$m^* G \leq m^* A_n + 1/n$ $\because$ [by (A)]

$m^* G \leq m^* A \longrightarrow (4)$ $[\because G \subset U_n]$

$G \supset A \Rightarrow m^* G \geq m^* A \longrightarrow (5)$

From (4) and (5) we get.

$m^* G = m^* A$.

Hence proved.

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1. Prove that if $m^*(A)=0$. Then $m^*(A\cup B)=m^*B$.

Proof: We have to prove $m^*(A\cup B)=m^*B$.

$$m^*(A\cup B)\le m^*A+m^*B.$$

$$m^*(A\cup B)\le m^*B \longrightarrow (1)$$

But $B\subset A\cup B$

$$\cancel{m^*B}\quad m^*B\le m^*(A\cup B)\longrightarrow (2)$$

From (1) and (2) we get

$$m^*(A\cup B)=m^*B.$$

2. Prove that m^* is translation invariant.

Proof: We have to prove that

$$m^*(A+x)=m^*A \quad \forall x\in R.$$

Given $\epsilon>0$ There exists a sequence I_n.

Such that $A\subset \cup I_n$ and $\sum l(I_n)\le m^*A+\epsilon \longrightarrow (1)$

Since $A\subset\cup I_n$.

$A+x\subset \cup I_n+x$ Then

$$m^*(A+x)\le m^*(\cup I_n+x)$$
$$\le \sum m^*(I_n+x)$$
$$\le \sum l(I_n+x)$$
$$\le \sum l(I_n)$$

$$m^*(A+x)\le m^*(A)+\epsilon.$$

Since ϵ is arbitrary

$$m^*(A+x) \le m^*A \longrightarrow (2)$$

Now $A = (A+x) - x$

$A = A + x + y$

$$m^*A = m^*\left[(A+x) + y\right]$$

$$m^*A \le m^*(A+x) \longrightarrow (3)$$

From (2) and (3) we get

$$m^*(A+x) = m^*A.$$

3. Let A be a set of all rational number between 0 and 1 let $\{I_n\}$ be the collection of the open intervals covering A. Then $\sum l(I_n) \geq 1$.

Proof: Let $I_1, I_2 \cdots I_n$ be a finite collection of open intervals such that,

$$A \subset I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_n.$$

$$A \subset \bigcup_{i=1}^{n} I_i$$

$$\Rightarrow \bar{A} \subset \bigcup_{i=1}^{n} \bar{I_i} \qquad \bar{A} = [0,1]$$

$$[0,1] \subset \bigcup_{i=1}^{n} \bar{I_i}$$

$$m^* [0,1] \leq \sum m^* (\bar{I_i})$$

$$l[0,1] \leq \sum m^* (\bar{I_i})$$

MESURABLE SETS AND LEBEGUE MEASURE:

MEASURABLE SETS:

A set E is said to be measurable if for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$.

NOTE: (i) Since lebegue outer measure m^* is used the measurable if for each set A we have sets are called measurable sets.

(ii) We have $A = (A \cap E) \cup (A \cap \tilde{E})$

$m^*A \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$ (since finite subadditivity)

We always have the result.

Hence E is measurable iff for each set $A \subset R$.

$$m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

(iii) Since the definition of measurability is symmetric in E and $\tilde{E}$. We have $\tilde{E}$ is measurable whenever E is measurable. Result, If $E = R$ we have.

$$m^*A = m^*(A \cap R) + m^*(A \cap \tilde{R})$$
$$= m^*(A \cap R) + m^*(A \cap \phi)$$
$$= m^*A + m^*\phi = m^*(A).$$

$\therefore R$ is measurable. Hence $\tilde{R} = \phi$ is also measurable.

LEMMA: 1 If $m^*E = 0$ then E is measurable.

Proof: Let A be any subset of $\mathbb{R}$. We have,

To prove: $m^*A \geq m^*(A\cap E) + m^*(A\cap \tilde{E})$.

(i) $A\cap E \subset E$

$\Rightarrow m^*(A\cap E) \leq m^*E$

$\Rightarrow m^*(A\cap E) = 0$.

$\Rightarrow m^*(A\cap E) = 0 \longrightarrow (1)$

(ii) $A\cap \tilde{E} \subset A$

$m^*(A\cap \tilde{E}) \leq m^*A$

add $m^*(A\cap E)$ on both sides.

$m^*(A\cap E) + m^*(A\cap \tilde{E}) \leq m^*A + m^*(A\cap E)$.

$m^*(A\cap E) + m^*(m\tilde{E}) \leq m^*A + 0$

$m^*(A\cap E) + m^*(A\cap \tilde{E}) \leq m^*A$

$\Rightarrow m^*A \geq m^*(A\cap E) + m^*(A\cap \tilde{E})$.

$\therefore$ E is measurable.

LEMMA: 2 If E_1 and E_2 are measurable. Then $E_1 \cup E_2$ is also measurable.

Proof! Let A be any subset of $\mathbb{R}$.

It is enough to prove that,

$m^*A \geq m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(\widetilde{E_1\cup E_2}))$

Since E_1 and E_2 are measurable. We have,

(16) $m^*A = m^*(A\cap E_1) + m^*(A\cap \tilde{E_1}) \longrightarrow (1)$

$m^*A = m^*(A\cap E_2) + m^*(A\cap \tilde{E_2}) \longrightarrow (2)$

Replacing A by $(A\cap \tilde{E_1})$ in equation (2)

$m^*(A\cap \tilde{E_1}) = m^*(A\cap(\tilde{E_1}\cap E_2)) + m^*(A\cap \tilde{E_1}\cap \tilde{E_2}) \longrightarrow (3)$

$m(E_1\cup E_2) = (A\cap E_1)\cup(A\cap E_2)$

$= (A\cap E_1)\cup(A\cap E_2\cap \mathbb{R})$.

$= (A\cap E_1)\cup[(A\cap E_2)\cap(E_1\cup \tilde{E_1})]$

$= (A\cap E_1)\cup[((A\cap E_2)\cap E_1)\cup((A\cap E_2)\cap \tilde{E_1})]$

$= (A\cap E_1)\cup(A\cap E_2\cap E_1)\cup(A\cap E_2\cap \tilde{E_1})$

$= (A\cap E_1)\cup(A\cap E_2\cap E_1)\cup(A\cap E_2\cap \tilde{E_1})$

$= (A\cap E_1)\cup(A\cap E_2\cap \tilde{E_1})$

~~$m^*(A\cap(E_1\cup E_2)) \leq m^*(A\cap E_1) + m^*(A_2\cap \tilde{E_1})$ Adding, $m^*(A\cap(\tilde{E_1}\cap \tilde{E_2}))$ on both sides.~~

~~$m^*(A\cap(\tilde{E_1}\cap \tilde{E_2})) + m^*(A\cap(\tilde{E_1}\cap \tilde{E_2})) \leq m^*(A\cap E_1) + m^*(A\cap E_2\cap \tilde{E_1}) + m^*(A\cap \tilde{E_1})$~~

$m^*(A\cap(E_1\cup E_2)) \leq m^*(A\cap E_1) + m^*(A_2\cap \tilde{E_1})$ Adding, $m^*(A\cap(\tilde{E_1}\cap \tilde{E_2}))$ on both sides.

$m^*(A\cap(E_1\cap E_2)) + m^*(A\cap(\tilde{E_1}\cap \tilde{E_2})) \leq m^*(A\cap E_1) + m^*(A\cap E_2\cap \tilde{E_1}) + m^*(A\cap(\tilde{E_1}\cap \tilde{E_2}))$

$m^*(A\cap(E_1\cap E_2)) + m^*(A\cap \tilde{E_1}\cap \tilde{E_2}) \leq m^*(A\cap E_1) + m^*(A\cap \tilde{E_1})$

$m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(\widetilde{E_1\cup E_2})) \le m^*A$ (17)

$\Rightarrow m^*A \ge m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(\widetilde{E_1\cup E_2}))$

COROLLARY: The family M of measurable set ~~at~~ is an algebra of sets.

Proof: If $E_1, E_2 \subset M$.

Then by the lemma. $E_1\cup E_2 \in M$ also if E_1 is measurable.

$\therefore \tilde{E}_1$ is measurable.

(i.e) $E_1 \in M$. $E_0 \in M$.

Then M is an algebra of sets.

NOTE: If $E_1, E_2 \dots E_n$ are measurable Then $E_1\cup E_2\cup\dots\cup E_n$ is also measurable set

i) AE_n.

ii) $A\tilde{E}_n$

iii) Measure (Replace).

LEMMA:3 Let A be any set and $E_1, E_2, \dots E_n$ be a finite sequence of disjoint measurable set

(2) $$m^*\left(A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A\cap E_i).$$

Proof: We prove this lemma by induction on "A" then $n=1$ the result is true.

Assume that the result is true for $(n-1)$ sets $n\ge 2$.

(i.e) $m^*\left(A\cap\left(\bigcup_{i=1}^{n} A\right)\right) = \sum_{i=1}^{n-1} m^*(A\cap E_i) \longrightarrow (1)$

We prove that the result is true for n sets. (18)

Since $E_1, E_2 \dots E_n$ are disjoint measurable sets.

i) $A\cap\left(\bigcup_{i=1}^{n} E_i\right)\cap E_n = A\cap E_n \longrightarrow (2)$

ii) $A\cap\left(\left(\bigcup_{i=1}^{n} E_i\right)\cap\tilde{E}_n\right) = A\cap\left[(E_1\cap\tilde{E}_n)\cup(E_2\cap\tilde{E}_n)\dots\cup(E_n\cap\tilde{E}_n)\right]$

$= A\cap\left[(E_1\cup E_2\cup\dots\cup E_{n-1})\cup\phi\right]$

$A\cap\left(\bigcup_{i=1}^{n} E_i\cap\tilde{E}_n\right) = A\cap\left(\bigcup_{i=1}^{n-1} E_i\right) \longrightarrow (3)$

Since E_n is measurable.

$m^*A = m^*(A\cap E_n) + m^*(A\cap\tilde{E}_n) \longrightarrow (4)$

Where A is any subset of R.

Replace A by $A\cap\bigcup_{i=1}^{n} E_i$ in (4)

$$m^*\left(A\cap\bigcup_{i=1}^{n} E_i\right) = m^*\left(A\cap\bigcup_{i=1}^{n} E_i\cap\tilde{E}_n\right)$$

$$= m^*\left(A\cap\bigcup_{i=1}^{n} E_i\cap E_n\right) + m^*\left(A\cup\left(\bigcup_{i=1}^{n} A\right)\right)$$

$$= m^*(A\cap E_n) + m^*\left(A\cap\bigcup_{i=1}^{n-1} E_i\right)$$

$$= m^*(A\cap E_n) + \sum_{i=1}^{n-1} m^*(A\cap E_i)$$

$$m^*\left(A\cap\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n-1} m^*(A\cap E_i) = \sum_{i=1}^{n} m^*(A\cap E_i)$$

Thus the result is true for any n.

Hence the proof.

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COROLLARY: Take $A = R$ in above inequality, we get

$$m^*\left(R \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(R \cap E_i)$$

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^* E_i.$$

THEOREM: 1. The collection M of measurable sets is a σ-algebra that is the complement of a measurable set is measurable and the union (and intersection) of a countable collection of measurable sets is measurable. Moreover, every set with outermeasure zero is measurable.

Proof: Let $A_n \in M$, $n = 1, 2, \ldots \infty$ and $E = \bigcup_{i=1}^{\infty} A_i$.

Claim: There exists a sequence $E_n \in M$ $n = 1, 2 \ldots \infty$. Such that,

$E_i \cap E_j = \phi$ for $i \neq j$.

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$

Proof of the claim: Set $E_1 = A_1$.

for $n > 1$, Define $E_n = A_n - (A_1 \cup A_2 \cup \ldots \cup A_{n-1})$

$= A_n \cap (\tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_{n-1})$

$E_n = A_n \cap (\tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_{n-1})$

Since M is an algebra.

$\tilde{A}_i \in M$ for $i = 1, 2 \ldots n-1$ and

$A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots \cap \tilde{A}_{n-1} \in M \longrightarrow (1)$

This $\Rightarrow$ that $E_n \in M$, $\forall n$ and $E_n \subset A_n$.

(20)

Let $m \neq n$.

Suppose that $m < n$.

We have $E_m \in A_m$

$E_m \cap E_n \subset A_m \cap E_n \longrightarrow (2)$

Now, $A_m \cap E_n = A_m \cap \{A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots \cap \tilde{A}_m \cap \ldots \cap \tilde{A}_{n-1}\}$

$A_m \cap E_n = A_n \cap \tilde{A}_1 \cap \ldots \cap \tilde{A}_m \cap \ldots \cap \tilde{A}_{n-1}$

$A_m \cap E_n = \phi \quad (\because m < n)$

$\therefore E_m \cap E_n = \phi$

Simillarly, We can prove for $m > n$ we have, $E_i \subset A_i$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} A_i \longrightarrow (3)$

Let $x \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow x \in A_i$ for some i.

Let n be smallest value of i such that $x \in A_i$. Suppose $x \notin A_i$ for some $i < n$.

$\therefore x \in A_n \cap \tilde{A}_{n-1} \cap \ldots \cap \tilde{A}_1$.

$\Rightarrow x \in E_n$.

$x \in \bigcup_{i=1}^{\infty} E_i$

$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} E_i \longrightarrow (4)$

From (3) and (4)

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$

Hence the claim.

Thus if E is the union of countable collection of (21)
measurable sets it must be the union of pairwise disjoint measurable set.

Let $F_n = \bigcup_{i=1}^{n} E_i$

Thus F_n is measurable. Also, $\tilde{F}_n \supset \tilde{E}$

$\tilde{F}_n \supset \tilde{E}$

$\Rightarrow A \cap \tilde{E}_n \supset A \cap \tilde{E} \longrightarrow (5)$.

Where A is any set we have,

$m^*A = m^*(A \cap E_n) + m^*(A \cap \tilde{E}_n)$

$\geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*\left(A \cap \tilde{E}\right)$

$m^*A \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$

Since the left side of the inequality is independent of n. We have,

$m^*A \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$

$\geq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*(A \cap \tilde{E})$

$m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$

$\therefore$ E is measurable.

$E = \bigcup_{i=1}^{\infty} E_i \in M$.

$\therefore$ M is σ-algebra.

Hence the proof.

LEMMA: 11 The interval (a, ∞) is measurable.

Proof: Let A be any set.

(22)

3 Let $A_1 = A \cap (a, \infty)$ and

$A_2 = A \cap (-\infty, a]$

It is enough to prove that

$m^*A \geq m^*(A_1) + m^*(A_2)$.

If $m^*A = \infty$ Then there is nothing to prove.

If $m^*A < \infty$ Then given $\epsilon > 0$ There is a countable collection $\{I_n\}$ of open intervals which cover A and for which such that,

$A \subset \cup I_n$ and $\sum l(I_n) \leq m^*A + \epsilon \longrightarrow (1)$

Let $I_n' = I_n \cap (a, \infty)$

$I_n'' = I_n \cap (-\infty, a]$

Here I_n' and I_n'' are intervals may be empty and

$l(I_n) = l(I_n') + l(I_n'')$

$m^*I_n = m^*I_n' + m^*I_n'' \longrightarrow (2)$

Since $A \subset \cup I_n$.

$A \cap (a, \infty) \subset \cup I_n \cap (a, \infty)$

Since $A_1 \subset I_n'$ we have,

$m^*A_1 \leq m^*(\cup I_n') \leq \sum m^*I_n'$.

$m^*A_1 \leq \sum l(I_n') \longrightarrow (3)$.

and since $A_2 \subset I_n''$ we have,

$m^*A_2 \leq m^*(\cup I_n'') \leq \sum m^* I_n''$

$m^*A_2 \leq \sum l(I_n'') \longrightarrow (4)$

From (3) and (4).

$$m^*A_1 + m^*A_2 \le \sum l(I_n') + \sum l(I_n'')$$
$$= \sum [l(I_n') + l(I_n'')]$$
$$= \sum l(I_n).$$

$\Rightarrow m^*A_1 + m^*A_2 \le m^*A + \epsilon$.

Since ϵ is arbitrary positive number

$\Rightarrow m^*A_1 + m^*A_2 \le m^*A$

(i.e) $m^*A \ge m^*A_1 + m^*A_2$.

$\therefore (a,\infty)$ is measurable.

Hence proved.

BOREL SET: The collection $\mathcal{B}$ of Boral sets is the smallest σ-algebra which contains all of the open sets.

THEOREM: 12 Every Borel set is measurable in particular each open sets, and each closed set is measurable.

(or) (i) Each open set m^* measurable. 4

(ii) Any closed set m^* measurable.

(iii) Every Borel set m^* is measurable.

Proof: It is enough to prove that M contains all open sets.

Since M is σ-algebra and $\mathcal{B}$ is the smallest σ-alge containing all open set are $\mathbb{R}$.

We get $\mathcal{B} \subset M$ and hence the proof follow.

Claim: Every open set $\mathbb{R}$ is measurable.

Consider (a,∞) for any $a \in \mathbb{R}$.

We have (a,∞) is measurable.

$\therefore (a,\infty)$ is $\in M$.

Since M is a σ-algebra.

(a,∞) (or) $[-\infty, a) \in M$.

(i.e) $[-\infty, a)$ is measurable. For $b \in \mathbb{R}$

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n).$$

Since each of $(-\infty, b)$ is measurable and since countable union of measurable set is measurable we get $(-\infty, b)$ is measurable.

$\therefore$ Any open interval (a,b) can be written as

$(a,b) = (-\infty, b) \cap (a,\infty)$.

Since both $(-\infty, b)$ and (a,∞),

$\therefore (a,b)$ is measurable.

Any closed set = complement open set measurable.

Already We know that each open set the union of countable number of open interval.

Hence each open set in $\mathbb{R}$ is measurable. Thus M is the σ-algebra containing set.

Hence every Borel set is measurable.

Hence proved.

LEBEGUE MEASURE:

The Lebegue M is the function from the family M of lebgue measurable set to the extended real line.

Define by $mE = m^*E$ (i.e) $m : M \to \mathbb{R}$.

Defined by $mE = m^*E \ \forall E \in M$.

PREPOSITION: 13

Let $\langle E_i \rangle$ be a sequence of measurable sets, Then $m(\cup E_i) \leq \sum m E_i$. If the sets E_n are pairwise disjoint, Then $m(\cup E_i) = \sum m E_i$.

Proof: $E_1, E_2, \ldots E_n$ be a finite sequence of measurable sets. Then by the lemma we have,

$$m\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(A \cap E_i) \longrightarrow (1).$$

Where A is any subset of R. Let $A \subset R$.

$$m\left(R \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(R \cap E_i)$$

$$m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i) \longrightarrow (2).$$

Thus, m is finitely additive. let sequence E_i be a infinite sequence of pairwise disjoint measurable sets.

$$\left(\bigcup_{i=1}^{\infty} E_i\right) \supset \left(\bigcup_{i=1}^{n} E_i\right)$$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m\left(\bigcup_{i=1}^{n} E_i\right)$$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{n} mE_i$$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{n} m(E_i).$$

Since the inequality independent of n we have,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m(E_i) \longrightarrow (3).$$

Reverse inequality follows countable additivity,

$$\therefore m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i) \longrightarrow (4).$$

(3) & (4) $\Rightarrow$ we get

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

PREPOSITION: 14

Let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets. That is, a sequence with $E_{n+1} \subset E_n$ for each n. Let mE_1 be finite then $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} mE_n$.

Proof: Given that sequence E_n is infinite.

Decreasing sequence of measurable sets.

(ie) $E_{n+1} \in E_n$ for each n.

Let $E = \bigcap_{i=1}^{\infty} E_i$ $\quad F_i = E_i \sim E_{i+1}$

$\therefore E_1 \sim E = \bigcup_{i=1}^{\infty} F_i \longrightarrow (1)$ where F_i are pairwise disjoined.

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Proof the claim:

Let $x \in E_1 \sim E$

$x \in E_1$ and $x \notin E = \bigcap_{i=1}^{\infty} E_i$

$x \in E_1$ and $x \in E_i$ for some i.

(i.e) $E_1 \sim E = E_1 - \bigcap_{i=1}^{\infty} E_i$

$= E_1 \cap \left(\widetilde{\bigcap_{i=1}^{\infty} E_i}\right)$

$= E_1 \cap \left(\bigcup_{i=1}^{\infty} \widetilde{E_i}\right)$

$= E_1 \cap (\widetilde{E_1} \cup \widetilde{E_2} \cup \cdots \cup \widetilde{E_i} \cdots)$

$= (E_1 \cap \widetilde{E_1}) \cup (E_1 \cap \widetilde{E_2}) \cup \cdots \cup (E_1 \cap \widetilde{E_i}) \cdots$

$= \phi \cup (E_1 \cap \widetilde{E_2}) \cup \cdots \cup (E_i \cap E_{i+1})$

$= (E_1 \cap \widetilde{E_2}) \cup \cdots \cup (E_{i-1} \cap \widetilde{E_i}) \cup \cdots$

$= (E_1 \sim E_2) \cup (E_2 \sim E_3) \cdots \cup (E_{i-1} \sim E_i) \cdots$

$= F_1 \cup F_2 \cup \cdots \cup F_{i-1} \cdots$

$E_1 \sim E = \bigcup_{i=1}^{\infty} F_i \quad [\because F_i = E_i - E_{i+1}]$

To prove: $\{F_i\}^{\infty}$ are pairwise disjoint.

$F_i \cap F_j = (E_i \sim E_{i+1}) \cap (E_j \sim E_{j+1})$

$= (E_i \cap \widetilde{E}_{i+1}) \cap (E_j \cap \widetilde{E}_{j+1})$

$= (E_i \cap E_j) \cap (\widetilde{E}_{i+1} \cap \widetilde{E}_{j+1})$

$= (E_{i+1} \cap E_{j+1})$.

$F_i \cap F_j \neq \phi$

(25) Then equation ①

$m(E_1 \sim E) = m\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} mF_i$

$= \sum_{i=1}^{\infty} m(E_i \sim E_{i+1}) \longrightarrow (2)$

$E = \bigcap_{i=1}^{\infty} E_i \subset E$.

$\Rightarrow E_i = E \cup (E_1 \sim E)$.

There exists $E_1 \sim E$ are disjoint

$\Rightarrow mE_1 = mE + m(E_1 \sim E) \longrightarrow (3)$ Also $E_{i+1} \subset E_i$

$\Rightarrow E_i = E_{i+1} \cup (E_i \sim E_{i+1})$.

$\Rightarrow E_{i+1}$ and $(E_i \sim E_{i+1})$ are disjoint.

$\Rightarrow mE_i = mE_i + 1 + m(E_i \sim E_{i+1}) \longrightarrow (4)$

Since $mE_1 \leq mE_1 < \infty$.

$(3) \Rightarrow m(E_1) - m(E) = m(E_1 \sim E)$.

$= \sum_1^{\infty} m(E_i \sim E_{i+1})$ (by (2))

$= \lim_{n \to \infty} \sum_1^{\infty} m(E_i - E_{i+1})$

$= \lim_{n\to\infty} [m(E_1 - E_2) + m(E_2 - E_3) + \cdots + m(E_n - E_{n+1})]$

$= \lim_{n\to\infty} (mE_1 - mE_{n+1})$

$mE_1 - mE = mE_1 - \lim_{n\to\infty} mE_{n+1}$

$m(E) = \lim_{n\to\infty} mE_{n+1}$

$m(E) = \lim_{n\to\infty} mE_{n+1} = \lim_{n\to\infty} mE_n$

$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$ Hence Proved.

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PREPOSITION: 15 Let E be a given set then the following five statements are equivalent. (29)

(i) E is measurable.

(ii) Given $\epsilon>0$. There is an open set $O \supset E$ with $m^*(O \sim E) <$

(iii) Given $\epsilon>0$, There is a closed set $F \subset E$ with $m^*(E \sim F) <$

(iv) There is a G in G_δ with $E \subset G$, $m^*(G \sim E)=0$.

(v) There is an F in F_σ with $F \subset E$. $m^*(E \sim F)=0$. If m^*E is finite, the above statements are equivalent to:

(vi) Given $\epsilon>0$, There is a finite union of open intervals such that $m^*(U \Delta E) < \epsilon$.

Proof: Assume that E is measurable.

Let $\epsilon>0$ be given that there exists a countable collection of open intervals. $\{I_n\}$ which covers E.

Such that $E \subset \cup \{I_n\} \longrightarrow (1)$

$$m^*E \leq m^*(\cup\{I_n\})$$
$$\leq \sum m^* I_n \quad [\because l(I_n) = m^*A].$$
$$\leq \sum l(I_n).$$
$$\leq m^*E + \epsilon \longrightarrow (2)$$

Let $O = \cup I_n$ be the open set.

$\therefore E \subset O$ [by (1)]

Since $O = \cup I_n$.

$$m^*O = m^*(\cup I_n).$$
$$= \sum m^* I_n.$$
$$= \sum l(I_n).$$

$m^*O < m^*E + \epsilon$

$m^*O - m^*E < \epsilon$ (30)

$m^*(O \sim E) < \epsilon$

To prove (ii) $\Longrightarrow$ (iv).

From (ii) $\forall n \in Z^+$ there exist an open set $O_n \supset E$ with $m^*(O_n \sim E) < 1/n$.

Let $G = \bigcap_{n=1}^{\infty} O_n \supset E$.

Intersection of countable collection of open set G_δ.

The countable union of closed sets.

Then $G \in G_\delta$ and $E \subset G$.

Since $G \subset O_n \forall n$.

$G \sim E \subset O_n \sim E$.

$m^*(G \sim E) \leq m^*(O_n \sim E)$.

$m^*(G \sim E) < 1/n = 0$

$m^*(G \sim E) = 0$

To prove (iv) $\Rightarrow$ (i):

From (iv) If $G \in G_\delta$. E is a subset of G and $m^*(G \sim E) = 0$.

W.k.t If $m^*E = 0$.

Then E is measurable.

We have $G \sim E$ is measurable.

$\Rightarrow G \in \tilde{E}$ is measurable.

$\Rightarrow$ G and $\tilde{E}$ is measurable.

$\therefore \Rightarrow E$ is measurable [$\because \tilde{E}$ is measurable].

(31) Hence (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

To prove: (i) $\Rightarrow$ (iii).

From (i) E is measurable.

$\Rightarrow \tilde{E}$ is measurable.

Let $\epsilon > 0$ be given then there exist an open set O $\ni$ $\tilde{E} \subset O$.

$m^*(O \sim \tilde{E}) < \epsilon$ (by (ii)).

Let $F = \tilde{O}$ $\longrightarrow$ (1)

Then F is closed. Since $\tilde{E} \subset O$, $E \supset \tilde{O}$.

$\Rightarrow E \supset F$

$\Rightarrow F \subset E$ (from (1)).

Since $m^*(O \sim \tilde{E}) < \epsilon$

$m^*(\tilde{F} \cap E) < \epsilon$

$m^*(E \cap \tilde{F}) < \epsilon$

~~$m^*(E \cap$~~ $m^*(E \sim F) < \epsilon$

To prove (iii) $\Rightarrow$ (v)

From (iii) $\forall n \in Z^+$.

$m^*(E \sim F) < \epsilon$.

5. **MEASURABLE FUNCTIONS:** An Extended real-Valued function f is said to be (Legbage) measurable if its domain is measurable and if it satisfies one of the following statements.

(i) for each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

(ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

(32) (iii) For each real number α the set $\{x : f(\alpha) < \alpha\}$ is measurable.

(iv) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

PREPOSITION: 18: Let f be an extended real-Valued function whose domain is measurable. Then the following statements are equivalent.

(i) For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

(ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable

(iii) For each real number α the set $\{x : f(\alpha) < \alpha\}$ is measurable.

(iv) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

These statements $\Rightarrow$ (v) For each extended real number α the set $\{x : f(x) = \alpha\}$ is measurable.

Proof: Let the domain of f be D.

$f(m)^{\alpha} \Rightarrow D$ is measurable set.

(1) $\Rightarrow$ (4) Already we know that

$\{x : f(x) \leq \alpha\} = D \sim \{x : f(x) > x\}$.

Since "The difference of two measurable set is measurable."

$\therefore$ The set $\{x : f(x) \leq \alpha\}$ is measurable.

$\therefore$ (1) $\Rightarrow$ (4) Similarly we get prove (4) $\Rightarrow$ (1) also

We can prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Next prove (1) $\Rightarrow$ (2).

We know that,

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$$

Each of the set $x \ni f(x) > \alpha - 1/n$ is measurable by (i) also the "intersection of a sequence of measurable set is measurable".

$\therefore$ The set $\{x : f(x) \geq \alpha\}$ is measurable.

Hence (1) $\Rightarrow$ (2).

Next to prove that (2) $\Rightarrow$ (1).

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + 1/n\}$$

Each of the set $\{x : f(x) \geq \alpha + 1/n\}$ is measurable by equation (ii) also union of sequence of measurable set is also measurable.

$\therefore$ The set $\{x : f(x) > \alpha\}$ is measurable.

(ii) $\Rightarrow$ (i) Simillarly (iii) $\Rightarrow$ (iv).

The four statment are equivalent.

We prove $\underline{II} = \underline{V}$

Now, $\{x : f(x) = \alpha\} = \{x : f(x) \leq \alpha\} \cap \{x : f(x) > \alpha\}$ $\forall \alpha$

Hence intersection of two measurable set is also measurable. We prove that (ii) (iv) $\Rightarrow$ (v).

Suppose $\infty = \infty$ then $\{x : f(x) = \infty\}$

$$= \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}.$$

(ii) $\Rightarrow$ (v) simillarly for $\infty = -\infty$

We have (v) $\Rightarrow$ (v) and we have (ii), (iv) $\Rightarrow$ (v).

Hence prooved.

PREPOSITION: 19 Let c be a constant and f and g two measurable real-valued functions defined on the same domain. The functions $f+c$, cf, $f+g$, $g-f$ and fg are also measurable.

Proof: Step: 1

To prove $f+c$ is measurable.

Using this condition.

$\{x / f(x) + c < \alpha\}$ is measurable.

Now, $\{x / f(x) + c < \alpha\} = \{x / f(x) < \alpha - c\}$ is measurable.

Since $f+c$ is measurable.

$\Rightarrow$ Hence $\{x / f(x) + c) < \alpha\}$ is measurable.

Step: 2 Claim: To prove cf is measurable.

Case (i) If $c = 0$ $cf = 0$.

which is a constant function.

constant function are measurable.

Case (ii) Suppose if $c \neq 0$ then.

$$\{x : cf(x) < \alpha\} = \begin{cases} x : f(x) < \alpha/c & \text{if } c > 0 \\ x : f(x) > \alpha/c & \text{if } c > 0 \end{cases}$$

If any case the right hand side is measurable.

$\{x : f(x) < \alpha\}$ is measure.

Hence cf is measure.

(35) **Step: 3** **Claim:** To prove $f+g$ is measurable.

consider $\{x : (f+g)(x) < \alpha\} = \{x : f(x) + g(x) < \alpha\}$

$= \{x : f(x) < \alpha - g(x)\}$.

Since between any two real numbers there exists a rational number say r such that $f(x) < r < \alpha - g(x)$.

$\{x : (f+g)(x) < \alpha\} = \bigcup_r \{r : f(x) < r\} \cap \{x : g(x) < \alpha\}$.

Since the rationals are countable and since each of $\{x : f(x) < r\}$ and $\{x : g(x) < \alpha\}$ are measurable for every r.

$\therefore$ We get $\{x : (f+g)(x) < \alpha\}$ is measurable.

(ie) $(f+g)$ is measurable.

Step: 4 **Claim:** $g-f$ is measurable.

$g - f = g + (-1) f$

Here $(-1)f$ is measurable.

[$\because$ cf is measurable]

$\therefore$ $-f$ is measurable.

Hence $g-f$ is measurable.

Step: 5 **Claim:** fg is measurable.

We shall prove that f^2 measure.

$\{x : f(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) > -\sqrt{\alpha}\}$

(36) each set in R.H.S is measurable. and $\alpha \geq 0$,

Hence $\{x : f^2(x) < \alpha\}$ is measurable.

$\therefore$ f^2 is measurable.

Now, $fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$

$\therefore$ fg is measurable.

DEFINITION: If $\{f_n\}$ is a sequence of function define its limit superior.

$$\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$$

We define the limit inferior by

THEOREM: 20 Let $\langle f_n \rangle$ be a sequence of measurable function (with the same domain of definition) Then the functions

i) $\sup\{f_1, f_2, \ldots f_n\}$. (ii) $\inf\{f_1, f_2 \ldots f_n\}$ (iii) $\sup_n f_n$

iv) $\inf_n f_n$ v) $\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$.

vi) $\underline{\lim} f_n = \sup_n \inf_{k \geq n} f_k$.

Proof: i) Let $h = \sup\{f_1, f_2 \ldots f_n\}$

$h(x) = \sup\{f_1(x), f_2(x), \ldots f_n(x)\}$

We shall show that $\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Let $h(x) > \alpha$

$\sup f_i(x) > \alpha$

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(ie) Then there exists i such that $f_i(x) > \alpha$.

$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$.

Now let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Then there exists some i such that $f_i(x) > \alpha$.

Since $h(x) \geq f_i(x), \forall i$.

$\Rightarrow h(x) > \alpha$

$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$.

Since R.H.S is a union of measurable sets.

Its measurable set $[x : h(x) > \alpha]$.

(ie) $h(x)$ is measurable.

(ii) Let $g = \inf \{f_1, f_2, \dots f_n\}$.

$g(x) = \inf\{f_1(x), f_2(x), \dots f_n(x)\}$

We show that,

$\{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} (x : f_i(x) < \alpha)$

Let $g(x) < \alpha$.

$\inf f_i(x) < \alpha$.

(ie) There exists i such that $f_i(x) < \alpha$.

$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$.

Then there exists some i such that

$f_i(x) < \alpha$.

Since $g(x) \leq f_i(x)$

$\Rightarrow g(x) < \alpha$.

$\therefore \{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$.

Since R.H.S is a union of measurable set its measurable set.

$\therefore \{x : g(x) < \alpha\}$.

(ie) $g(x)$ is measurable.

(iii) Let $g_1(x) = \sup_n f_n(x)$.

Then $\{x : g_1(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$.

$\Rightarrow g_1$ is measurable.

(iv) Let $g_2(x) = \inf_n f_n(x)$

Then $\{x : g_2(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) < \alpha\}$

$\Rightarrow g_2$ is measurable.

v) $\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$.

W.K.T $\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$.

$\inf_n \sup \{f_n, f_{n+1}, \dots\}$.

$\Rightarrow \overline{\lim} f_n$ is measurable.

vi) Similarly $\underline{\lim} f_n$ is measurable.

Hence proved.

(19) DEFINITION: ALMOST EVERYWHERE:

A property is said to hold almost every where (a.e) if the set points where it fails to hold is a set of measure zero.

In particular we say that $f = g$ a.e if f and g have the same domain and

$$m\{x : f(x) \neq g(x)\} = 0.$$

PREPOSITION: 21 If f is a measurable function and $f = g$ a.e then g is measurable.

proof: Let α be a positive real number

Let $E = \{x : f(x) \neq g(x)\}$ By hypothesis, $mE = 0$. If $g(x) > \alpha$.

$x \in E$ and $g(x) > \alpha$ (or) $x \notin E$ and $f(x) > \alpha$

Then $\{x : g(x) > \alpha\} = \{x \in E : g(x) > \alpha\} \cup \{x \notin E : f(x) > \alpha\}$

$= \{x \in E : g(x) > \alpha\} \cup \{x : f(x) > \alpha\} \sim \{x \in E : g(x) \leq \alpha\}$.

$= \{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} \sim \{x \in E : g(x) \leq \alpha\}$.

Since f is measurable function.

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable.

The sets $\{x \in E : g(x) > \alpha\}$ $\{x \in E : g(x) \leq \alpha\}$ are measurable. Since they are subsets of E and $mE = 0$

$\therefore \{x : g(x) > \alpha\}$ is measurable for each α.

(20) Hence g is measurable.

CHARACTERSTIC FUNCTION: If A is any set, we define the characterstic function φ_A of the set A to be the function given by,

$$\varphi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

NOTE: The function χ_A is measurable iff A is measurable

SIMPLE: A real-valued function ϕ is called simple. It is measurable and assumes only a finite number of values. If ϕ is simple and has the values $\alpha_1, \alpha_2, \dots \alpha_n$. Then $\phi = \sum_{i=1}^{n} \alpha_i A_i$. Where $A_i = \{x : \phi(x) = \alpha_i\}$

NOTE: The sum, product and difference of two simple functions are simple.

BOREL MEASURABILITY: A function f is said to be Borel measurable if for each α the set $\{x : f(x) > \alpha\}$ is a borel set.

NOTE: Every Borel measurable function is lebesge measurable.

LITTLEWOOD'S THREE PRINCIPLES:

PREPOSITION: 23 (X) Let E be a measurable set of finite measure and $\langle f_n \rangle$ a sequence of measurable.

function defined on E, Let f be a real-valued (41)
function such that for each x in E.

We have $f_n(x) \to f(x)$. Then given $\epsilon > 0$ and $\delta > 0$, there is a measure set $A \subset E$ with $mA < \delta$ and an integer N, such that for all $\notin A$ and all $n \geq N$.

$|f_n(x) - f(x)| < \epsilon$.

Proof: Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$ and

set $E_N = \bigcup_{n=N}^{\infty} G_n$.

$= \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$ for some $n \geq 1$.

and $E_{N+1} = \bigcup_{n=N}^{\infty} G_n$ we have $E_{N+1} \subset E_N$.

Since $f_n(x) \to f(x)$ $\forall x$ in E.

We get for each $x \in E$ there must be some E_N.

Such that $x \notin E_N$.

Hence $\cap E_N = \phi$.

[If $\cap E_N \neq \phi$ there exists a $x \in E$ such that $x \in E_N$ for every N].

Which is contradiction to our assumption.

Each G_n is measurable set.

$\therefore E_n$ is measurable for all N.

Since E is of finite measure and we have E_N's are finite.

(42)

Then by proposition (5).

"Since $\{E_n\}$ is a finite decreasing sequence of measurable sets, we have,

$$m\left(\bigcap_{n=1}^{\infty} E_N\right) = \lim_{n \to \infty} mE_N\text{"}.$$

$\Rightarrow \lim_{n \to \infty} mE_N = m(\phi) = 0$.

$\Rightarrow \lim_{n \to \infty} mE_N = 0$.

Hence Given $\delta > 0$ there exists N, such that.

$mE_N < \delta$, $\forall n \geq N$.

(ie) $m\{x \in E : |f_n(x) - f(x)| \geq \epsilon, n \geq N\} < \delta$.

Let this E_N be denoted by A.

Then $mA < \delta$ and

$\tilde{A} = \{x \in E : |f_n(x) - f(x)| < \epsilon, \forall n \geq N$.

Hence the proof.

——— x ——— x ———