UNIT-II [I - RIEMANN INTEGRAL]

Let f be a bounded real valued function defined on the interval $[a,b]$ and let $a=\xi_0<\xi_1<\xi_2<\dots\leq\xi_n=b$. be a subdivision of $[a,b]$.

Then for each subdivision, we can defined the sums.

$$S=\sum_{i=n}^{n}(\xi_i-\xi_{i-1})M_i$$

$$s=\sum_{i=1}^{n}(\xi_i-\xi_{i-1})m_i$$

Where $M_i=\sup f(x)$

$\xi_i<x\leq\xi_{i-1}$

Then we define the upper riemann integral of f by $R\overline{\int_a^b} f(x)dx=\inf S$

Where the inf taken over all the possible subdivision of $[a,b]$. Simillarly, we define the lower integral

$$R\int_{\underline{a}}^{b} f(x)dx=\sup s.$$

We know that $R\overline{\int_a^b} f(x)dx\leq\int_{\underline{a}}^{b} f(x)dx$

If the two integrals all equal, we say that f is Riemann integral and the common value is called Riemann integral of f.

(44) (i.e) $R\overline{\int_a^b} f(x)dx=R\int_{\underline{a}}^{b} f(x)dx=R\int_a^b f(x)dx$

Step function: By a step function we mean a function "ψ" which has the form.

$\psi(x)=c_i$ where $\xi_{i-1}<x<\xi_i$ for some subdivision of $[a,b]$ and some set of constants c_i.

$$\therefore\int_a^b \psi(x)dx=\sum_{i=1}^{n}\epsilon_i(\epsilon_i-\epsilon_{i-1})$$

$\therefore$ We can see that,

$$R\overline{\int_a^b} f(x)dx=\inf\int_a^b \psi(x)dx \text{ for all step functions}$$

$\psi(x)\geq f(x)$ simillarly,

$$R\int_{\underline{a}}^{b} f(x)dx=\sup\int_a^b \phi(x)dx. \text{ for all step functions. } (1)$$

$\phi(x)\leq f(x)$.

PROBLEM: 1. Show that if $d(x)=\begin{cases}0, & x \text{ is irrational}\\ 1, & x \text{ is rational}\end{cases}$

Then $R\overline{\int_a^b} f(x)dx=b-a$, $R\int_{\underline{a}}^{b} f(x)dx=0$.

Soln: Consider a subdivision of $[a,b]$

$a=x_0<x_1<\dots<x_n=b$.

$m_i=\inf\limits_{x\in[x_{i-1},x]} f(x)$

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$m_i = 0$

Since each subinterval contains both rationals and irrationals.

$$S = \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i = 0.$$

$$R\int_{a}^{b} f(x)\,dx = \sup S = 0$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$M_i = 1$.

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) = b-a$$

$$R\int_{a}^{b} f(x)\,dx = \inf S = b-a.$$

CANONICAL FORM OF A SIMPLE FUNCTION:

The canonical form of a simple function ϕ is the expression. $\phi(x) = \sum_{i=1}^{n} a_i \psi_{E_i}(x)$

Where $a_1, a_2, \ldots a_n$ are distinct non-zero real numbers and $E_1, E_2, \cdots E_n$ are non-empty disjoint measurable set. whose union is E (The domain is ϕ).

LEBEGUE INTEGRAL OF A SIMPLE FUNCTION:

Consider a simple function ϕ, Then ϕ is measurable and it takes only finite number of values consider the canonical form of ϕ, which is given by.

$$\phi(x) = \sum_{i=1}^{n} a_i \psi_{A_i}(x).$$

Where $A_i = \{x : \phi(x) = a_i\}$

If ϕ vanishes outside a set of finite measure Then the define the integral of ϕ by

$$\int \phi\, dx = \sum_{i=1}^{n} a_i\, m A_i$$

LEMMA 1: Let $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}$ with $E_i \cap E_j = \phi$ for $i \neq j$ Suppose each set E_i is a measurable set of finite measure then $\int \phi = \sum_{i=1}^{n} a_i\, m E_i$

proof: Given that $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}$ and

$E_i \cap E_j = \phi$ for $i \neq j$. and E_i's are measurable.

The set $A_a = \{x : \phi(x) = a\} = \bigcup_{a_i = a} E_i$

$\therefore \phi = \sum a\, \psi_{A_a}$ is a canonical representation of ϕ.

$$\therefore \int \phi = \sum a_m A_a \longrightarrow (1)$$

(47) Since $A_a = \bigcup_{a_i = a} E_i$

$$mA_a = m\bigcup_{a_i=a} E_i = \sum_{a_i=a} mE_i$$

$$a_m A_a = a \sum_{a_i=a} mE_i$$

$$\sum_a a_m A_a = \sum_a a\left(\sum_{a_i=a} mE_i\right)$$

$$\sum a_m A_a = \sum_{i=1}^{n} a_i mE_i$$

$$\therefore \int \phi = \sum_{i=1}^{n} a_i mE_i.$$

PROPOSITION: 2 Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then $\int (a\phi + b\psi) = a\int\phi + b\int\psi$. and if $\phi = \psi$ a.e Then $\int\phi \geq \int\psi$.

Proof: I$\Rightarrow$ Given that ϕ and ψ are simple function.

$\therefore$ We can represent both ϕ and ψ in canonical form.

Let $\{A_i\}$ and $\{B_i\}$ be the sets occuring in canonical representation on ϕ and ψ respectively.

Let A_0 and B_0 be the sets where ϕ and ψ vanish

Let E_k be the set obtained by taking the intersection $A_i \cap B_i$.

(48) Then E_k's are disjoint measurable set. We can write the simple functions ϕ and ψ as,

$$\phi = \sum_{k=1}^{N} a_k \psi_{E_k}.$$

$$\psi = \sum_{k=1}^{N} b_k \psi_{E_k}.$$

$$\therefore a\phi + b\psi = \sum_{k=1}^{N} (aa_k + bb_k)\psi_{E_k}.$$

by lemma: 1,

"Let $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}$, where E_i's are disjoint measurable sets of finite measure then $\int\phi = \sum_{i=1}^{n} a_i mE_i$."

We have,

$$\int (a\phi + b\psi) = \sum_{i=1}^{k} (aa_k + bb_k) M E_k$$

II$\Rightarrow$ Let $\phi \geq \psi$ almost everywhere. Then $\phi - \psi \geq 0$ almost everywhere.

In part (i) Let $a=1, b=1$.

$\therefore \int (\phi - \psi) = \int\phi - \int\psi$ but $\phi - \psi \geq 0$ almost everywhere by definition of the integral.

$$\int(\phi - \psi) \geq 0 \quad \text{(ie)} \quad \int\phi - \int\psi \geq 0.$$

$$\int\phi \geq \int\psi$$ Hence the proof.

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PROPOSITION: 3 (8) (49)

Let f be defined and bounded on a measurable set E with mE finite in order that $\inf_{f\le\psi}\int_E \psi(x)dx$

$$\inf_{f\le\psi}\int_E \psi(x)dx = \sup_{f\ge\phi}\int_E \phi(x)dx$$ for all simple functions ϕ and ψ it is necessary and sufficient that f be measurable.

(or)

A bounded function f defined on a measurable set E of finite measure is lebgue integrable iff f is measurable.

proof: NECESSARY PART:

Assume that f is measurable.

To Prove: $\inf_{f\le\psi}\int_E \psi(x)dx = \sup_{f\ge\phi}\int_E \phi(x)dx$.

Given that f is a bounded function on E.

Let M be the bounded of f.

(ie) $-M\le f(x)\le M$, for every $x\in E$

Divide the interval $[-M,M]$ into $2n$ equal parts and consider the sets

$$E_k=\left\{x : \frac{kM}{n}\ge f(x)>\frac{(k-1)M}{n}\right\} \quad (-n\le k\le n)$$

(50)

Clearly, E_k's are measurable and $\{E_k : -n\le k\le n\}$ is a countable collection of pairwise disjoint measurable sets and this union is E.

(i.e) $E=\bigcup_{k=-n}^{n} E_k$, $E_i\cap E_j=\phi$ for $i\ne j$

$$mE = m\bigcup_{k=-n}^{n} E_k$$

$$mE=\sum_{k=-n}^{n} mE_k \longrightarrow (1)$$

We define two simple functions $\phi_n(x)$ and $\psi_n(x)$ as,

$$\phi_n(x)=\frac{M}{n}\sum_{k=-n}^{n}(k-1)\psi_{E_k}(x)$$

$$\psi_n(x)=\frac{M}{n}\sum_{k=-n}^{n}k\,\psi_{E_k}(x)$$

clearly, $\phi_n(x)\le f(x)\le\psi_n(x)$

Now, $$\int_E \psi_n(x)dx=\frac{M}{n}\sum_{k=-n}^{n} k\,mE_k \longrightarrow (2).$$

$$\int_E \phi_n(x)dx=\frac{M}{n}\sum_{k=-n}^{n}(k-1)\,mE_k \longrightarrow (3)$$

W.k.T $\inf_{f\le\psi}\int_E \psi(x)dx \le \int_E \psi_n(x)dx$

(51) Now by (2)

$$\inf_{f\le\psi}\int_E \psi(x)dx \le \frac{M}{n}\sum_{k=-n}^{n} k\, mE_k$$

Similarly, W.K.T

$$\sup_{f\ge\phi}\int_E \phi(x)dx \ge \int_E \phi_n(x)\,dx$$

$$\sup_{f\ge\phi}\int_E \phi(x)dx \ge \frac{M}{n}\sum_{k=-n}^{n}(k-1)\, mE_k$$

Hence, $0\le \inf\int_E \psi(x)dx - \sup\int_E \phi(x)dx$

$$\le \frac{M}{n}\sum_{k=-n}^{n} k\,mE_k - \frac{M}{n}\sum_{k=-n}^{n}(k-1)\,mE_k$$

$$= \frac{M}{n}\sum_{k=-n}^{n} mE_k$$

(ie) $0\le \inf\int_E \psi(x)dx - \sup\int_E \phi(x)dx$

$$\le \frac{M}{n}\sum_{k=-n}^{n} mE_k$$

Since n is arbitrary we have

$$\inf\int_E \psi(x)dx - \sup\int_E \phi(x)dx = 0.$$

(52) (ie) $\inf_{f\le\psi}\int_E \psi(x)\,dx = \sup_{f\ge\phi}\int_E \phi(x)\,dx$

SUFFICIENT PART: Suppose that $\inf_{f\le\psi}\int_E \psi(x)dx = \sup_{f\ge\phi}\int_E \phi(x)dx$ for all simple function ϕ and ψ.

(2.5m)

To prove: "f" is measurable.

Given: "n" there are simple functions ϕ_n and ψ_n such that, $\phi_n(x)\le f(x)\le \psi_n(x)$ and

$$\int_E \psi_n(x)dx - \int_E \phi_n(x)dx < 1/n$$

Then the functions $\psi^* = \inf \psi_n$

$\phi^* = \sup \phi_n$ are measurable and

$\phi^*(x)\le f(x)\le \psi^*(x)$. Define the set

$$\Delta_\nu = \{x : \phi^*(x) < \psi^*(x) - 1/\nu\} \text{ for each } \nu.$$

Then $\Delta = \{x : \phi^*(x) \le \psi^*(x)\}$.

But each Δ_ν is contained in the set

$$\{x : \phi_n(x) < \psi_n(x) - 1/\nu\}$$

$$\therefore m\{x : \phi_n(x) < \psi_n(x) - 1/\nu\} < \nu/n$$

Since "n" is arbitrary.

$m\{x : \phi_n(x) < \psi_n(x) < 1/\nu\} = 0.$

$m\Delta_\nu \le m\{x : \phi_n(x) < \psi_n(x) - 1/\nu\} = 0$

(i.e) $m\Delta_\nu = 0$

$\Rightarrow m\Delta = 0$

Thus $\phi^* = \psi^*$ except on a set of measure zero.

(i.e) $\phi^* = f$ except on a set of measure zero.

(i.e) $\phi^* = f$ almost everywhere and ϕ^* is measurable.

Hence f is measurable.

LEBEGUE INTEGRAL:

If f is a bounded measurable function defined on a measurable set E with mE finite, we define the lebegue integral of f over E by,

$$\int_E f(x)\,dx = \inf \int_E \psi(x)\,dx \text{ for all simple}$$

functions $\psi \le f$.

NOTE:

1. If $E = [a,b]$ we write $\int_E f = \int_a^b f$
2. If f is a bounded measurable function that vanishes outside a set E of finite measure, we write, $\int f = \int_E f$

Also, $\int_E f = \int_V F V F E$

PROPOSITION: 4 Let f be a bounded function defined on $[a,b]$, If f is Riemann integrable on $[a,b]$. Then it is measurable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

Proof:

Given f is bounded on $[a,b]$ and f is Riemann integrable on $[a,b]$.

To prove: f is measurable and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

Since every step function is also a simple function, we have,

$$R\int_{-a}^{b} f(x)\,dx \le \sup_{\phi \le f} \int_a^b \phi(x)\,dx \le \inf_{\psi \ge f} \int_a^b \psi(x)\,dx \le R\int_a^{-b} f(x)\,dx$$

Since f is Riemann integrable.

$$R\int_a^{-b} f(x)\,dx = R\int_{-a}^{b} f(x)\,dx = R\int_a^b f(x)\,dx$$

$\therefore$ The inequality becomes,

$$\sup_{\phi \le f} \int_a^b \phi(x)\,dx \le \inf_{\psi \ge f} \int_a^b \psi(x)\,dx = R\int_a^b f(x)\,dx$$

Then by the proposition,

(55) " Let f be defined and bounded on a measurable set E with mE finite."

In order that,

$$\inf_{f\le\psi}\int_E \psi(x)\,dx = \sup_{f\ge\phi}\int_E \phi(x)\,dx$$ for all simple functions ϕ and ψ. It is necessary and sufficient that, f is measurable.

We have f is measurable and

$$\int_a^b f(x)\,dx = R\int_a^b f(x)\,dx$$

Hence the proof.

PROPOSITION: 5 If f and g are bounded measurable function defined on a set E of finite measure. Then

1. $\int_E (af+bg) = a\int_E f + b\int_E g$.
2. If $f=g$ almost everywhere. Then $\int_E f = \int_E g$
3. If $f\le g$ a.e. Then $\int_E f \le \int_E g$ hence $\left|\int_E f\right| \le \int_E |f|$.
4. If $A\le f(x)\le B$ Then $A\,mE \le \int_E f \le B\,mE$
5. If A and B are disjoint measurable of finite measure. Then $\int_{A\cup B} f = \int_A f + \int_B f$

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(56) Proof :(1) If ψ is a simple function. Then $a\psi$ is also a simple function for $a>0$ $\int_E af = \inf_{\psi\ge f}\int_E a\psi(x)\,dx$

$$= a\inf_{\psi\ge f}\int_E \psi(x)\,dx$$

$$= a\int_E f \qquad \left[\because \int_E f = \inf_{\psi\ge f}\psi(x)\,dx\right]$$

If $a<0$ $\int_E af = \sup_{\phi\le f}\int_E a\phi = a\sup_{\psi\le f}\int_E \phi \longrightarrow (I)$

$$\int_E af = a\int_E f.$$

If ψ_1 is a simple function such that $f\le\psi_1$ and ψ_2 is a simple function such that $g\le\psi_2$. Then $\psi_1+\psi_2$ is a simple function.

$$f+g\le\psi_1+\psi_2$$

$$\therefore \int_E f+g \le \inf_{f+g\le\psi_1+\psi_2}\int_E(\psi_1+\psi_2)\le\int_E\psi_1+\int_E\psi_2$$

$$\therefore \int_E f+g \le \int_E \psi_1 + \int_E \psi_2$$

Taking infimum on right side.

$$\int_E f+g \le \inf_{\psi_1\ge f}\int_E\psi_1 + \inf_{\psi_2\ge g}\int_E\psi_2 = \int_E f + \int_E g$$

$$\therefore \int_E f+g \le \int_E f + \int_E g \longrightarrow ①$$

on the other hand.

$\phi_1 \le f$ and $\phi_2 \le g \Rightarrow (\phi_1+\phi_2)$ is a simple function.

such that $\phi_1+\phi_2 \le f+g$

$$\therefore \int_E f+g \ge \int_E \phi_1+\phi_2 = \int_E \phi_1 + \int_E \phi_2$$

Taking sup on right side,

$$\int_E (f+g) \ge \sup_{\phi_1 \le f} \int_E \phi_1 + \sup_{\phi_2 \le g} \int_E \phi_2 = \int_E f + \int_E g$$

$$\therefore \int_E (f+g) \ge \int_E f + \int_E g \longrightarrow ②$$

from ① and ②

$$\int_E (f+g) = \int_E f + \int_E g \longrightarrow II$$

from I and II

$$\int_E (af+bg) = a\int_E f + b\int_E g$$

(ii) to prove $\int_E f = \int_E g$

It is sufficient to prove that $\int_E (f-g) = 0$. Since $f-g=0$ almost everywhere.

It follows that if $\psi \ge f-g$, then $\psi \ge 0$ a.e.

$\therefore \int_E \psi \ge 0$. (i.e) $\int_E f-g \ge 0$.

Similarly $\int_E (f-g) \le 0$.

hence $\int_E f-g = 0$.

(i.e) $\int_E f = \int_E g$ when $f=g$ a.e.

(iii) Given that $f \le g$ a.e

(i.e) $f-g \le 0$ a.e

if $\phi \le f-g$.

$\phi \le 0$ a.e

$\therefore \int_E \phi \le 0$ a.e

$$\sup_{\phi \le f-g} \int_E \phi \le 0$$

$$\int_E f-g \le 0 \Rightarrow \int_E f \le \int_E g$$

W.K.T $f \le |f|$

$$\therefore \int_E f \le \int_E |f|$$

Similarly, $-|f| \le f \Rightarrow -\int_E |f| \le \int_E f$

(ie) $-\int_E |f| \le \int_E f \le \int_E |f|$

hence $\left|\int_E f\right| \le \int_E |f|$

(iv) Note that, $\int_E 1 = mE$

Given that $A \le f(x) \le B$

$\therefore \int_E A \le \int_E f(x) \le \int_E B \Rightarrow A\int_E 1 \le \int_E f(x) \le B\int_E 1$

$\Rightarrow AmE \le \int_E f(x) \le BmE$

(v) Since A and B are disjoint. characterstic.

$\varphi_{A\cup B} = \varphi_A + \varphi_B$

$\therefore \int_{A\cup B} f = \int f\ \varphi_{A\cup B}$

$= \int f\varphi_A + \int f\varphi_B$

$\therefore \int_{A\cup B} f = \int_A f + \int_B f$

Hence the proof.

BOUNDED CONVERGENCE THEOREM:

Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \le M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E Then $\int_E f = \lim \int_E f_n$.

Proof: The conclusion of the Preposition would be trivial. If $\{f_n\}$ convergence to f uniformly by little wood's three principle.

"Let E be a measurable set of finite measure and $\{f_n\}$ be a sequence of measurable functions defined on E."

Let f be a real valued functions.

such that for each x in E. We have $f_n(x) \to f(x)$.

Then given $\epsilon > 0$ and $\delta > 0$. There is a measurable set $A \subset E$ with $mA < \epsilon/4m$ such that for $n \ge N$ and $x \in E \sim A$. We have,

$|f_n(x) - f(x)| < \epsilon/2mE$ Thus

$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right|$

$\le \int_E |f_n - f|$

$= \int_{E\sim A} |f_n - f| + \int_A |f_n - f|$

$< \int_{E\sim A} \epsilon/2mE + \int_A 2m$

$$= \frac{\epsilon_1}{2mE} m(E\sim A) + 2m \cdot mA$$

$$\leq \frac{\epsilon_1}{2mE} mE + 2m \frac{\epsilon_1}{4m}$$

$$\leq \epsilon_1/2 + \epsilon_1/2 < \epsilon_1$$

(ie) $\left| \int_E f_n - \int_E f \right| < \epsilon_1$ for every $n \geq N$.

$$\therefore \lim \int_E f_n = \int_E f$$

Hence the proof

PROPOSITION: 7

A bounded function f on $[a,b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

THE INTEGRAL OF NON-NEGATIVE FUNCTIONS:

If f is a non-negative measurable function defined on a measurable on E. We define,

$$\int_E f = \sup_{h \leq f} \int_E h$$

Where h is a bounded measurable function such that $m\{x : h(x) \neq 0\}$ is finite.

PROPOSITION: 8 If f and g are non-negative measurable functions. Then i) $\int_E cf = c\int_E f$ $c>0$.

(ii) $\int_E f+g = \int_E f + \int_E g$

(iii) If $f \leq g$ a.e then $\int_E f \leq \int_E g$.

Proof:

(i) If h is bounded measurable function. Then ch is also a bounded measurable function. for $c>0$ $\int_E cf = \sup_{h \leq cf} \int_E ch$

$$= c \sup_{h \leq cf} \int_E h \Rightarrow c\int_E f$$

(ii) Given that f and g are non-negative measurable function such that,

$$h(x) \leq f(x) \text{ and } k(x) \leq g(x)$$

$$\therefore h(x) + k(x) \leq f(x) + g(x)$$

$$\Rightarrow \int_E h + \int_E k \leq \int_E f+g$$

Taking supremum

$$\sup_{h \leq f} \int_E h + \sup_{k \leq g} \int_E k \leq \int_E f+g$$

$$\int_E f + \int_E g \leq \int_E f+g \longrightarrow (1)$$ on the otherhand,

Let l be a bounded measurable function which vanishes outside a set of finite measure and which is not grater than $f+g$. (ie) $l \leq f+g$.

(63) Then we define the functions h and k by setting,

$h(x) = \min\{f(x), l(x)\}$ and

$k(x) = l(x) - h(x)$

$\therefore h(x) \leq f(x)$ and $k(x) \leq g(x)$.

Where h and k are bounded by bounded for "l" and vanish, where l vanishes.

$$\therefore \int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g$$

$$\int_E l \leq \int_E f + \int_E g$$

Taking supremum

$$\sup_{l \leq f+g} \int_E l \leq \int_E f + \int_E g.$$

$$\int_E f+g \leq \int_E f + \int_E g \longrightarrow (2)$$

from (1) and (2)

$$\int_E f+g = \int_E f + \int_E g.$$

(iii) Given $f \leq g$ a.e

If $\varphi = f - g$ Then $\varphi \leq 0$ a.e

(64) Let h be a bounded measurable function such that $h \leq f - g \leq 0$

$$\Rightarrow \int_E \varphi \leq 0$$

$$\Rightarrow \sup_{\varphi \leq f-g} \int_E \varphi \leq 0 \Rightarrow \int_E f - g \leq 0$$

$$\Rightarrow \int_E f \leq \int_E g$$

Hence the proof.

FATON'S LEMMA: (5)(5M) If $\langle f_n \rangle$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, Then $\int_E f \leq \lim \int_E f_n$.

Proof: Given that let $\{f_n\}$ be a sequence of non-negative measurable function and $f_n \to f$ a.e on E then we have to show that $\int_E f \leq \lim_{n\to\infty} \int_E f_n$

without loss of generality we may assume that the sequence $\{f_n\}$.

convergence to f everywhere on E.

Since the integrals over the sets of measurable zero are zero.

Let h be a bounded measurable function. such that (65)
$h \leq f$ and which vanishes outside a set E' of finite measure. Where.

$$E' = m(\{x \in E : h(x) \neq 0\}) < \infty .$$

define a function h_n by setting.

$$h_n(x) = \min \{h(x), f_n(x)\}.$$

Then it is clear that each h_n is bounded by bound of h and vanish outside E'. Moreover,

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\{h(x), f_n(x)\}$$

$$= \min\{h(x), \lim_{n\to\infty} f_n(x)\}$$

$$= \min\{h(x), f(x)\}$$

$$= h(x) \quad x \in E$$

Thus $\{h_n\}$ is a uniformly bounded sequence of measurable function such that.

$$h_n \leq h \text{ on } E'.$$

$\therefore$ By the bounded convergence theorem

"If $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n \text{"}.$$

(66)

$$\int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n$$

$$\Rightarrow \int_E h = \int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n \leq \underline{\lim}_{n\to\infty} \int_E f_n$$

Hence, taking the supremum over all $h \leq f$. We get

$$\int_E f \leq \underline{\lim} \int_E f_n$$

Hence proved.

MONOTONE CONVERGENCE THEOREM:

(5 M) Let $\langle f_n \rangle$ be an increasing sequence of nonnegative measurable functions and Let $f = \lim f_n$ almost everywhere Then $\int f = \lim \int f_n$.

Proof: By Theorem 9 We have,

$$\int f \leq \underline{\lim} \int f_n .$$ But for each n We have $f_n \leq f$ and so $\int f_n \leq \int f$. But this implies,

$$\overline{\lim} \int f_n \leq \int f .$$

Hence $\int f = \lim \int f_n$.]

COROLLARY: Let $\{u_n\}$ be a sequence of non-negative measurable functions and let $f = \sum_{n=1}^{\infty} u_n$ then $\int f = \sum_{n=1}^{\infty} \int u_n$ then $\int f = \sum_{n=1}^{\infty} u_n$.

proof: . If f and g are non-negative measurable functions, then $\int f+g = \int f + \int g \longrightarrow (1)$

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67 Consider the n^{th} partial sum of the series $\sum_{n=1}^{\infty} u_i$

(ie) $S_n = \sum_{n=1}^{n} u_i \longrightarrow (2)$

By (1) $\int S_n = \int \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} \int u_i \longrightarrow (3)$

Since $\sum_{n=1}^{\infty} u_n = f$, $S_n \to f$ as $n \to \infty$.

Since $\{u_n\}$ is a sequence of non-negative measurable functions.

$\{S_n\}$ is an increasing sequence and $S_n \to f$.

By monotone convergence theorem,

$$\int f = \lim \int S_n$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int u_i \quad (\text{by } (3))$$

$$\int f = \sum_{i=1}^{\infty} \int u_i$$

(ie) $\int f = \sum_{n=1}^{\infty} \int u_n$.

THEOREM: 12 Let f be a non-negative function a disjoint sequence of measurable sets let $E = \cup E_i$ then $\int_E f = \sum \int_{E_i} f$.

(3 sm)

Proof: Let $u_i = f \psi E_i$

Then $\sum u_i = \sum f \psi E_i$

68 $= f[\psi_{E_1} + \psi_{E_2} + \cdots]$

$= f[\psi_{E_1 \cup E_2 \cup \cdots}]$

$\sum u_i = f \psi_E$

$\int f \psi_E = \int \sum u_i$

$= \sum \int u_i$

$\int f \psi_E = \sum \int f \psi_{E_i}$

$\Rightarrow \int_E f = \sum \int_{E_i} f$.

~~DEFINITION: INTEGRABLE: A non-negative measurable functions. If f is integrable over F and $g(x) < f(x)$ on E. Then g is also integrable on E and $\int_E f - g = \int_E f - \int_E g$.~~

~~Pro~~

DEFINITION: INTEGRABLE: A non-negative measurable function f is called integrable over the measurable set E, if $\int_E f < \infty$.

PREPOSITION: 13 Let f and g be two non-negative measurable functions. If f is integrable over E and $g(x) < f(x)$ on E. Then g is also integrable on E and $\int_E f - g = \int_E f - \int_E g$.

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Proof: Given that f is integrable over E.

(ie) $\int_E f < \infty$.

also given that $g(x) < f(x)$ on E.

$\therefore g(x) - f(x) < 0$, $f(x) - g(x) > 0$.

$(f-g)(x) > 0$ in E.

$\therefore (f-g)$ is non-negative.

Now, $\int_E f = \int_E (f-g+g) = \int_E (f-g) + \int_E g \longrightarrow (1)$

The L.H.S of (1) is finite,

$\therefore$ the R.H.S of (1) is also finite.

$\therefore \int_E (f-g) < \infty$ and $\int_E g < \infty$.

$\therefore g$ is integrable over E.

Now (1) $\Rightarrow \int_E f - \int_E g = \int_E f-g$.

PREPOSITION:14: Let f be a non-negative function which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \epsilon$.

Proof: Suppose that f is bounded on E.

(ie) $|f(x)| \le M$ for every $x \in E$. for given $\epsilon > 0$ take $\delta < \epsilon/M$ if $mA < \delta$. Then,

$$\therefore \int_A f \le \int_A M = M\, mA$$

$$\le M\delta < M < \epsilon/M < \epsilon$$

$$\therefore \int_A f < \epsilon.$$

Now assume that f may not be bounded define a new function f_n by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) < n \\ n, & \text{otherwise.} \end{cases}$$

Then each f_n is bounded. since $|f_n(x)| \le n$ for every x

Also $f_n \le f_{n+1}$ $\forall n$.

$\therefore \{f_n\}$ is an increasing sequence of non-negative functions such that $f_n \to f$ at each point.

By monotone convergent theorem,

$$\int_E f = \lim \int_E f_n.$$

$\therefore$ For given $\epsilon > 0$ $\exists$ a positive integer N such that

$$\left| \int_E f_n - \int_E f \right| < \epsilon, \quad \forall n \ge N.$$

In particular, $\left| \int_E f_N - \int_E f \right| < \epsilon/2$

$$\Rightarrow -\epsilon/2 < \int_E f_N - \int_E f < \epsilon/2$$

$$\Rightarrow -\epsilon/2 < \int_E (f_N - f)$$

$\Rightarrow \int_E (f - f_n) < \epsilon/2$

Take $\delta < \epsilon/2N$, If $mA < \delta$.

Then $\int_A f = \int_A (f - f_N + f_N)$

$$\int_A f = \int_A f - f_n + \int_A f_n$$

$$< \epsilon/n + \int_A N < \epsilon/2 + N \cdot mA$$

$$< \epsilon/2 + N \cdot \delta = \epsilon/2 + N \cdot \epsilon/2N$$

$$= 2\,\epsilon/2$$

$\therefore \int_A f < \epsilon$

Hence proved.

PREPOSITION: 15 Let f and g be integrable over E then

(i) The function cf is integrable over E and $\int_E cf = c\int_E f$.

(ii) The function $f+g$ is integrable over E and $\int_E f+g = \int_E f + \int_E g$

(iii) If $f \le g$ a.e then $\int_E f \le \int_E g$.

(iv) If A and B are disjoint measurable sets contained in E then $\int_{A\cup B} f = \int_A f + \int_B f$.

Proof: To prove (i)

We know that $cf = (cf)^+ - (cf)^-$ where $(cf)^+$ and $(cf)^-$ are positive and negative parts of cf which are non-negative.

$\therefore (cf)^+$ and $(cf)^-$ are integrable over E.

$\therefore \int_E (cf)^+ < \infty$ and $\int_E (cf)^- < \infty$.

$\therefore \int_E (cf) = \int_E (cf)^+ - \int_E (cf)^- < \infty \quad (\because \int_E (cf) < \infty)$.

$\therefore cf$ is integrable over E.

Also, $\int_E cf = c\int_E f^+ - c\int_E f^-$

$$= c\left[\int_E f^+ - \int_E f^-\right] = c\int_E f$$

To Prove: (ii)

Given that f and g are integrable.

$\therefore$ The non-negative function (f^+), (f^-), (g^+) and (g^-) are integrable.

$\therefore \int_E f = \int_E f^+ - \int_E f^- < \infty$. And

$\int_E g = \int_E g - \int_E g^- < \infty$.

(Now $f+g = (f+g)^+ - (f+g)^- = f^+ + g^+ - f^- - g^-$

$$= (f^+ - f) + g^+ - g$$

hence $\int_E (f+g)^+ = \int_E f^+ + \int_E g^+ < \infty$

$$\int_E (f+g)^- = \int_E f^- + \int_E g^- < \infty.$$

$(f+g)^+$ and $(f+g)^-$ are integrable over E.

$$\therefore \int_E (f+g) = \int_E (f+g)^+ - \int_E (f+g)^- < \infty.$$

$\therefore f+g$ is integrable over E.

$$\therefore \int_E (f+g) = \int_E (f^+ + g^+) - \int_E (f^- + g^-)$$

$$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$\int_E (f+g) = \int_E f + \int_E g$$

TO prove(iii):

If $f \le g$ a.e then $g - f \ge 0$ a.e

(ie) $g-f$ is non-negative.

Since the integrable of a non-negative measurable function is also non-negative.

$$\int_E g - f \ge 0$$

$$\Rightarrow \int_E g - \int_E f \ge 0$$

$$\therefore \int_E f \le \int_E g.$$

TO prove: (iv)

$$\int_{A \cup B} f = \int_A f + \int_B f$$

W.K.T $\displaystyle\int_{A\cup B} f = \int f \, \psi_{A\cup B}$

$$= \int f \psi_A + \int f \psi_B$$

$$= \int_A f + \int_B f.$$

THE GENERAL LEBEGUE INTEGRAL:

Consider the function f the positive part f^+ of f is defined by $f^+ = f \vee 0$.

(ie) $f^+ = \max\{f(x), 0\}$.

$$f^+(x) = \begin{cases} f(x), & f(x) \ge 0 \\ 0, & f(x) < 0 \end{cases}$$

Simillarly, We define the negative part f^- of f is defined by $f^- = -f(x) \vee 0$.

(ie) $f^-(x) = \max\{-f(x), 0\}$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \ge 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

If f is measurable then f^+ and f^- are also measurable we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

DEFINITION: A measurable function f is said to be integrable over E. If f^+ and f^- are both integrable over E. In this case we define $\int_E f = \int_E f^+ - \int_E f^-$.

THEOREM: [LEBESGUE CONVERGENCE THEOREM].

Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable function such that $|f_n| \le g$ on E and for almost all x in E. We have $f(x) = \lim f_n(x)$

then $\int_E f = \lim \int_E f_n$.

Proof: Given that $|f_n| \le g$

$\Rightarrow -g \le f_n \le g$.

(i) $f_n \le g \therefore g - f_n \ge 0$.

The function $g - f_n$ is non-negative. By Fatou's Lemma.

$$\int_E g - f \le \underline{\lim} \int_E (g - f_n) \longrightarrow (1)$$

Since $|f_n| \le g$, we have $|f| \le g$.

Since g is integrable, f is also integrable.

$$\therefore (1) \Rightarrow \int_E g - \int_E f \le \int_E g - \underline{\lim} \int_E f_n$$

$$\Rightarrow -\int_E f \le -\underline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \ge \overline{\lim} \int_E f_n \longrightarrow (2)$$

Consider, $-g \le f_n$

$g + f_n \ge 0$.

(ii) $\therefore g + f_n$ is non-negative.

By the Fatou's lemma.

$$\int_E f + g \le \underline{\lim} \int_E f_n + g$$

$$\Rightarrow \int_E f + \int_E g \le \underline{\lim} \int_E f_n + \int_E g.$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \longrightarrow (3)$$

from (2) and (3)

$$\int_E f \le \underline{\lim} \int_E f_n \le \overline{\lim} \int_E f_n \le \lim \int_E f_n$$

$$\therefore \int_E f = \lim \int_E f_n.$$

THEOREM: Let $\{g_n\}$ be a sequence of integral function which converges a.e to an integrable function g. Let $\{f_n\}$ be a sequence of measurable function such that $|f_n| \le g_n$ and $\{f_n\}$ converges to f a.e if $\int g = \lim \int g_n$ then $\int f = \lim \int f_n$.

Proof: Given $\{g_n\}$ be a sequence of integral functions which convergence to g are such that $\{g_n(x)\}$ convergence to $g(x)$ a.e and g is integrable also f_n be a sequence of measurable function such that

$|f_n| \le g_n$ and f_n converges to f a.e also given (77)

$\int g = \lim \int g_n$

$\overline{\lim} \int g_n \le \int g \le \underline{\lim} \int g_n$

to prove $\int f = \lim \int f_n$

since $|f_n| \le g_n$

Then $-g_n \le f_n \le g_n$

$g_n - f_n \ge 0$ and $g_n + f_n \ge 0$.

Consider the function $(g_n - f_n)$ is non-negative.

Since $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$.

Then by the Factor's lemma.

$\int f \le \underline{\lim} \int f_n$

We have $\int g - f \le \underline{\lim} \int g_n - f_n$

$\int g - \int f \le \underline{\lim} \int g_n - \underline{\lim} \int f_n$

$\int g - \int f \le \int g - \underline{\lim} \int f_n$

$-\int f \le -\lim \int f_n$

$\int f \ge \overline{\lim} \int f_n \longrightarrow (1)$

Similarly consider the function

$g_n + f_n$ is non-negative.

$\int g + f \le \underline{\lim} \int g_n + f_n$

$\int g + \int f \le \underline{\lim} \int g_n + \underline{\lim} \int f_n$.

$\int g + \int f \le \int g + \underline{\lim} \int f_n$ (78)

$\int f \le \underline{\lim} \int f_n \longrightarrow (2)$

from (1) and (2)

$\overline{\lim} \int f_n \le f_n < \underline{\lim} \int f_n$

$\therefore \int f = \lim \int f_n$.

THEOREM: Show that if f is integrable over E then so is $|f|$ and $\left|\int_E f\right| \le \int_E |f|$ does the integrability of $|f|$ simply that of f.

Proof: W.K.T $f = f^+ - f^-$

and $|f| = f^+ + f^-$

since f is integrable over E. f^+ and f^- are also integrable over E.

Hence $|f| = f^+ - f^-$ is also integrable.

since $f \le |f|$

$\int_E f \le \int_E |f|$

Also $f \ge -|f|$

$\int_E f \ge -\int_E |f|$

$-\int_E |f| \le \int_E f \le \int_E |f|$

$\Rightarrow \left|\int_E f\right| \le \int_E |f|$

Let $|f|$ be integrable
Since $f \le |f|$, f is also integrable.
Hence the integrability $|f|$ implies that of f.
Hence proved.

UNIT-III [DIFFERENTIATION AND INTEGRATION]

DIFFERENTIATION OF MONOTONE FUNCTION:

Let $\mathcal{J}$ be a collection of intervals then we say that $\mathcal{J}$ covers a set E in the sense of Vitali, if for each $\epsilon > 0$ and any x in E, there is an interval such that $x \in I$ and $l(I) < \epsilon$

NOTE: (i) $\mathcal{J}$ is called as Vitalli's cover of the set E
(ii) the intervals may be open, closed (or) half open.
But we do not allow degenerate intervals consisting of only one point.

VITALI'S LEMMA: Let E be a set of finite outer measure and $\mathcal{J}$ be a collection of intervals that cover E in the sense of Vitali, then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, I_2, \ldots I_N\}$ of intervals in $\mathcal{J}$ such that

$$m^* \left\{E \sim \bigcup_{n=1}^{N} I_n\right\} < \epsilon.$$

Proof: It is sufficient to prove the lemma in the case that each interval $\mathcal{J}$ is closed, for otherwise